LECTURE 23 USING LAPLACE TRANSFORM TO SOLVE DIFFERENTIAL EQUATIONS

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Laplace transform and Its properties.

- Definition:
$$\mathcal{L}\{f\}(s) := F(s) := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{1}$$
Whether $\mathcal{L}\{f\}(s)$ or $F(s)$ is used depends on which symbol is more convenient in the context.

- Domain: Usually the above integral is not defined for all s. The set of s where it is defined is called the "domain" of the transform.
  If $|f| \leqslant K\, e^{at}$ for all t, then $\mathcal{L}\{f\}(s)$ is defined for $s > a$. Or equivalently, the domain of $\mathcal{L}\{f\}$ contains the set $s > a$.

- Properties:
  - Linearity. Let a, b be constants. Then
$$\mathcal{L}\{a\, f + b\, g\} = a\,\mathcal{L}\{f\} + b\,\mathcal{L}\{g\}. \tag{2}$$
  - Transform of derivatives.
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\,\mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$
  - Transform of products.
    - $\mathcal{L}\{e^{at} f\} = F(s-a)$. Here $F(s) = \mathcal{L}\{f\}(s)$.
    - $\mathcal{L}\{t^n f\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$. Again $F(s) = \mathcal{L}\{f\}(s)$.

- Laplace transform of simple functions.
  It all starts from $\mathcal{L}\{1\} = \frac{1}{s}$.
  Then using the above properties, one can get Laplace transforms of $e^{at}\, t^n \cos b\, t$, $e^{at}\, t^n \sin b\, t$ and furthermore linear combinations of these functions.

Example 1. Calculate $\mathcal{L}\{e^{at}\}$, $\mathcal{L}\{\cos b\, t\}$ and $\mathcal{L}\{\sin b\, t\}$

Recall $\mathcal{L}\{e^{at} f\} = F(s-a)$. Thus if we set $F(s) = \mathcal{L}\{1\}$, then
$$\mathcal{L}\{e^{at}\} = \mathcal{L}\{e^{at} \cdot 1\} = F(s-a). \tag{4}$$
As we know $F(s) = \frac{1}{s}$, we have
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}. \tag{5}$$
For cos and sin, the easiest way is the following. First recall
$$e^{ibt} = \cos b\, t + i \sin b\, t. \tag{6}$$
Therefore due to linearity
$$\mathcal{L}\{e^{ibt}\} = \mathcal{L}\{\cos b\, t\} + i\,\mathcal{L}\{\sin b\, t\}. \tag{7}$$
On the other hand, setting $a = i\, b$ in $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ we have
$$\mathcal{L}\{\cos b\, t\} + i\,\mathcal{L}\{\sin b\, t\} = \frac{1}{s - i\, b} = \frac{s + i\, b}{s^2 + b^2} = \frac{s}{s^2 + b^2} + i\,\frac{b}{s^2 + b^2}. \tag{8}$$
Thus
$$\mathcal{L}\{\cos b\, t\} = \frac{s}{s^2 + b^2}; \qquad \mathcal{L}\{\sin b\, t\} = \frac{b}{s^2 + b^2}. \tag{9}$$

Example 2. Calculate $\mathcal{L}\{t^2\, e^t \cos 3\, t\}$.

For this one we have to first use $\mathcal{L}\{t^n f\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$ with $f = e^t \cos 3t$, then use $\mathcal{L}\{e^{at} f\} = F(s-a)$ with $f = \cos 3t$.

We have

$$\begin{aligned}\mathcal{L}\{t^2 e^t \cos 3t\} &= (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{L}\{e^t \cos 3t\} \\ &= (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left(\frac{s-1}{(s-1)^2+9} \right) \\ &= \frac{2(s-1)(s^2-2s-26)}{[(s-1)^2+9]^3}. \end{aligned} \tag{10}$$

Using Laplace transform to solve equations.

- 3 Steps:
  1. Transform the equation: left hand side, right hand side;
  2. Get $Y(s) = \mathcal{L}\{y\}(s)$.
  3. Figure out y such that $y(t) = \mathcal{L}^{-1}\{Y\}$.
- The last step is usually the most difficult.
- A remark about computing $\mathcal{L}^{-1}$: There is a formula. But it involves integration in $\mathbb{C}$. Consequently for inverse transforming simple functions (such as ratios of polynomials) it is more convenient to use partial fraction combined with knowledge of simplest cases of transforms (exponentials, polynomials, cos and sin).

Examples.

Example 3. $y'' - y' - 6y = 0, \qquad y(0) = 1, y'(0) = -1.$

1. Transform the equation:

$$\mathcal{L}\{y'' - y' - 6y\} = [s^2 Y - s\,y(0) - y'(0)] - [s\,Y - y(0)] - 6Y = (s^2 - s - 6)\,Y - s + 2 \tag{11}$$

(I think I copied "$y'(0) = 1$" in the lecture?)

$\mathcal{L}\{0\} = 0$.

So the transformed equation is

$$(s^2 - s - 6)\,Y - s + 2 = 0. \tag{12}$$

2. Solve Y. We get

$$Y = \frac{s-2}{s^2 - s - 6}. \tag{13}$$

3. Find $y = \mathcal{L}^{-1}\{Y\}$. To do this we use partial fraction.

First factorize the denominator:

$$s^2 - s - 6 = (s-3)(s+2). \tag{14}$$

Then

$$\frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}. \tag{15}$$

Therefore

$$s - 2 = A(s+2) + B(s-3) = (A+B)\,s + 2A - 3B. \tag{16}$$

This can only be true if

$$A + B = 1; \qquad 2A - 3B = -2. \tag{17}$$

Solving this we get

$$A = \frac{1}{5}, \qquad B = \frac{4}{5}. \tag{18}$$

Now we have

$$y = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{1/5}{s-3} + \frac{4/5}{s+2}\right\} = \frac{1}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{4}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{e^{3t}}{5} + \frac{4\,e^{-2t}}{5}. \tag{19}$$

Example 4. $y'' + 2\,y' + y = 4\,e^{-t}, \qquad y(0) = 0, y'(0) = -1.$

1. Transform the equation

$$\mathcal{L}\{y'' + 2\,y' + y\} = [s^2\,Y - s\,y(0) - y'(0)] + 2\,[s\,Y - y(0)] + Y = (s^2 + 2\,s + 1)\,Y + 1. \tag{20}$$

$$\mathcal{L}\{4\,e^{-t}\} = 4\,\mathcal{L}\{e^{-t}\} = \frac{4}{s+1}. \tag{21}$$

So the transformed equation is

$$(s^2 + 2\,s + 1)\,Y + 1 = \frac{4}{s+1}. \tag{22}$$

2. Solve Y.

$$Y = \frac{4}{(s+1)(s^2+2\,s+1)} - \frac{1}{(s^2+2\,s+1)}. \tag{23}$$

3. Find $y = \mathcal{L}^{-1}(Y)$. We have

$$y = \mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s^2+2\,s+1)} - \frac{1}{(s^2+2\,s+1)}\right\} = 4\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+2\,s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2\,s+1)}\right\}. \tag{24}$$

Now notice that $s^2 + 2\,s + 1 = (s+1)^2$ so

$$y = 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}. \tag{25}$$

Recall

$$\mathcal{L}\{e^{at}\,f(t)\} = F(s-a) \Longrightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at}\,f(t). \tag{26}$$

So

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}\,e^{-t}. \tag{27}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-t}\,t. \tag{28}$$

Put these together,

$$y = 2\,t^2\,e^{-t} - t\,e^{-t}. \tag{29}$$

Partial Fractions.

Remark 5. There are a bunch of online videos explaining this topic. For example

- Khan Academy: http://youtu.be/S-XKGBesRzk
- MIT Open Course Partial fraction (Just search this).

The basic idea is to write $\frac{P}{Q}$ into the sum of functions of the type $\frac{A}{(s-r)^m}$. As

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{A}{(s-r)^m}\right\} &= A\,e^{rt}\,\mathcal{L}^{-1}\left\{\frac{1}{s^m}\right\} \\
&= A\,e^{rt}\,\mathcal{L}^{-1}\left\{(-1)^{m-1}\,\frac{1}{(m-1)!}\,\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left(\frac{1}{s}\right)\right\} \\
&= A\,e^{rt}\,\frac{1}{(m-1)!}\,t^{m-1}
\end{aligned} \tag{30}$$

we can then invert each term and obtain $\mathcal{L}^{-1}\{P/Q\}$.

To carry out this plan, we need to first factorize Q:

$$Q(s) = (s - r_1) \cdots (s - r_n) \tag{31}$$

where n is the degree of Q. Clearly there are three cases:

1. All r_i's are real and distinct;
2. Some r_i's are real and repeated;
3. Some r_i's are complex.

We discuss them one by one.

1. All r_i's are real and distinct.

   In this case we can find appropriate constants $A_1, \ldots, A_n$ such that

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}. \tag{32}$$

Example 6. Compute

$$\mathcal{L}^{-1}\left\{\frac{6\,s^2 - 13\,s + 2}{s\,(s-1)\,(s-6)}\right\}. \tag{33}$$

Solution. First we check that the degree of the denominator is indeed higher than the degree of the nominator. Thus we can write

$$\frac{6\,s^2 - 13\,s + 2}{s\,(s-1)\,(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6}. \tag{34}$$

Summing the RHS gives

$$\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6} = \frac{A\,(s-1)\,(s-6) + B\,s\,(s-6) + C\,s\,(s-1)}{s\,(s-1)\,(s-6)} \tag{35}$$

We need to find A, B, C such that

$$A\,(s-1)\,(s-6) + B\,s\,(s-6) + C\,s\,(s-1) = 6\,s^2 - 13\,s + 2. \tag{36}$$

Naïvely, one may want to expand the LHS into

$$(A + B + C)\,s^2 + (-7\,A - 6\,B - C)\,s + 6\,A \tag{37}$$

and then solve

$$A + B + C = 6 \tag{38}$$

$$-7\,A - 6\,B - C = -13 \tag{39}$$

$$6\,A = 2. \tag{40}$$

However there is a much simpler way. The key observation is that when we set $s = 0, 1, 6$, exactly two of the three terms vanish. In other words, when we set $s = 0, 1, 6$, exactly one unknown is left in the equation – one equation, one unknown, linear: the simplest equation possible!

- Setting $s = 0$, we have

$$A\,(0-1)\,(0-6) = 2 \Longrightarrow A = 1/3. \tag{41}$$

- Setting $s = 1$, we have

$$B\,(1-6) = -5 \Longrightarrow B = 1. \tag{42}$$

- Setting $s = 6$, we have

$$C\,6\,(6-1) = 216 - 78 + 2 = 140 \Longrightarrow C = 14/3. \tag{43}$$

Thus the solution is

$$A = \frac{1}{3}, \quad B = 1, \quad C = \frac{14}{3}. \tag{44}$$

Therefore we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{6\,s^2-13\,s+2}{s\,(s-1)\,(s-6)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/3}{s}+\frac{1}{s-1}+\frac{14/3}{s-6}\right\}\\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{14}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}\\ &= \frac{1}{3}+e^t+\frac{14}{3}e^{6t}. \end{aligned} \tag{45}$$

2. Some r_i's are real and repeated.

   In this case, for this particular r_i, we need to put int

$$\frac{A_{i1}}{s-r_i}+\cdots+\frac{A_{im}}{(s-r_i)^m} \tag{46}$$

where m is the multiplicity of this particular r_i.

Example 7. Compute

$$\mathcal{L}^{-1}\left\{\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}\right\}. \tag{47}$$

Solution. Again, we first check that the nominator's degree is lower.

Next we write the function into partial fractions:

$$\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1}. \tag{48}$$

Calculating the RHS, we have

$$\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1} = \frac{A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2}{(s+3)^2\,(s+1)}. \tag{49}$$

We need A, B, C such that

$$A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2=5\,s^2+34\,s+53. \tag{50}$$

Setting $s=-3$, we have

$$B\,(-3+1)=45-102+53=-4\Longrightarrow B=2. \tag{51}$$

Setting $s=-1$, we have

$$C\,(-1+3)^2=5-34+53=24\Longrightarrow C=6. \tag{52}$$

To determine A, we pick $s=0$ to obtain

$$3\,A+B+9\,C=53\Longrightarrow A=-1. \tag{53}$$

Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}\right\} &= \mathcal{L}^{-1}\left\{(-1)\,\frac{1}{s+3}+2\,\frac{1}{(s+3)^2}+6\,\frac{1}{s+1}\right\}\\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}+2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}+6\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= -e^{-3t}+2\,t\,e^{-3t}+6\,e^{-t}. \end{aligned} \tag{54}$$

Remark 8. As we have seen, when there are repeated roots, some of the unknowns cannot be determined easily. To determine these constants, we can either pick arbitrary values of s, or use the following idea. Let's say we need to determine A, B, C in

$$f(s)=A+B\,(s+1)+C\,(s+1)^2. \tag{55}$$

Setting $s=-1$ we have

$$A=f(-1). \tag{56}$$

To determine B, C we can either pick two arbitrary values of s, say $s=0,2$ and solve

$$A+B+C = f(0) \tag{57}$$
$$A+3\,B+9\,C = f(2) \tag{58}$$

or we can differentiate the equation:

$$f'(s)=B+C\,(s+1)\Longrightarrow B=f'(-1), \tag{59}$$

$$f''(s)=C\Longrightarrow C=f''(-1). \tag{60}$$

Is this "differentiation" method a simpler approach than setting arbitrary values of s? I guess it differs from case to case, and from person to person.

3. Some r_i's are complex.

   Let's say $r_i=\alpha+i\,\beta$, with multiplicity m. It can be shown that there must be $r_j=\alpha-i\,\beta$ with the same multiplicity. Then corresponding to r_i, r_j we introduce

$$\frac{C_1\,s+D_1}{(s-\alpha)^2+\beta^2}+\frac{C_2\,s+D_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{C_m\,s+D_m}{[(s-\alpha)^2+\beta^2]^m}. \tag{61}$$

Equivalently, we can factorize Q into first order factors $s-r_i$ and second order factors $s^2+p\,s+q$ where $s^2+p\,s+q=0$ would give conjugate complex roots. Then suppose there is a factor of $(s^2+p\,s+q)^m$, the corresponding partial fractions are

$$\frac{C_1\,s+D_1}{s^2+p\,s+q}+\cdots+\frac{C_m\,s+D_m}{(s^2+p\,s+q)^m}. \tag{62}$$

Example 9. Compute

$$\mathcal{L}^{-1}\left\{\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}\right\}. \tag{63}$$

Solution. Again, the degree of the nominator is lower. Check.

We write

$$\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}=\frac{A}{s-2}+\frac{B\,s+C}{s^2+2\,s+5}=\frac{A\,(s^2+2\,s+5)+(B\,s+C)\,(s-2)}{(s-2)\,(s^2+2\,s+5)}. \tag{64}$$

We need to find A, B, C such that

$$A\,(s^2+2\,s+5)+(B\,s+C)\,(s-2)=7\,s^2+23\,s+30. \tag{65}$$

Setting $s=2$ we have

$$A\,(4+4+5)=28+46+30=104\Longrightarrow A=8. \tag{66}$$

To find B, C, we need to set s to values different from 2 and obtain equations for B, C. There is a minor trick here that can make the equations simple. We notice that the B disappears if we set $s=0$. Setting $s=0$ we have

$$5\,A-2\,C=30\Longrightarrow C=5. \tag{67}$$

Finally comparing the s^2 terms (or setting s to yet another value) we have

$$A+B=7\Longrightarrow B=-1. \tag{68}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{8}{s-2}\right\}+\mathcal{L}^{-1}\left\{\frac{-s+5}{s^2+2\,s+5}\right\}. \tag{69}$$

The first term leads to $8\,e^{2t}$. To compute the second term, we further simplify

$$\begin{aligned}\frac{-s+5}{s^2+2\,s+5} &= -\frac{s-5}{(s+1)^2+4}\\ &= -\frac{s+1}{(s+1)^2+4}+3\,\frac{2}{(s+1)^2+2^2}.\end{aligned} \tag{70}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{-s+5}{s^2+2s+5}\right\} = -\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3\,\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \quad (71)$$

$$= -e^{-t}\cos 2t+3\,e^{-t}\sin 2t. \quad (72)$$

Summarizing, we have

$$\mathcal{L}^{-1}\left\{\frac{7s^2+23s+30}{(s-2)\,(s^2+2s+5)}\right\}=8\,e^{2t}-e^{-t}\cos 2t+3\,e^{-t}\sin 2t. \quad (73)$$

Remark 10. We summarize the method of partial fractions. The method represents a complicated ratio P/Q by a sum of simple ratios in which only simple polynomials of degrees no more than 2 are involved through the following procedure.

1. Factorize Q:

$$Q(s)=(s-r_1)\cdots(s-r_n). \quad (74)$$

2. Go through $r_1,\ldots,r_n$ and write down the terms of the RHS sum of

$$\frac{P}{Q}=\sum\cdots \quad (75)$$

according to the following rules:

i. If r_i is a single real root, write down

$$\frac{A_i}{s-r_i}. \quad (76)$$

ii. If r_i is a repeated real root, say with multiplicity m, write down

$$\frac{A_{i1}}{s-r_i}+\frac{A_{i2}}{(s-r_i)^2}+\cdots+\frac{A_{im}}{(s-r_i)^m}. \quad (77)$$

After this, discard those other copies of r_i from the list $r_1,\ldots,r_n$ and move on to the next root. Note that the previous "single root" case is actually contained in this case.

iii. If $r_i=\alpha+i\,\beta$ is complex root with multiplicity m, then there must be another $r_j=\alpha-i\,\beta$ with the same multiplicity. Write down

$$\frac{C_{i1}\,s+D_{i1}}{(s-\alpha)^2+\beta^2}+\cdots+\frac{C_{im}\,s+D_{im}}{[(s-\alpha)^2+\beta^2]^m}. \quad (78)$$

For example, if

$$Q(s)=(s-1)(s-3)^3\,(s+i)\,(s-i), \quad (79)$$

we have six roots (counting multiplicity) $1,3,3,3,-i,i$. Now to form the RHS, we go through this list one by one:

$$1\!:\text{Single real root} \implies \frac{A}{s-1}; \quad (80)$$

$$3\!:\text{repeated real root with multiplicity 3} \implies \frac{B}{s-3}+\frac{C}{(s-3)^2}+\frac{D}{(s-3)^3}; \quad (81)$$

$$\text{Ignore the remaining two 3's.} \quad (82)$$

$$-i\!:\text{Complex root with multiplicity 1} \implies \frac{E\,s+F}{s^2+1}; \quad (83)$$

$$\text{Ignore the complex conjugate } i. \quad (84)$$

3. Determine the constants using the following procedure: We use the above example

$$Q(s)=(s-1)(s-3)^3\,(s+i)\,(s-i), \quad (85)$$

which gives

$$\frac{P}{Q}=\frac{A}{s-1}+\frac{B}{s-3}+\frac{C}{(s-3)^2}+\frac{D}{(s-3)^3}+\frac{E\,s+F}{s^2+1} \quad (86)$$

leading to

$$P(s) = A\,(s-3)^3\,(s^2+1) + B\,(s-1)\,(s-3)^2\,(s^2+1) + C\,(s-1)\,(s-3)\,(s^2+1) + D\,(s-1)\,(s^2+1) + (E\,s+F)\,(s-1)\,(s-3)^3. \tag{87}$$

i. Set s to be each of the single real roots. This would immediately give all the constants corresponding to those single roots.

   In our example, we see that setting $s=1$ immediately gives A.

ii. Set s to be the repeated real roots. This would immediately give all the constants in the last terms of the terms corresponding to those repeated roots.

   In our example, setting $s=3$ immediately gives D.

   - At this stage, you may want to try the "differentiation method". In our example, differentiating once we obtain

$$\begin{aligned} P'(s) \;=\; & A\,[2\,(s-3)\,(s^2+1)+(s-3)^2\,(2\,s)] \\ & +B\,[(s-3)^2\,(s^2+1)+2\,(s-1)\,(s-3)\,(s^2+1)+2\,s\,(s-1)\,(s-3)^2] \\ & +C\,[(s-3)\,(s^2+1)+(s-1)\,(s^2+1)+2\,s\,(s-1)\,(s-3)] \\ & +D\,[s^2+1+2\,s\,(s-1)] \\ & +E\,[(s-1)\,(s-3)^3]+(E\,s+F)\,[(s-3)^3+3\,(s-1)\,(s-3)^2]. \end{aligned} \tag{88}$$

     Looks very complicated, but as soon as we substitute $s=3$, only C and D remain. As we have already found D, determining C is easy.

     Differentiate again and then set $s=3$, we obtain one equation for B, C, D. Since we already know C, D, B is immediately determined.

iii. Set $s=0$.

iv. If there are still some constants need to be determined, compare the coefficient for the highest power term s^n of the RHS. Note that as P has lower degree, we always have $0=\cdots$. In our example,

$$P(s) = A\,(s-3)^3\,(s^2+1) + B\,(s-1)\,(s-3)^2\,(s^2+1) + C\,(s-1)\,(s-3)\,(s^2+1) + D\,(s-1)\,(s^2+1) + (E\,s+F)\,(s-1)\,(s-3)^3. \tag{89}$$

   The higher order term on the RHS is s^5. Assuming

$$P(s) = p_5\,s^5 + \cdots \tag{90}$$

   we have

$$p_5 = A + B + E. \tag{91}$$

   Note that this is equivalent to setting $s=\infty$.

v. Let's say there are k constants still need to be determined. Set s to be k arbitrary values. You will obtain k equations for these k costants, solve them.

   In our example, $k=0$ if we have used the "differentiation method", $k=2$ if we haven't.