Lecture 25 Review for Midterm 2

11/07/2011

Main Topics. (Easy – Hard)

- Euler Equations: (Lecture 20)
  - $a\,x^2\,y'' + b\,x\,y' + c\,y = 0$
  - Indicial/Characteristic equation:
  $$a\,r\,(r-1) + b\,r + c = 0 \tag{1}$$
  Can be obtained by setting $y = x^r$.
  - General solution: 3 Cases.
    - $C_1\,x^{r_1} + C_2\,x^{r_2}$;
    - $C_1\,x^r + C_2\,x^r \ln x$;
    - $C_1\,x^\alpha \cos(\beta \ln x) + C_2\,x^\alpha \sin(\beta \ln x)$.
- Higher order constant coefficient linear homogeneous equations
  - Equation.
  $$a_0\,y^{(n)} + \cdots + a_n\,y = 0. \tag{2}$$
  - Find general solution. (Lectures 13, 14)
    1. Solve the characteristic equation
    $$a_0\,r^n + \cdots + a_n = 0 \tag{3}$$
    and get a list of roots.
    2. Order the roots: real roots first, then complex conjugate pairs. Write down the fundamental set $y_1, \ldots, y_n$ according to the following rules:
       - A real root r, repeated k times, yields k solutions in the fundamental set: e^{rt}, $t\,e^{rt}, \ldots, t^{k-1}\,e^{rt}$ (Note that when r is a single root, this automatically give only e^{rt}.
       - A pair of complex roots $\alpha \pm i\,\beta$, repeated k times, yields $2\,k$ solutions in the fundamental set: $e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, \ldots, t^{k-1}\,e^{\alpha t} \cos \beta t, t^{k-1}\,e^{\alpha t} \sin \beta t$.
    3. $y = C_1\,y_1 + \cdots + C_n\,y_n$. Solve the characteristic equation
    $$a_0\,r^n + \cdots + a_n = 0 \tag{4}$$
    and get $r_1, r_2, \ldots, r_n$.
  - Technical issues:
    1. Factorization of polynomials;
    2. Solving $r^n = a$. (Lecture 14)
       a. Write $a + b\,i = R\,e^{i(\theta_0 + 2k\pi)}$;
       b. Write
       $$(a + b\,i)^{1/n} = R^{1/n} \exp\left[i\,\frac{\theta_0 + 2\,k\,\pi}{n}\right]. \tag{5}$$
       c. Set $k =$ n consecutive numbers (for example $0, 1, \ldots, n-1$, or $-\frac{n}{2} + 1, \ldots, 0, \ldots, \frac{n}{2}$ when n is even and similarly when n is odd. Each value of k gives one root.

d. Simplify if possible.

Remark 1. Idea: From $(r\,e^{i\phi})^n = r^n\,e^{in\phi}$, we see that to find the n roots of a, all we need are n pairs (r,ϕ) such that $r^n\,e^{in\phi} = a$ or equivalently

$$r^n\,e^{in\phi} = R\,e^{i(\theta_0 + 2k\pi)}. \tag{6}$$

- Undetermined coefficient for higher order equations (Lecture 15)
  1. Solve the homogeneous equation. Get the list of roots $r_1, \ldots, r_n$ for the characteristic equation.
  2. Check $g(t)$:
     - If $g(t) = e^{rt}\,(a_0 + \cdots + a_n\,t^n)$, then
       $$y_p = t^s\,e^{rt}\,(A_0 + \cdots + A_n\,t^n); \tag{7}$$
       $s = \#$ of times r appears in the list $r_1, \ldots, r_n$.
     - If $g(t) = e^{\alpha t}\cos\beta t\,(a_0 + \cdots + a_n\,t^n) + e^{\alpha t}\sin\beta t\,(b_0 + \cdots + b_m\,t^m)$, then
       $$y_p = t^s\,e^{\alpha t}\cos\beta t\,(A_0 + \cdots + A_k\,t^k) + t^s\,e^{\alpha t}\sin\beta t\,(B_0 + \cdots + B_k\,t^k) \tag{8}$$
       $s = \#$ of times $\alpha + \beta i$ appears in the list $r_1, \ldots, r_n$. Here k equals the larger of n, m.
     - If $g(t)$ is not of either type but can be written as $g_1 + \cdots + g_l$ with each g_i falling in one of the above two types, then obtain y_{p_i} for each g_i and write
       $$y_p = y_{p1} + \cdots + y_{pl}. \tag{9}$$
  3. Substitute y_p obtained above into the equation to fix all the coefficients.
  4. Write down the general solution. $y = C_1\,y_1 + \cdots + C_n\,y_n + y_p$.
  5. If initial value problem, use initial conditions to get $C_1, \ldots, C_n$.
- Power series method at ordinary points
  - The method: (Lectures 16, 18, 19)
    1. Identify x_0;
    2. Write $y =$ power series;
    3. Substitute into the equation;
    4. Simplify; Shift indices where necessary;
    5. Get recurrence relation;
    6. Depending on the question asked, find a general formula for a_n, or compute a_n one by one until satisfactory.
  - Technical issues:
    – Radius of convergence of general power series: (Lecture 17)
    $$\rho^{-1} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \tag{10}$$
    when the limit exists.

    – **Lower bound of** radius of convergence of solution without solving the equation:
    $$\rho \geqslant \text{distance of } x_0 \text{ to the nearest } \textbf{complex} \text{ singular point of the equation.} \tag{11}$$

    – Ordinary and singular points: A point is singular for the equation (**in standard form!**)
    $$y'' + p(x)\,y' + q(x)\,y = 0 \tag{12}$$
    if either p or q (or both) is not analytic at at this point; Otherwise it's called "ordinary".

- Checking analyticity. (Lecture 19)
  1. $e^x, \sin x, \cos x$ and polynomials are analytic for all x;
     $\ln(x)$ is **not** analytic at 0.
  2. If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then the composite function $g(f(x))$ is analytic at x_0. For example, e^{x^2} is analytic everywhere.
  3. If $f(x)$ and $g(x)$ are both analytic at x_0, then $f \pm g$ and $f g$ are analytic at x_0;
  4. If $f(x)$ and $g(x)$ are analytic at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is analytic at x_0.

- Power series method at regular singular points (Frobenius method) (Lecture 21)
  - Regular singular point: x_0 is "regular singular" point if
    - x_0 is singular;
    - $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at x_0.
  - The method: Consider the equation

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{13}$$

at an regular singular point x_0. Let ρ be no bigger than the radius of convergence of either $(x - x_0)\, p$ or $(x - x_0)^2\, q$. Let r_1, r_2 solve the indicial equation

$$r\,(r-1) + p_0\, r + q_0 = 0. \tag{14}$$

Then

1. If $r_1 - r_2$ is not an integer, then the two linearly independent solutions are given by

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \quad y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} \bar{a}_n\,(x - x_0)^n. \tag{15}$$

The coefficients a_n and $\bar{a}_n$ should be determined through the recursive relation

$$[(n+r)\,(n+r-1) + (n+r)\, p_0 + q_0]\, a_n + \sum_{m=0}^{n-1} [(m+r)\, p_{n-m} + q_{n-m}]\, a_m = 0. \tag{16}$$

2. If $r_1 = r_2$, then y_1 is given by the same formula as above, and y_2 is of the form

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{r_1} \sum_{n=1}^{\infty} d_n\,(x - x_0)^n. \tag{17}$$

3. If $r_1 - r_2$ is an integer, then take r_1 to be the larger root (More precisely, when r_1, r_2 are both complex, take r_1 to be the one with larger real part, that is $\mathrm{Re}(r_1) \geqslant \mathrm{Re}(r_2)$). Then y_1 is still the same, while

$$y_2(x) = c\, y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=0}^{\infty} e_n\,(x - x_0)^n. \tag{18}$$

Note that c may be 0.

- Solving general 2nd order linear equations (Reduction of order, variation of parameters) (Lecture 12)
  - Solve the corresponding homogeneous equation.
    - If coefficients are constants, use formulas.
    - If it's Euler equation, use formulas.
    - Otherwise,
      1. Find out one solution by guessing. Denote it as y_1.

2. Write the equation into standard form:

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{19}$$

3. Obtain y_2 from the "reduction of order" formula:

$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{20}$$

- Obtain the particular solution y_p through

$$y_p = u_1\, y_1 + u_2\, y_2,\ \ u_1 = \int \frac{-y_2\, g}{W[y_1, y_2]},\ \ u_2 = \int \frac{y_1\, g}{W[y_1, y_2]}. \tag{21}$$

  $W[y_1, y_2] = y_1\, y_2' - y_1'\, y_2$ is the Wronskian of y_1, y_2.