(Oct. 7, 12:00pm – 12:50pm, DP6069)

Math 334 2010 Midterm 1 Solutions

Name _________________________

ID# _________________________

Signature _________________________

- Only pen/pencil/eraser are allowed. Scratch papers will be provided.
- Please **write clearly**, with intermediate steps to **show sufficient work** even if you can solve the problem in "one go". Otherwise you may not receive full credit.
- Please box, underline, or highlight the most important parts of your answers.

Problem	Points	Score
1	25	
2	25	
3	15	
4	15	
5	15	
6	5	
Total	100	

Problem 1. (25 pts) Solve the initial value problem

$$y'' + 4\,y = 0, \qquad y(0) = 0, \;\; y'(0) = 1. \tag{1}$$

Solution. This is 3.3 17.

- Characteristic equation:

$$r^2 + 4 = 0 \Longrightarrow r_1 = 2\,i, r_2 = -2\,i. \tag{2}$$

- General solution

$$y = C_1 \cos 2\,t + C_2 \sin 2\,t. \tag{3}$$

- Fix C_1, C_2:

$$y' = -2\,C_1 \sin 2\,t + 2\,C_2 \cos 2\,t. \tag{4}$$

  Thus

$$y(0) = 0 \Longrightarrow C_1 = 0; \tag{5}$$

$$y'(0) = 1 \Longrightarrow 2\,C_2 = 1 \Longrightarrow C_2 = 1/2. \tag{6}$$

- Final answer:

$$y = \frac{\sin 2\,t}{2}. \tag{7}$$

Grading scheme etc:

- *Know the procedure (9 pts)*
  - *Characteristic equation (3 pts) -> General solution (3 pts) -> Use IV (3 pts)*
- *Detailed solution (16 pts)*
  - *Correct characteristic equation (3 pts)*
  - *Correct roots (2 pts)*
  - *Correct general solution (2 pts)*
  - *Correct y' (2 pts)*
  - *Correct C_1, C_2 (2+2 pts)*
  - *Correct final answer (3 pts).*
- *Remarks:*
  - *If characteristic equation is conceptually wrong, 9/25.*
  - *Common mistake:*
    - $r^2 + 4\,r = 0$. *The power of r corresponds to the number of derivatives. So y'' (two derivatives) gives r^2 while $4\,y$ (zero derivatives) should give 4 instead of $4\,r$.*
    - $\cos 0 = 0$.
    - $C_2 = \frac{1}{2}$, *so* $\frac{1}{2}\cos 2\,t$.

Problem 2 (25 pts) Find the general solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\,x^3 + 1}{y\,(2 + 3\,y)}. \tag{8}$$

Solution. This is Chapter 2 Problem 7.

- First spot that it is separable.
- Move terms:

$$y\,(2 + 3\,y)\,\mathrm{d}y = (4\,x^3 + 1)\,\mathrm{d}x. \tag{9}$$

- Integrate:

$$y^2 + y^3 = x^4 + x + C. \tag{10}$$

- Solution is given by

$$y^2 + y^3 - x^4 - x = C. \tag{11}$$

- It's also OK to work along the "exact equations" procedure.

Grading scheme etc:

- *Know the procedure (9 pts): Separable (3 pts) -> Move x terms to one side and y terms to the other (3 pts) -> Integrate (3 pts).*
- *Details (16 pts)*
  - *Correct equation $y\,(2 + 3\,y)\,\mathrm{d}y = (4\,x^3 + 1)\,\mathrm{d}x$ (4 pts)*
  - *Integrations: Correct y term integration (3 pts), correct x term integration (3 pts), remember to add C (3 pts).*
  - *Correct answer (3 pts).*
- *Remarks*
  - *If wrongly add $y = 0, \ldots, -1$. Note that here we are multiplying, not dividing, both sides by $y\,(2 + 3\,y)$ so there is no need to check the zeroes of this function.*
  - *If working along "exact equation" line, then: Correct M, N (2+2), $u = \int \ldots + g(y)$ (3), evaluation of the integral (3), get $g(y)$ (3), final answer (3).*

Problem 3. (15 pts) Solve

$$y'' + 9\,y = \sin 3\,t, \qquad y(0) = 2;\ \ y'(0) = -1. \tag{12}$$

Solution.

- We should use undetermined coefficients.
- First solve the homogeneous equation

$$y'' + 9\,y = 0 \tag{13}$$

whose characteristic equation is $r^2 + 9 = 0 \Longrightarrow r_{1,2} = \pm 3\,i$. So

$$y_1 = \cos 3\,t; \qquad y_2 = \sin 3\,t. \tag{14}$$

- The right hand side is of the form $e^{\alpha t} \sin \beta\, t\,(A_0 + \cdots + A_n\, t^n)$ with $\alpha = 0, \beta = 3, n = 0$. So guess

$$y_p = t^s\,[A \cos 3\,t + B \sin 3\,t]. \tag{15}$$

Since $\alpha + i\,\beta = 3\,i$ is indeed a solution to the characteristic equation, we take $s = 1$. So

$$y_p = t\,[A \cos 3\,t + B \sin 3\,t]. \tag{16}$$

- Substitute into the equation:

$$t\,[-9\,A \cos 3\,t - 9\,B \sin 3\,t] + 2\,[-3\,A \sin 3\,t + 3\,B \cos 3\,t] + 9\,t\,[A \cos 3\,t + B \sin 3\,t] = \sin 3\,t \tag{17}$$

which simplifies to

$$6\,B \cos 3\,t - 6\,A \sin 3\,t = \sin 3\,t \tag{18}$$

so

$$B = 0, A = -\frac{1}{6}. \tag{19}$$

Thus we have

$$y_p = -\frac{t\cos 3\,t}{6}. \tag{20}$$

- General solution:

$$y = C_1 \cos 3\,t + C_2 \sin 3\,t - \frac{t\cos 3\,t}{6}. \tag{21}$$

- Use initial values to get C_1, C_2:

  Preparation: $y' = -3\,C_1 \sin 3\,t + 3\,C_2 \cos 3\,t + \frac{t\sin 3\,t}{2} - \frac{\cos 3\,t}{6}$.

  Now

$$y(0) = 2 \Longrightarrow C_1 = 2; \tag{22}$$

$$y'(0) = -1 \Longrightarrow 3\,C_2 - \frac{1}{6} = -1 \Longrightarrow C_2 = -\frac{5}{18}. \tag{23}$$

- Final answer:

$$y = 2\cos 3\,t - \frac{5}{18}\sin 3\,t - \frac{t\cos 3\,t}{6}. \tag{24}$$

- It's OK if you use variation of parameters to get the general solution.

Grading Scheme etc:

- *Know the procedure (5 pts): Solve homogeneous equation (2); Undetermined coefficients (2); Use IV (1).*
- *Details (10 pts):*
  - *Solution of homogeneous equation: 2 pts;*
  - *Correct form of y_p: 3 pts: t^s (1); $s = 1$ (1); Both $\sin$ and $\cos$ (1);*
  - *Correct A, B: 2 pts;*
  - *Determine C_1, C_2: 2 pts;*
  - *Answer 1 pt.*
- *Mistakes:*
  - *Try to determine C_1, C_2 before y_p is obtained.*
  - *Characteristic equation: $r^2 + 9\,r = 0$.*
  - *$A = -1/6$ so $y_p = -\frac{1}{6}\,t\sin 3\,t$.*

Problem 4 (15 pts) Find an integrating factor for and solve

$$(3\,x^4+y)\,\mathrm{d}x+(2\,x^2\,y-x)\,\mathrm{d}y=0. \tag{25}$$

Solution.

- Getting the μ equation:

$$M=3\,x^4+y \Longrightarrow \frac{\partial M}{\partial y}=1; \qquad N=2\,x^2\,y-x \Longrightarrow \frac{\partial N}{\partial x}=4\,x\,y-1. \tag{26}$$

  So

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=4\,x\,y-2=2\,(2\,x\,y-1). \tag{27}$$

  The μ equation is

$$(3\,x^4+y)\,\frac{\partial \mu}{\partial y}-(2\,x^2\,y-x)\,\frac{\partial \mu}{\partial x}=2\,(2\,x\,y-1)\,\mu. \tag{28}$$

- Guess μ.

  - Guess $\mu=\mu(x)$:

$$-(2\,x^2\,y-x)\,\mu'=2\,(2\,x\,y-1)\,\mu \Longrightarrow -x\,\mu'=2\,\mu. \tag{29}$$

    So

$$\frac{\mu'}{\mu}=-\frac{2}{x} \Longrightarrow \mu=x^{-2}. \tag{30}$$

- Multiply the equation by μ. We get

$$\left(3\,x^2+\frac{y}{x^2}\right)\mathrm{d}x+\left(2\,y-\frac{1}{x}\right)\mathrm{d}y=0. \tag{31}$$

- Check exactness:

$$\frac{\partial(3\,x^2+y/x^2)}{\partial y}=\frac{1}{x^2}; \qquad \frac{\partial\left(2\,y-\frac{1}{x}\right)}{\partial x}=-\left(-\frac{1}{x^2}\right)=\frac{1}{x^2}. \tag{32}$$

- Solve the transformed equation:

$$\left(3\,x^2+\frac{y}{x^2}\right)\mathrm{d}x+\left(2\,y-\frac{1}{x}\right)\mathrm{d}y=0. \tag{33}$$

Compute

$$u(x,y)=\int\left(3x^2+\frac{y}{x^2}\right)\mathrm{d}x+g(y)=x^3-\frac{y}{x}+g(y). \tag{34}$$

Take $\frac{\partial}{\partial y}$:

$$\frac{\partial u}{\partial y}=-\frac{1}{x}+g'(y). \tag{35}$$

Compare with $\left(2y-\frac{1}{x}\right)$ we have

$$g'(y)=2y\Longrightarrow g(y)=y^2. \tag{36}$$

- So the solution is

$$x^3-\frac{y}{x}+y^2=C. \tag{37}$$

Grading Scheme etc:

- *Procedure (5 pts): Find μ (2); Multiply the equation by μ (1); Integrate the resulting exact equation (2).*
- *Details (10 pts):*
  - *Correct equation for μ: 2 pts;*
  - *Correct equation for $\mu=\mu(x)$: 1 pt;*
  - *Correct simplification: 1 pt;*
  - *$\mu=1/x^2$: 1 pt.*
  - *Correct transformed equation. 1 pt;*
  - *$u=\int\cdots+g(x)$: 1 pt.*
  - *Evaluation of $\int\cdots$: 1 pt.*
  - *Obtain $g(x)$: 1 pt.*
  - *Final answer: 1 pt.*
- *Common mistakes:*
  - *Unable to simplify $-(2x^2y-x)\mu'=(4xy-2)\mu$.*
  - *Sloppy writing: For example $-(2x^2y-x)$ becomes $-2x^2y-x$ in the very next line, which then naturally ruins everything.*
- *Remarks*
  - *Ability to carry out all the calculation efficiently is crucial to the solution of such problems.*

Problem 5 (15 pts)

a) **(7 pts)** Show that if the equation $M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0$ is such that

$$\frac{x^2}{x\,M+y\,N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=F\left(\frac{y}{x}\right) \tag{38}$$

then an integrating factor is given by

$$\mu(x,y)=\exp\left\{\int F(u)\,\mathrm{d}u\right\},\qquad u=\frac{y}{x}. \tag{39}$$

b) **(8 pts)** Use the result in a) to solve

$$(2\,x-y+2\,x\,y-y^2)\,\mathrm{d}x+(x+x^2+x\,y)\,\mathrm{d}y=0. \tag{40}$$

Note that you can work on b) even if you cannot do a).

Solution.

a) If $\mu(x,y)=\mu\left(\frac{y}{x}\right)$, then (letting $u=y/x$)

$$\frac{\partial\mu}{\partial y}=\frac{1}{x}\,\mu'(u),\qquad \frac{\partial\mu}{\partial x}=-\frac{y}{x^2}\,\mu'(u). \tag{41}$$

Now the equation for μ:

$$M\,\frac{\partial\mu}{\partial y}-N\,\frac{\partial\mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu \tag{42}$$

becomes

$$M\left[\frac{1}{x}\,\mu'(u)\right]-N\left[-\frac{y}{x^2}\,\mu'(u)\right]=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu(u) \tag{43}$$

which simplifies to

$$\left[\frac{M}{x}+\frac{N\,y}{x^2}\right]\mu'(u)=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu(u) \tag{44}$$

which is just

$$\frac{x\,M+y\,N}{x^2}\,\mu'(u)=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu(u)\Longleftrightarrow\frac{\mu'(u)}{\mu(u)}=\frac{x^2}{x\,M+y\,N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right). \tag{45}$$

Now if

$$\frac{x^2}{x\,M+y\,N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=F\left(\frac{y}{x}\right) \tag{46}$$

the μ equation becomes

$$\frac{\mu'(u)}{\mu(u)}=F(u) \tag{47}$$

whose solution is

$$\mu(u)=\exp\left\{\int F(u)\,\mathrm{d}u\right\}. \tag{48}$$

- Note: It's also OK to substitute $\mu(u)=\exp\{\int F(u)\,\mathrm{d}u\}$ into the equation for μ and show that it is indeed a solution.

b) Since we are told to use a), just compute

$$\frac{\partial N}{\partial x}=\frac{\partial(x+x^2+x\,y)}{\partial x}=1+2\,x+y;\qquad \frac{\partial M}{\partial y}=(-1+2\,x-2\,y) \tag{49}$$

so

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=3\,y+2. \tag{50}$$

and

$$\frac{x^2}{x\,M+y\,N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=1. \tag{51}$$

So

$$\mu(u)=e^{\int 1}=e^{u} \tag{52}$$

and consequently

$$\mu(x,y)=e^{y/x}. \tag{53}$$

Now multiply the equation by the integrating factor we obtained:

$$\left[e^{y/x}\,(2\,x-y+2\,x\,y-y^{2})\right]\mathrm{d}x+\left[e^{y/x}\,(x+x^{2}+x\,y)\right]\mathrm{d}y=0. \tag{54}$$

Compute

$$\begin{aligned} u(x,y) &= \int\left[e^{y/x}\,(x+x^{2}+x\,y)\right]\mathrm{d}y+g(x)\\ &= (x+x^{2})\int e^{y/x}\,\mathrm{d}y+x\int y\,e^{y/x}\,\mathrm{d}y+g(x)\\ &= (x+x^{2})\,x\,e^{y/x}+x\left[x\int y\,\mathrm{d}e^{y/x}\right]+g(x)\\ &= (x^{2}+x^{3})\,e^{y/x}+x^{2}\left[y\,e^{y/x}-\int e^{y/x}\,\mathrm{d}y\right]+g(x)\\ &= (x^{2}+x^{3})\,e^{y/x}+x^{2}\,y\,e^{y/x}-x^{3}\,e^{y/x}+g(x)\\ &= (x^{2}+x^{2}\,y)\,e^{y/x}+g(x). \end{aligned} \tag{55}$$

Now compute

$$\frac{\partial u}{\partial x}=(2\,x+2\,x\,y)\,e^{y/x}+(x^{2}+x^{2}\,y)\left(-\frac{y}{x^{2}}\right)e^{y/x}+g'(x)=(2\,x+2\,x\,y-y-y^{2})\,e^{y/x} \tag{56}$$

and compare with $e^{y/x}\,(2\,x-y+2\,x\,y-y^{2})$ we see that $g'(x)=0$ so can take $g(x)=0$.

So the general solution is given by

$$(x^{2}+x^{2}\,y)\,e^{y/x}=C. \tag{57}$$

Grading scheme etc. For "advanced" and "challenge" problems, no "Procedure" points anymore.

- *Part (a)*
  - *Equation for μ: 2 pts.*
  - *Correct calculation of $\frac{\partial\mu}{\partial x},\frac{\partial\mu}{\partial y}$: 2 pts.*
  - *Equation for $\mu(u)$: 1 pt*
  - *Get μ: 2 pts.*
- *Part (b)*
  - *Calculation of $\frac{\partial M}{\partial y},\frac{\partial N}{\partial x}$: 2 pts;*
  - *Check (a): 1 pt*
  - *Get μ: 1 pt*
  - *Multiply the equation by μ: 1 pt*
  - *$\int\cdots$ and $g(x)$: 2 pts*
  - *Answer: 1 pt.*
- *Common mistakes:*
  - *Didn't notice "Use the result in a)" and wasted time guess $\mu=\mu(x),\mu=\mu(y),\ldots$*

Problem 6 (5 pts) If the roots of the characteristic equation are real, show that a solution of $a\,y''+b\,y'+c\,y=0$ is either everywhere zero or else can take on the value zero at most once.

Proof. Note that there are two cases: Distinct roots and repeated roots. We discuss them one by one.

- Case 1. Distinct roots.

  In this case all we need to show is that if $r_1 \neq r_2$ are real, then

  $$C_1\,e^{r_1 t}+C_2\,e^{r_2 t}=0 \tag{58}$$

  can have at most one zero unless $C_1=C_2=0$.

  Assume the contrary. If we have C_1, C_2 not both zero, and $t_1 \neq t_2$ such that

  $$C_1\,e^{r_1 t_1}+C_2\,e^{r_2 t_1} = 0 \tag{59}$$
  $$C_1\,e^{r_1 t_2}+C_2\,e^{r_2 t_2} = 0 \tag{60}$$

  Multiply the first equation by $-e^{r_1(t_2-t_1)}$ and add to the second equation, we get

  $$C_2\left[e^{r_2 t_2}-e^{r_1(t_2-t_1)}\,e^{r_2 t_1}\right]=0 \tag{61}$$

  which simplifies to

  $$C_2\,e^{r_2 t_2}\left[1-e^{r_1(t_2-t_1)}\,e^{r_2(t_1-t_2)}\right]=0 \tag{62}$$

  and then

  $$C_2\,e^{r_2 t_2}\left[1-e^{(r_1-r_2)(t_2-t_1)}\right]=0. \tag{63}$$

  As $r_1 \neq r_2$, $t_1 \neq t_2$, we have $1-e^{(r_1-r_2)(t_2-t_1)} \neq 0$ and consequently $C_2=0$.

  But the whole argument still works if we replace all 2 by 1 and 1 by 2. So $C_1=0$. Contradiction.

- Case 2. Repeated roots. Denote the root by r. We need to show that

  $$C_1\,e^{rt}+C_2\,t\,e^{rt}=0 \tag{64}$$

  can have at most one zero unless $C_1=C_2=0$.

  Now that

  $$C_1\,e^{rt}+C_2\,t\,e^{rt}=0 \Longrightarrow C_1+C_2\,t=0. \tag{65}$$

  If there are $t_1 \neq t_2$ such that

  $$C_1+C_2\,t_1=0, \qquad C_1+C_2\,t_2=0 \tag{66}$$

  Taking the difference we get $C_2=0$. Substitute back into either equation we get $C_1=0$. Contradiction. □

Grading scheme etc.

- *Two cases of general solutions for "both roots real": 2 pts*
- *Analyze case 1 (distinct roots): 2 pts*
- *Analyze case 2 (repeated roots): 1 pt.*