Lecture 31 Vectors, Matrices, Determinants, Eigenvalues/Eigenvectors

11/25/2011

Review.

- General first order system:

$$\begin{aligned} \dot{x}_1 &= F_1(t, x_1, \ldots, x_n) && (1)\\ \dot{x}_2 &= F_2(t, x_1, \ldots, x_n) && (2)\\ \vdots & \;\vdots\; \vdots \\ \dot{x}_n &= F_n(t, x_1, \ldots, x_n) && (3) \end{aligned}$$

  - Existence and uniqueness:
    Let R be a region in the t-x_1-x_2-$\cdots$-x_n space (which is $n+1$ dimensional) containing the point $(t_0, x_1^0, \ldots, x_n^0)$. If all the partial derivatives $\frac{\partial F_i}{\partial x_j}$ $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ remain bounded in R, then there **is** a **unique** solution $(x_1(t), \ldots, x_n(t))$ satisfying both the equations and the initial condition.

- General linear first order system:

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n + g_1(t) && (4)\\ \vdots & \;\vdots\; \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n + g_n(t). && (5) \end{aligned}$$

  - General solution:

$$c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t) + \boldsymbol{x}_p(t) \tag{6}$$

    with $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ a fundamental set for the homogeneous system, and $\boldsymbol{x}_p$ a "particular solution", that is a solution to the non-homogeneous system.

- General linear homogeneous first order system:

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n && (7)\\ \vdots & \;\vdots\; \vdots \\ \dot{x}_n &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n. && (8) \end{aligned}$$

  - General solution:

$$c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t) \tag{9}$$

    with $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ a fundamental set for the homogeneous system.

- Wronskian:

$$W\big[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}\big] = \det \big(\ \boldsymbol{x}^{(1)} \ \cdots \ \boldsymbol{x}^{(n)}\ \big) = \det \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}. \tag{10}$$

  - $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ linearly independent $\Longleftrightarrow$ $W \neq 0$ at t_0 (or any other time).

- First order linear homogeneous constant-coefficient system:

$$\begin{aligned} \dot{x}_1 &= a_{11}\, x_1 + \cdots + a_{1n}\, x_n && (11)\\ \vdots & \;\vdots\; \vdots \\ \dot{x}_n &= a_{n1}\, x_1 + \cdots + a_{nn}\, x_n && (12) \end{aligned}$$

  - We can solve them completely.

- Preparation: Vectors, matrices, determinants, eigenvalues/eigenvectors.

Vectors and Matrices.

- Vectors
  - Several numbers in a row (or column), treat as a unit, subject to natural rules of operation.
  - Notation:
    $$(x_1, \ldots, x_n) \text{ or } \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{13}$$
    Shorthand: $\boldsymbol{x}$, $\vec{x}$, or simply x.[1]
  - Operations on vectors:
    - Addition and subtraction between vectors;
    - Multiplication with scalars;
    - Multiplication between a row and a column vector **of the same size**:
      $$(x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1\, y_1 + \cdots + x_n\, y_n. \tag{14}$$
    - Transpose:
      $$(x_1, \ldots, x_n)^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T = (x_1, \ldots, x_n). \tag{15}$$
      So the row-column product can be written as
      $$\boldsymbol{x}^T \boldsymbol{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1\, y_1 + \cdots + x_n\, y_n. \tag{16}$$
    - Conjugate:
      $$\overline{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \begin{pmatrix} \bar{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix} \tag{17}$$
      where $\bar{\cdot}$ is the complex conjugate.
    - Adjoint:
      $$\boldsymbol{x}^* = \overline{(\boldsymbol{x}^T)} = (\bar{\boldsymbol{x}})^T = (x_1^*, \ldots, x_n^*). \tag{18}$$
    - Inner product:
      $$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^*\, \boldsymbol{y}. \tag{19}$$
      Remark. The difference between inner product and row-column product is that the inner product is that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geqslant 0$ for all $\boldsymbol{x}$.
    - Length/Norm:
      $$\|\boldsymbol{x}\| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sqrt{\boldsymbol{x}^*\, \boldsymbol{x}}. \tag{20}$$
- Matrices
  - Several numbers in a rectangular array:
    $$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}. \tag{21}$$

1. The first notation often appear in science literature, the second elementary mathematics, the third higher mathematics (where the readers are required to figure out what every symbol really means...).

Also denoted: $A = (a_{ij})$ with the understanding that a_{ij} is the number at the intersection of the i-th row and the j-th column.

For example the "Hilbert matrix":

$$A = \left(\frac{1}{i+j} \right) \tag{22}$$

means the matrix reads

$$\begin{pmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \cdots \\ \frac{1}{2+1} & \ddots & \\ \vdots & & \ddots \end{pmatrix}. \tag{23}$$

- A matrix with m rows and n columns is said to be an $m \times n$ (read: m by n) matrix.
- If we denote column vectors:

$$\boldsymbol{a}_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}, \tag{24}$$

then A is a row of such column vectors:

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{pmatrix}; \tag{25}$$

We can also denote row vectors

$$\boldsymbol{b}_i^T := \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}. \tag{26}$$

Then A is a column of such row vectors:

$$A = \begin{pmatrix} \boldsymbol{b}_1^T \\ \vdots \\ \boldsymbol{b}_m^T \end{pmatrix}. \tag{27}$$

Note. From this understanding we can treat column vectors as $n \times 1$ matrices, while row vectors as $1 \times n$ matrices. As we will see soon this gives a unified treatment of matrix-matrix and matrix-vector products.

- Matrix operations:
  - Addition and subtraction between matrices of the **same size**.
  - Multiplication by scalar;
  - Multiplcation between matrices of consistent sizes:

$$A\,B \text{ requires: number of columns of } A = \text{number of rows of } B; \tag{28}$$

If $A = \begin{pmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_m^T \end{pmatrix}$, $B = \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix}$, then $A\,B = (\boldsymbol{a}_i^T \boldsymbol{b}_j)$ (that is the number at the intersection of the i-th row and the j-th column is the product of the i-th row vector in A with the j-th column vector in B. This is only possible when $\boldsymbol{a}_i$ and $\boldsymbol{b}_j$ are of the same size, in other words: number of columns of $A =$ number of rows of B.

Remark. Note that this automatically defined multiplication between matrices and vectors.

  - Matrix multiplication is associative but in general not commutative.

$$A\,(B\,C) = (A\,B)\,C; \quad A\,B \neq B\,A. \tag{29}$$