Week 02: Method of Characteristics

From now on we will study one by one classical techniques of obtaining solution formulas for PDEs. The first one is the method of characteristics, which is particularly useful when solving first order equations.

1. Classification of first-order equations.

The general form of first-order PDE (in $\mathbb{R}^2$):

$$F(x, y, u, u_x, u_y) = 0, \qquad (x, y) \in D \subset \mathbb{R}^2. \tag{1}$$

or in $\mathbb{R}^3$:

$$F(x, y, z, u, u_x, u_y, u_z) = 0. \tag{2}$$

Often the following notation is used in 2D:

$$p = u_x, q = u_y \tag{3}$$

thus the equation can be written as

$$F(x, y, u, p, q) = 0. \tag{4}$$

The linear equations can be classificed into the following cases, from easier to more difficult:

1. Linear:
$$F(x, y, u, u_x, u_y) = a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u - d(x, y). \tag{5}$$
2. Semi-linear:
$$F(x, y, u, u_x, u_y) = a(x, y)\, u_x + b(x, y)\, u_y - c(x, y, u). \tag{6}$$
3. Quasi-linear:
$$F(x, y, u, u_x, u_y) = a(x, y, u)\, u_x + b(x, y, u)\, u_y - c(x, y, u). \tag{7}$$
4. General case.

2. Method of characteristics.

We try to find a method to solve the general first-order quasi-linear equation

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u). \tag{8}$$

Let's start by thinking geometrically. Consider the 3 dimensional space with coordinates $(x,\ y,\ u)$. Assume that $u = u(x, y)$ is a solution to the equation, it is clear that it represents a surface in the (x, y, u) space.

Now we explore the geometrical meaning of the equation. Observe that the equation can be written in the form of an inner product

$$\begin{pmatrix} a(x, y, u) \\ b(x, y, u) \\ c(x, y, u) \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} a(x, y, u) \\ b(x, y, u) \\ c(x, y, u) \end{pmatrix} \perp \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix}. \tag{9}$$

Therefore all we need to do is to understand the relation between the vector $\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix}$ and the solution.

First introduce a new function $\Psi \colon \mathbb{R}^3 \mapsto \mathbb{R}$ through

$$\Phi(x, y, u) = u - u(x, y). \tag{10}$$

Note that in the RHS of the above, the first u is a variable, the second u is a function. For example, suppose $u(x, y) = x^2 + y^2$, then the corresponding $\Phi(x, y, u) = u - \left(x^2 + y^2\right)$.

Now we easily see that

$$\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = -\begin{pmatrix} \Phi_x \\ \Phi_y \\ \Phi_u \end{pmatrix} = -\nabla\Phi. \tag{11}$$

Recall that geometrically, $\nabla\Phi$ (and also $-\nabla\Phi$) is a normal vector of the surface $\Phi = 0$. As a consequence $\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix}$ is perpendicular to the solution surface $u = u(x, y)$.

On the other hand, from the equation we know that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix}$ which means $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ must be tangent to the surface $u = u(x, y)$.

Now we summarize. We have shown that the equation is equivalent to the geometrical requirement that the vector $\begin{pmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{pmatrix}$ is tangent to the solution surface $u = u(x, y)$. As a consequence, any integral curve of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, that is any $\begin{pmatrix} X(s) \\ Y(s) \\ U(s) \end{pmatrix}$ satisfying

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X, Y, U) \tag{12}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(X, Y, U) \tag{13}$$

$$\frac{\mathrm{d}U}{\mathrm{d}s} = c(X, Y, U) \tag{14}$$

must be contained in one of the solution surfaces. Conversely, any surface "woven" by such integral curves is a solution surface.

The above understanding leads to the following "method of characteristics" due to Lagrange.

Theorem 1. *The general solution of a first-order, quasi-linear PDE*

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u) \tag{15}$$

satisfies

$$F(\phi, \psi) = 0, \tag{16}$$

where F is an arbitrary function of $\phi(x, y, u)$ and $\psi(x, y, u)$, and any intersection of the level sets of ϕ and ψ is a solution of the characteristic equations

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c}. \tag{17}$$

Remark 2. As we will see soon, ϕ, ψ are obtained through solving the characteristic equations. And each F gives a solution to the original equation.

Remark 3. The curves mentioned above are called the families of *characteristic curves* of the equation.

Remark 4. As we will see, the main technique in getting ϕ and ψ is

$$\frac{a}{b} = \frac{c}{d} \Longrightarrow \frac{a \pm c}{b \pm d} = \frac{a}{b} = \frac{c}{d}. \tag{18}$$

In the following, we will show how to apply this method. We start with the simplest case.

3. Solving linear first-order equations.

3.1. Equations with constant coefficients.

We start from the simplest case, where a, b, c and d are just constants.

Example 5. (§2.8, 3(b)) Find the general solution of the equation

$$a\, u_x + b\, u_y = 0; \qquad a, b \text{ are constant.} \tag{19}$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{0}. \tag{20}$$

What we need are two functions $\phi(x, y, u)$ and $\psi(x, y, u)$ such that $\mathrm{d}\phi = 0$, $\mathrm{d}\psi = 0$ along the characteristics.

Obviously we can take $\phi = u$. For ψ, notice that

$$\mathrm{d}(a\,y - b\,x) = a\,\mathrm{d}y - b\,\mathrm{d}x = 0, \tag{21}$$

thus we can take

$$\psi = a\,y - b\,x. \tag{22}$$

As a consequence, the solution satisfies

$$F(a\,y - b\,x, u) = 0 \tag{23}$$

for any function F. This means

$$u = f(a\,y - b\,x). \tag{24}$$

for an arbitrary function f.

Example 6. (Cauchy problem, §2.8, 5(a)) Oftentimes, the value of the solution along some specific curve in the plane is prescribed. For example, solve

$$3\,u_x + 2\,u_y = 0, \qquad u(x, 0) = \sin x. \tag{25}$$

Solution. From the above example we know that the general solution takes the form

$$u = f(2\,x - 3\,y). \tag{26}$$

Now substituting this into the initial condition, we obtain

$$f(2\,x) = \sin x \Longrightarrow f(x) = \sin\frac{x}{2}. \tag{27}$$

Therefore

$$u(x, y) = \sin\frac{2\,x - 3\,y}{2}.$$

Example 7. ($c, d \neq 0$) What happens when the c, d are not 0? The method still works. We write down the characteristic equations

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{d - c\,u}. \tag{28}$$

We have

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} \Longrightarrow \mathrm{d}(b\,x - a\,y) = 0, \tag{29}$$

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}u}{d - c\,u} \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = a^{-1}\,d - a^{-1}\,c\,u \Longrightarrow u = C\,e^{-a^{-1}cx} + c^{-1}\,d \Longrightarrow \mathrm{d}\left(e^{a^{-1}cx}\left(u - c^{-1}\,d\right)\right) = 0 \tag{30}$$

As a consequence

$$\phi = b\,x - a\,y, \quad \psi = e^{a^{-1}cx}\left(u - c^{-1}\,d\right) \tag{31}$$

and

$$F(\phi, \psi) = 0 \tag{32}$$

gives

$$u = c^{-1}\,d + e^{-a^{-1}cx}\,f(b\,x - a\,y). \tag{33}$$

Remark 8. In the above, one may be tempted to conclude

$$\mathrm{d}\left(u - C\,e^{-a^{-1}cx}\right) = 0 \tag{34}$$

and try to use $u - C\,e^{-a^{-1}cx}$ as ψ. This is obviously wrong as d somehow disappeared. One should keep in mind that neither ϕ or ψ can involve arbitrary constants. It is their values that are arbitrary constants.

Example 9. Some times people use the following "method of characteristics":

First solve

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = b \tag{35}$$

then solve

$$u_s + c\,u = d \tag{36}$$

to obtain the solution u in the form $u = u(s, t)$. Finally represent s, t by x, y and obtain the solution. This is equivalent to our method but considerably more complicated to use. Anyone who does not believe this should try using this method to the following examples with non-constant coefficients.

Remark 10. One may wonder, how to find out ϕ and ψ efficiently? Unfortunately there may not be any short-cut. One way to systematically find ϕ and ψ is the following. The characteristic equations consists of three equations. Pick any two of them. If you can find general solutions, then you have ϕ and ψ. But this fails when any coefficient involves all other variables.

3.2. Equations with non-constant coefficients.

Example 11. (§2.8, 3(h)) Find the general solution of

$$y\,u_y - x\,u_x = 1. \tag{37}$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{-x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{1}. \tag{38}$$

Using

$$\frac{\mathrm{d}x}{-x} = \frac{\mathrm{d}y}{y} \tag{39}$$

we obtain

$$y\,\mathrm{d}x + x\,\mathrm{d}y = 0 \implies \mathrm{d}(x\,y) = 0. \tag{40}$$

Thus

$$\phi = x\,y; \tag{41}$$

On the other hand, from

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{1} \tag{42}$$

we obtain

$$\mathrm{d}u = \mathrm{d}\log y \implies \mathrm{d}(u - \log y) = 0. \tag{43}$$

As a consequence we can take

$$\psi = u - \log y. \tag{44}$$

Putting these together we obtain

$$F(x\,y, u - \log y) = 0 \tag{45}$$

which gives

$$u = \log y + f(x\,y). \tag{46}$$

Example 12. (§2.8, 5(c)) Find the solution of the following Cauchy problem:

$$x\,u_x + y\,u_y = 2\,x\,y, \qquad \text{with } u = 2 \text{ on } y = x^2. \tag{47}$$

Solution. We need to first find the general solution, then using the value on $y = x^2$ to determine the arbitrary function involved.

- Find the general solution

  The characteristic equations are

  $$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{2\,x\,y}. \tag{48}$$

  Using

  $$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} \tag{49}$$

we obtain

$$\mathrm{d}\left(\frac{y}{x}\right)=0 \Longrightarrow \text{can take } \phi=\frac{y}{x}. \tag{50}$$

On the other hand, we have

$$\mathrm{d}u=2\,x\,\mathrm{d}y=2\,y\,\mathrm{d}x=x\,\mathrm{d}y+y\,\mathrm{d}x=\mathrm{d}(x\,y) \Longrightarrow \mathrm{d}(u-x\,y)=0 \tag{51}$$

therefore

$$\psi=u-x\,y. \tag{52}$$

The general solution satisfies

$$F\left(\frac{y}{x},u-x\,y\right)=0 \Longrightarrow u=x\,y+f\left(\frac{y}{x}\right). \tag{53}$$

– Determine the solution.

We have $u=2$ along $y=x^2$, that is

$$u\left(x,x^2\right)=2. \tag{54}$$

Using the formula for the general solution, we have

$$x^3+f(x)=2 \Longrightarrow f(x)=2-x^3. \tag{55}$$

As a consequence

$$u(x,y)=x\,y+2-\left(\frac{y}{x}\right)^3. \tag{56}$$

4. Solving semi-linear first-order equations.

Example 13. (§2.8, 3(g)) Find the general solution of the following equation:

$$y^2\,u_x-x\,y\,u_y=x\,(u-2\,y). \tag{57}$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{y^2}=\frac{\mathrm{d}y}{-x\,y}=\frac{\mathrm{d}u}{x\,(u-2\,y)}. \tag{58}$$

From

$$\frac{\mathrm{d}x}{y^2}=\frac{\mathrm{d}y}{-x\,y} \tag{59}$$

we have

$$\mathrm{d}\left(x^2+y^2\right)=0 \Longrightarrow \phi=x^2+y^2. \tag{60}$$

On the other hand, we have

$$\frac{\mathrm{d}y}{-x\,y}=\frac{\mathrm{d}u}{x\,(u-2\,y)} \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}y}=-\frac{u}{y}+2 \Longrightarrow \frac{\mathrm{d}(u-y)}{\mathrm{d}y}=-\frac{u-y}{y} \Longrightarrow \mathrm{d}((u-y)\,y)=0. \tag{61}$$

Thus we take

$$\psi=y\,(u-y). \tag{62}$$

Now

$$F\left(x^2+y^2,y\,(u-y)\right)=0 \tag{63}$$

gives

$$u=y+y^{-1}\,f\left(x^2+y^2\right). \tag{64}$$

Example 14. (§2.8, 5(g)) Solve

$$x\,u_x+y\,u_y=u+1 \qquad \text{with } u(x,y)=x^2 \text{ on } y=x^2. \tag{65}$$

Solution. The characteritic equations are

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}u}{u+1} \tag{66}$$

which easily lead to

$$\phi = \frac{y}{x}, \quad \psi = \frac{u+1}{x}. \tag{67}$$

Thus

$$u = x\, f\left(\frac{y}{x}\right) - 1. \tag{68}$$

Now the Cauchy data implies

$$x\, f(x) - 1 = u\left(x, x^2\right) = x^2 \tag{69}$$

thus

$$f(x) = x + x^{-1}. \tag{70}$$

As a consequence

$$u(x,y) = x\left(\frac{y}{x}\right) + x\left(\frac{y}{x}\right)^{-1} - 1 = y + \frac{x^2}{y} - 1. \tag{71}$$

5. Solving quasi-linear first-order equations.

Example 15. (§2.8, 3(f)) Find the general solution of

$$(y+u)\, u_x + y\, u_y = x - y. \tag{72}$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{y+u} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{x-y}. \tag{73}$$

From this we have

$$\frac{\mathrm{d}x}{y+u} = \frac{\mathrm{d}(y+u)}{x} \implies \mathrm{d}\left(x^2 - (y+u)^2\right) = 0. \tag{74}$$

Thus we can take

$$\phi = x^2 + (y+u)^2. \tag{75}$$

On the other hand, we have

$$\frac{\mathrm{d}(x-y)}{u} = \frac{\mathrm{d}u}{x-y} \implies \mathrm{d}\left(u^2 - (x-y)^2\right) = 0. \tag{76}$$

As a consequence, the solution is given by

$$F\left(x^2 - (y+u)^2, u^2 - (x-y)^2\right) = 0. \tag{77}$$

Remark 16. Note that, in the above we can also use

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}(x+u)}{x+u} \implies \mathrm{d}\left(\frac{x+u}{y}\right) = \text{const}. \tag{78}$$

Thus the formula may not be unique.

Example 17. (§2.8, 5(h)) solve

$$u\, u_x - u\, u_y = u^2 + (x+y)^2 \qquad \text{with } u = 1 \text{ on } y = 0. \tag{79}$$

Solution. The characteristic equation are

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{-u} = \frac{\mathrm{d}u}{u^2 + (x+y)^2}. \tag{80}$$

We have

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{-u} \implies \mathrm{d}(x+y) = 0 \implies \phi = x + y. \tag{81}$$

Then we have

$$\frac{\mathrm{d}y}{-u} = \frac{\mathrm{d}u}{u^2 + \phi^2} \implies \psi = e^{2y}\left(u^2 + \phi^2\right). \tag{82}$$

Now from the Cauchy data we have

$$F\big(\phi, 1+\phi^2\big) \equiv 0. \tag{83}$$

Therefore effectively F has to be

$$F(\phi, \psi) = \psi - \big(\phi^2 + 1\big) \tag{84}$$

and the solution satisfies

$$e^{2y}\left(u^2 + (x+y)^2\right) - \left(1 + (x+y)^2\right) = 0 \tag{85}$$

which leads to

$$u = \pm\left[\left\{1 + (x+y)^2\right\} e^{-2y} - (x+y)^2\right]^{1/2}. \tag{86}$$

6. Equations with more than two variables.

The method of characteristics can be applied to higher dimensional problems with no difficulty in principle – it indeed becomes more difficult in practice!

Example 18. (§2.8, 8(d)) Solve the following equation

$$y\,z\,u_x - x\,z\,u_y + x\,y\left(x^2 + y^2\right) u_z = 0. \tag{87}$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{y\,z} = \frac{\mathrm{d}y}{-\,x\,z} = \frac{\mathrm{d}z}{x\,y\,(x^2+y^2)} = \frac{\mathrm{d}u}{0}. \tag{88}$$

This time we need three invariants, let's denote them by ϕ, ψ and η.

Clearly one can take

$$\phi = u. \tag{89}$$

For the second invariant, we observe

$$\frac{\mathrm{d}x}{y\,z} = \frac{\mathrm{d}y}{-\,x\,z} \implies \frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{-\,x} \implies \mathrm{d}\big(x^2 + y^2\big) = 0 \implies \psi = x^2 + y^2. \tag{90}$$

The last invariant can be obtained through

$$\frac{\mathrm{d}y}{-\,x\,z} = \frac{\mathrm{d}z}{x\,y\,\phi} \implies \frac{\mathrm{d}y}{-\,z} = \frac{\mathrm{d}z}{y\,\phi} \implies \mathrm{d}\big(z^2 + y^2\,\phi\big) = 0 \implies \eta = z^2 + y^2\,\phi. \tag{91}$$

Therefore the solutions are obtained by setting

$$F\big(u, x^2 + y^2, z^2 + y^2\left(x^2 + y^2\right)\big) = 0 \tag{92}$$

which gives

$$u = f\big(x^2 + y^2, z^2 + y^2\left(x^2 + y^2\right)\big) \tag{93}$$

for arbitrary f.[1]

1. Keep in mind that this f will be determined once some Cauchy data is given.