WEEK 01: INTRODUCTION

1. What are partial differential equations.

- An equation:
  - Marriam-Webster Online:

    A usually formal statement of the equality or equivalence of mathematical or logical expressions

  - In mathematics, an equation is such a formal statement involving unknowns. That is, $2^2 = 4$ is not usually referred to as an equation.

- Depending on what kind of quantities the unknowns represent, we can have differential kinds of equations. To mention a few:

  1. An *algebraic equation* is an equation in which the unknowns represent numbers. For example,

  $$x^2 - 3\,x + 5 = 0. \tag{1}$$

  2. A *functional equation* is an equation in which the unknowns represent functions, but not their derivatives or integrals. For example

  $$u(x+y) = u(x)\,u(y). \tag{2}$$

  3. A *differential equation* (henceforth will sometimes be referred to as *DE*) is an equation in which the unknowns represent functions and their derivatives.

     a) An *ordinary differential equation* (*ODE*) is a differential equation in which the underlying space is one-dimensional. In other words all the functions involved depend on one single variable (often denoted by t, as the largest source of ODEs are mechanical systems and the natural variable is time).

     b) A *partial differential equation* (*PDE*) is a differential equation in which at least two independent variables are involved. As a consequence, the derivatives in the equation are partial derivatives, and hence the name.

  Example 1. The equation

  $$u_t = u^2 \tag{3}$$

  is an ODE, while the equation

  $$u_t = u_x^2 \tag{4}$$

  is a PDE.

  4. An *integral equation* is an equation in which the unknowns represent functions and their integrals. For example

  $$u(t) = u(0) + \int_0^t u(s)\,\mathrm{d}s. \tag{5}$$

  5. And many other types ...

2. History and connections to other fields.

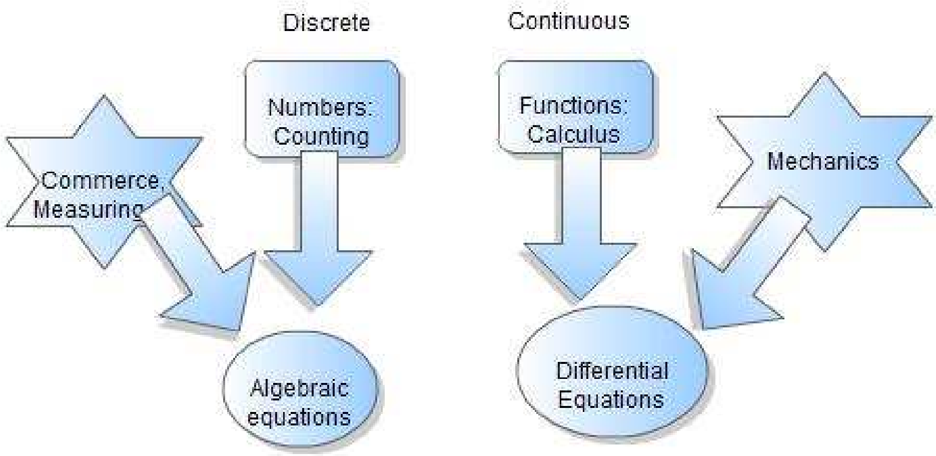

Figure 1. How does PDE come into being

Remark 2. Most of the algebraic equations studied by ancient mathematicians are actually from ancient geometry – they describe relations between length, area, volume, etc.; In comparison, the largest source of modern DEs is also geometry, if we identify a large part of theoretical physics as geometry.

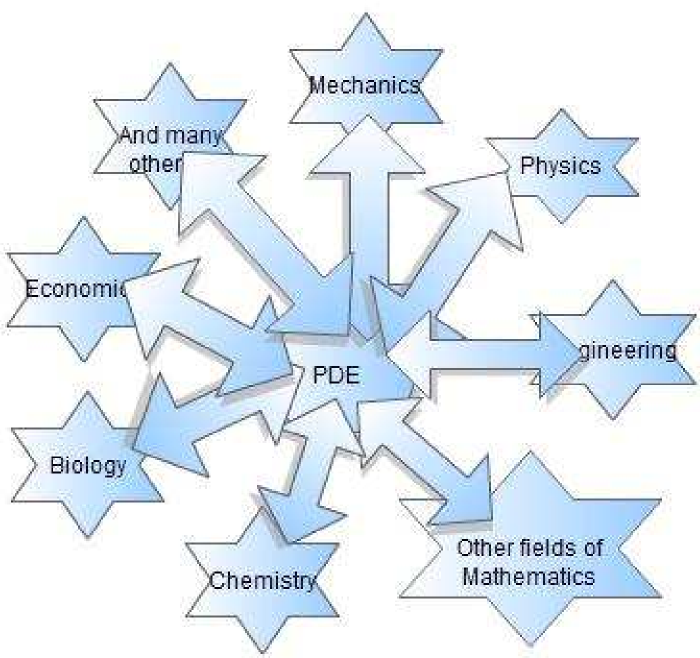

Figure 2. The role of PDE in Science and Engineering

3. Examples of PDEs.

3.1. Earliest PDEs.

As we have seen, PDEs arose in the 18th and 19th century mainly from the fast development of mechanics. What is fascinating is that, these earliest PDEs turned out to be either ubiquitous – they are important in fields totally unrelated to their origins – or very hard!

Example 3. (Poisson equation) The n-dimensional Poisson equation is an equation of one unknown function of n variables $u(x_1, \ldots, x_n)$.

$$\triangle u \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = f(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in \Omega \subseteq \mathbb{R}^n \tag{6}$$

When $f = 0$, it's called the *Laplace equation*. The particular operator

$$\triangle \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{7}$$

is called the *Laplacian*.

When $n = 2$ or 3, instead of x_1, x_2, x_3 people often use x, y, z.

The derivation of this equation is discussed in §3.6 of the textbook.

Example 4. (Heat equation) The n-dimensional[1] heat equation reads

$$u_t - k \triangle u = 0. \tag{8}$$

The derivation of this equation is discussed in §3.5 in the textbook.

Example 5. (Wave equation) The n-dimensional wave equation reads

$$u_{tt} - c^2 \triangle u = 0. \tag{9}$$

This equation is discussed in detail in §3.2–3.4 in the textbook.

Remark 6. The above equations have already been well-understood. However it turned out that they are far more useful than their origins indicate. These three equations are currently serving as the "base" for scientists and engineers when they set out to explore more complicated equations.

In contrast, the following equations, also proposed in the 18th and 19th century, are still poorly understood. In fact, whoever solves the 3D Navier-Stokes equations will be awarded $1,000,000 by the Clay institute!

Example 7. (3D Navier-Stokes equations) The 3D Navier-Stokes equations read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{10}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{11}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{13}$$

Or in compact form:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u} \tag{14}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{15}$$

Here the "divergence" operator is defined as

$$\nabla \cdot \boldsymbol{u} = \operatorname{div} \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{16}$$

1. We see that in fact $n+1$ variables are involved. The n-dimensional wave equation is in the same situation.

3.2. Other examples.

Later many more PDEs arose from many fields in science and engineering. For example:

- Modern physics:

  Schrödinger's equation, Maxwell's equations, Einstein's equation, Yang-Mills equation, etc.
- Engineering:

  Conservation laws (modeling phenomena from traffic jams to flood flows to the flight of missles)
- Economics:

  Black-Scholes equation.
- Chemistry and biology:

  Reaction-diffusion equations.

One can check Chapter 3 of the textbook and/or search in wiki to get some idea of these equations.

4. What do people do after obtaining a PDE through mathematical modelling.

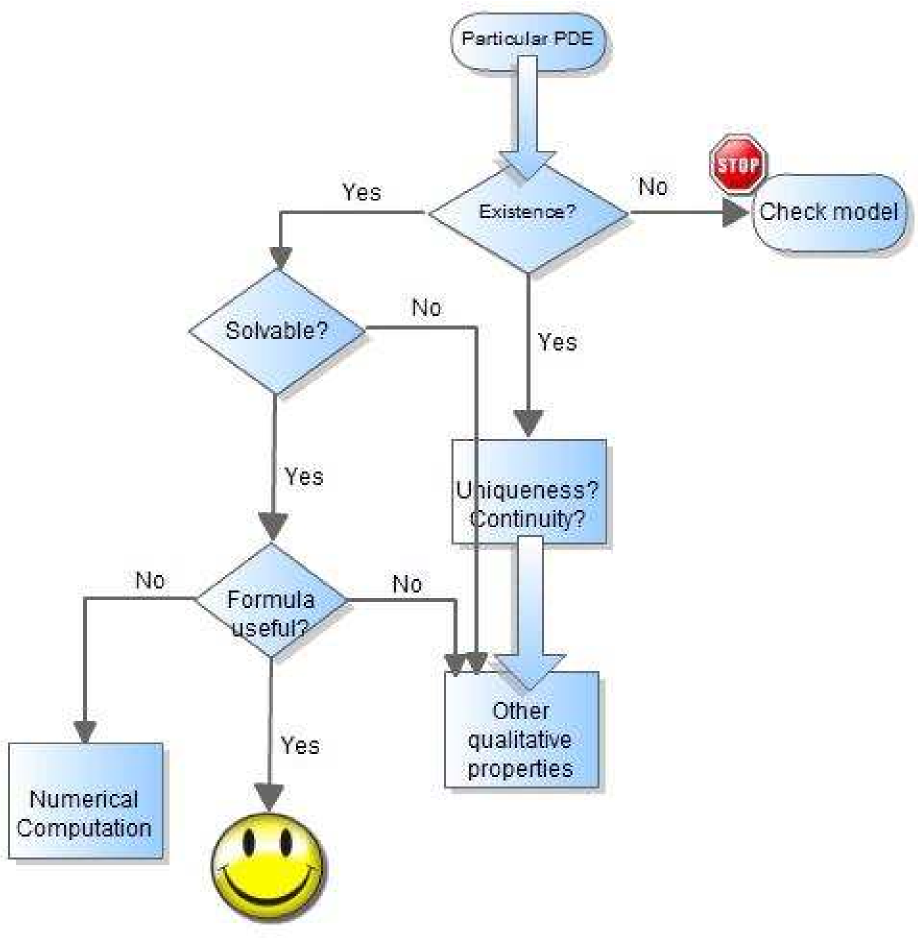

Figure 3. The Process of Studying a PDE

- Existence: There is at least one solution.
- Uniqueness: There is at most one solution.
- Continuity: The solution depends continuously on the data.

In general, we need extra knowledge besides the equation itself to solve the equation. For example for the heat equation,

$$u_t - u_{xx} = 0, \qquad 0 < x < l, \; t > 0 \tag{17}$$

it's existence, uniqueness, and continuity will be satisfactorily settled once we add

- Initial condition (IC):

  $$u(x,0) = \sin x \text{ (or other functions)}, \quad 0 \leqslant x \leqslant l, \tag{18}$$

  and

- boundary conditions (BC):

  $$u(0,t) = 0, \qquad u(l,t) = 0, \qquad t \geqslant 0. \tag{19}$$

  Here the 0's can be replaced by other functions in t.

Course Goal. This course (337) is about solving PDEs, that is finding formulas for the solutions. A *solution* is a function which makes the equation an identity. In other words, this course is about the solvability question only.[2] More precisely, this course is about solving PDEs with at most 4 variables.

Example 8. (§1.6, 2 - 4)

1. Verify that the function

   $$u(x,y) = e^x \sin y \tag{20}$$

   is (one of) the solution(s) of the equation

   $$u_{xx} + u_{yy} = 0. \tag{21}$$

2. Show that $u = f(x\,y)$, where f is an arbitrary differentiable function satisfies

   $$x\,u_x - y\,u_y = 0. \tag{22}$$

   As a consequence, $\sin(x\,y),\ \log(x\,y),\ e^{xy}$ are all solutions.

3. Show that $u = f(x)\,g(y)$ where f, g are arbitrary twice differentiable functions satisfies

   $$u\,u_{xy} - u_x\,u_y = 0. \tag{23}$$

   That is, any u that can be written as $f(x)\,g(y)$ is a solution to the equation.

5. Solving PDEs – Direct integration.

Example 9. Find the general solution of

$$u_{xy} = 0. \tag{24}$$

Solution. Since

$$(u_x)_y = 0 \tag{25}$$

we have

$$u_x = h(x) \tag{26}$$

for some arbitrary function h. Now integrate again we obtain

$$u(x,y) = \int h(x) + g(y) = f(x) + g(y) \tag{27}$$

where f, g are arbitrary functions.

Example 10. (§1.6, 6) Find the general solution of

$$u_{xx} + u_x = 0 \tag{28}$$

2. At this stage, we do not care about whether the formula obtained is useful or not!

by setting $u_x = v$.
Solution. Setting $v = u_x$ we obtain the following equation for v:

$$v_x + v = 0. \tag{29}$$

This can be solved by multiplying both sides by e^x and notice that

$$e^x (v_x + v) = \frac{\partial}{\partial x}(e^x v). \tag{30}$$

Therefore we have

$$u_x = v = e^{-x} g(y). \tag{31}$$

Integrate, we obtain

$$u(x, y) = f(y) - e^{-x} g(y)$$

Example 11. Find the general solution of

$$u\, u_{xy} - u_x\, u_y = 0. \tag{32}$$

Solution. We notice that

$$\frac{\partial}{\partial y}\left(\frac{u_x}{u}\right) = \frac{u_{xy}\, u - u_x\, u_y}{u^2} = 0. \tag{33}$$

As a consequence (assuming $u > 0$)

$$\frac{\partial}{\partial x}(\log u) = \frac{u_x}{u} = g(x). \tag{34}$$

Solving this equation we obtain

$$\log u = \int g(x) + h(y) \implies u(x, y) = e^{\int g(x)}\, e^{h(y)}. \tag{35}$$

In fact, one can check that $u(x, y) = f(x)\, g(y)$ is a solution for any x, y.

6. Solving PDEs – Change of variables.

In most cases, it's impossible to integrate directly. However one can still solve the equations by introducing appropriate change of variables.

Example 12. (§1.6, 3) Solve

$$x\, u_x - y\, u_y = 0. \tag{36}$$

Solution. We introduce new variables[3]

$$\xi = x\, y, \qquad \eta = x, \qquad v(\xi, \eta) = u(x, y) \tag{37}$$

Then

$$u_x = v_\xi\, \xi_x + v_\eta\, \eta_x = v_\xi\, y + v_\eta, \tag{38}$$

$$u_y = v_\xi\, \xi_y + v_\eta\, \eta_y = v_\xi\, x. \tag{39}$$

Substituting back into the equation we have

$$x\, u_\eta = 0 \text{ which leads to}^4\ v_\eta = 0 \Longrightarrow v(\xi, \eta) = f(\xi) \implies u(x, y) = f(x\, y) \tag{40}$$

for some arbitrary function f.

Example 13. (§1.6, 9) Show that the general solution of

$$u_{tt} - c^2\, u_{xx} = 0 \tag{41}$$

is $u(x, t) = f(x - c\, t) + g(x + c\, t)$, where f and g are arbitrary twice differentiable functions.

3. In the future, we will not bother to introduce $v(\xi, \eta) = u(x, y)$ anymore and just write u_ξ, u_η, etc.

4. The cancellation of x is legitimate because we are considering classical solutions here, which means v_η is a continuous function. One can prove that any continuous functions $f(x, y)$ satisfying $x\, f(x, y) = 0$ must satisfy $f(x, y) = 0$.

Solution. We do the change of variables:

$$\xi = x + c\,t, \qquad \eta = x - c\,t. \tag{42}$$

Then let

$$v(\xi, \eta) = u(x, t). \tag{43}$$

In other words, $v(x + c\,t, x - c\,t) = u(x, t)$.

Now according to the chain rule, we have

$$u_{xx} = (u_x)_x = (u_\xi\,\xi_x + u_\eta\,\eta_x)_x = u_{\xi\xi}\,\xi_x^2 + 2\,u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}, \tag{44}$$

$$u_{tt} = (u_t)_t = (u_\xi\,\xi_t + u_\eta\,\eta_t)_t = u_{\xi\xi}\,\xi_t^2 + 2\,u_{\xi\eta}\,\xi_t\,\eta_t + u_{\eta\eta}\,\eta_t^2 + u_\xi\,\xi_{tt} + u_\eta\,\eta_{tt}. \tag{45}$$

Therefore we have

$$-\,4\,c^2\,v_{\xi\eta} = 0. \tag{46}$$

And the solution has to be

$$v(\xi, \eta) = g(\xi) + f(\eta) \Longrightarrow u(x, t) = g(x + c\,t) + f(x - c\,t) \tag{47}$$

for arbitrary functions f and g.

Example 14. (§1.6, 19) Show that $u(x, y) = f\big(2\,y + x^2\big) + g\big(2\,y - x^2\big)$ is a general solution of the equation

$$u_{xx} - \frac{1}{x}\,u_x - x^2\,u_{yy} = 0. \tag{48}$$

Solution. The equation becomes

$$u_{\xi\xi} - u_{\eta\eta} = 0 \tag{49}$$

after setting $\xi = x^2$, $\eta = 2\,y$.

Remark 15. We have seen that the crucial steps in the above three examples are the changes of variables. One natural question is how do we figure these out? We will introduce systematic ways of doing these in the following few weeks.

7. Classifications of PDEs.

A general PDE can be written as

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_nx_n}, \ldots) = 0. \tag{50}$$

F has finitely many arguments. For example, for

$$u_{xx} + u_{yy} = f. \tag{51}$$

We can use

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = u_{xx} + u_{yy} - f(x, y). \tag{52}$$

As we can see now, all kinds of complicated functions can be used for F, creating all kinds of complicated PDEs. In practice, F is sometimes non-smooth or even implicitly defined (that is, one cannot write down the formula of F!). Among myriad of equations, only the simplest ones (basically those of first order, or the simpler ones of linear second order equations) can be solved explicitly, using the method introduced in the following weeks. Nevertheless, it is still worth doing so due to the following reasons:

1. **(Practical)** These simplest equations are already very good approximations of "real world" physical (chemical, biological, etc.) processes.
2. **(Theoretical)** These simplest equations help a lot in understanding more complicated ones.
3. **(Psychological)** It is important to develop some sense of the behaviors of general PDEs. Often the best way to obtain such sense is through explicitly solving the simplest equations.

In the following, we first make clearer which equations will be studied in this course.

7.1. Linear or nonlinear.

The equation is

– *linear* if the function F is an affine function. More specifically, the equation

$$F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n},\ldots)=0 \tag{53}$$

is a linear PDE if F can be written as

$$G(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n},\ldots)+H(x_1,\ldots,x_n) \tag{54}$$

where G is a linear function at every point $(x_1,\ldots,x_n)$.

For example, $f(x,y)=x+y$ is linear, while $f(x,y,z)=x\,y+z$ is not. In particular, one should note that "f is a linear function" is not the same as "f is linear in every variable". For example, $f(x,y)=x\,y$ is linear in both x and y, but it's not a linear function.

– *nonlinear* otherwise.

  → Quasilinear: if F is a linear function w.r.t. to the highest ordered derivatives. For example,

$$u\,u_{xx}+u_x^2+u_y^3\,u_{yy}=x^2+y^2 \tag{55}$$

  is quasilinear.

  → Fully nonlinear: if the equation is nonlinear but not quasilinear, then it said to be fully nonlinear.

7.2. First order, second order, third order, ...

The order refers to the order of the highest order derivative(s) of the unknown function appearing in F. For example,

$$u_x^2+2\,u_{xy}+3\,e^{u_y}=0 \tag{56}$$

is of 2nd order.

7.3. Homogeneous or nonhomogeneous.

If all the terms in F involves u or its derivatives, the equation is said to be homogeneous, otherwise it's said to be nonhomogeneous.

7.4. Examples.

Example 16. (§1.6, 1 (a),(b)) What are the types of the following PDEs.

a) $u_{xx}+x\,u_y=y$.

b) $u\,u_x-2\,x\,y\,u_y=0$.

8. Linear operators.

In this course, we will study linear and quasi-linear first order equations, and linear second order equations. Recall that an equation is linear when it can be written

$$G(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n},\ldots)=f(x_1,\ldots,x_n) \tag{57}$$

where G is a linear function of $(u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n},\ldots)$ at each point $(x_1,\ldots,x_n)$. A compact way of saying the same thing is to introduce an "operator"

$$L[u]\equiv G(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n},\ldots). \tag{58}$$

Thus the equation becomes

$$L[u]=f. \tag{59}$$

Such a viewpoint is important in the understanding of more advanced study of PDEs.

8.1. Definitions.

- An *operator* is a mathematical rule which, when applied to a function, produces another function.
- An *operator* is said to be linear if it satisfies the following
  1. $L[c\,u]=c\,L[u]$.

2. $L[u_1 + u_2] = L[u_1 + u_2]$.

One can show that

$$L\left[\sum_{j=1}^{k} c_j\, u_j\right] = \sum_{j=1}^{k} c_j\, L[u_j]. \tag{60}$$

Examples of operators and checking linearity of operators.

8.2. The algebra of linear operators.

- If L, M are linear operators, so is $L + M$. Furthermore we have
  1. $L + M = M + L$
  2. $(L + M) + N = L + (M + N)$
- If L, M are linear operators, so is $L\, M$. Note that this "product" is not the usual product. Definition (relation to matrix product!)
  1. $(L\, M)\, N = L\, (M\, N)$
  2. $L\, M = M\, L$ in general does not hold. But when L, M are differential operators with constant coefficients, it is correct. Examples.
- $L\, (c_1\, M + c_2\, N) = c_1\, L\, M + c_2\, L\, N$.

8.3. Principle of linear superposition.

Consider a linear equation

$$L[u] = G. \tag{61}$$

If we know u_1, u_2 such that

$$L[u_1] = G_1, \quad L[u_2] = G_2, \tag{62}$$

Then the linearity of L gives:

$$L[u_1 + u_2] = L[u_1 + u_2] = G_1 + G_2 = G. \tag{63}$$

More generally, for any $u_1, \ldots, u_k$ solving

$$L[u_i] = G_i \tag{64}$$

and any constants $c_1, \ldots, c_k$, we have

$$L\left[\sum_{1}^{k} c_i\, u_i\right] = \sum_{i=1}^{k} c_i\, G_i. \tag{65}$$

Now suppose we would like to solve

$$L[u] = G. \tag{66}$$

All we need to do is to find appropriate functions G_i and appropriate constants c_i, such that

1. For each $i = 1, \ldots, k$, the equation
$$L[u_i] = G_i \tag{67}$$
is easy to solve;
2. We have
$$G = \sum_{i=1}^{k} c_i\, G_i. \tag{68}$$

However, suppose we are given a equation, for example

$$u_{xx} + u_{yy} = e^{-(x^2 + y^2)}, \tag{69}$$

it is not at all clear how we could find such c_i's and G_i's. In fact it is often impossible.

On the other hand, the situation changes drastically when we allow $k = \infty$ or even be a continuous parameter. We will see how this works in a few weeks.

Example 17. (§1.6, 14) Show that

$$u(x,y;k)=e^{-ky}\sin(kx), \quad x\in\mathbb{R}, \quad y>0 \tag{70}$$

is a solution of the equation

$$u_{xx}+u_{yy}=0 \tag{71}$$

for any real parameter k. Verify that

$$u(x,y)=\int_0^\infty c(k)\,e^{-ky}\sin(kx)\,\mathrm{d}k \tag{72}$$

is also a solution of the above equation.

Remark 18. The superposition principle can also be applied to boundary and initial conditions. For example, the solution u of

$$u_{xx}+u_{yy}=e^{-(x^2+y^2)} \qquad x^2+y^2<1; \qquad u(x,y)=\cos\theta \quad x^2+y^2=1 \tag{73}$$

where $(x,y)\to(r,\theta)$ can be otained by adding up the solutions of

$$v_{xx}+v_{yy}=e^{-(x^2+y^2)} \qquad x^2+y^2<1; \qquad v(x,y)=0 \quad x^2+y^2=1 \tag{74}$$

and

$$w_{xx}+w_{yy}=0 \qquad x^2+y^2<1; \qquad w(x,y)=\cos\theta \quad x^2+y^2=1. \tag{75}$$