Lecture 21 Power Series Method at Singular Points – Frobenius Theory

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Review.

- The usual power series method, that is setting $y=\sum_{n=0}^{\infty} a_n (x-x_0)^n$, breaks down if x_0 is a singular point. Here "breaks down" means "cannot find all solutions".

- It's possible to completely solve one class of DE

$$a\,x^2\,y''+b\,x\,y'+c\,y=0 \tag{1}$$

  – Euler equations – at singular points. The solutions involve x^r (r not $0,1,2,\ldots$) and $\ln x$, which are not analytic at $x_0=0$ and therefore cannot be represented as $\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

  Remark 1. Note that $x^{\alpha+i\beta}=x^{\alpha}(\cos\beta\ln x+\sin\beta\ln x)$.

- It turns out, once we include these two new ingredients into our ansatz, we can solve equations at singular points as long as those singular points are "regular singular":

  x_0 is "regular singular" point if

  - x_0 is singular;
  - $(x-x_0)\,p(x)$ and $(x-x_0)^2\,q(x)$ are analytic at x_0.

The Method of Frobenius.

The method of Frobenius is a modification to the power series method guided by the above observation. This method is effective at regular singular points. The basic idea is to look for solutions of the form

$$(x-x_0)^r\sum_{n=0}^{\infty} a_n (x-x_0)^n. \tag{2}$$

Consider the equation

$$y''+p(x)\,y'+q(x)\,y=0. \tag{3}$$

Let x_0 be a regular singular point. That is

$$p(x)\,(x-x_0)=\sum_{n=0}^{\infty} p_n (x-x_0)^n, \qquad q(x)\,(x-x_0)^2=\sum_{n=0}^{\infty} q_n (x-x_0)^n \tag{4}$$

with certain radii of convergence.

To make the following discussion easier to read, we assume $x_0=0$.

Substitute the expansion

$$y=x^r\sum_{n=0}^{\infty} a_n\,x^n \tag{5}$$

into the equation we get

$$\left(x^r\sum_{n=0}^{\infty} a_n\,x^n\right)''+p(x)\left(x^r\sum_{n=0}^{\infty} a_n\,x^n\right)'+q(x)\,x^r\sum_{n=0}^{\infty} a_n\,x^n=0. \tag{6}$$

Now compute

$$\left(x^r \sum_{n=0}^{\infty} a_n x^n\right)'' = \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}. \tag{7}$$

$$\begin{aligned} p(x)\left(x^r \sum_{n=0}^{\infty} a_n x^n\right)' &= p(x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)' \\ &= p(x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) \\ &= (p(x)\, x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-2}\right) \\ &= \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-2}\right) \\ &= \sum_{n=0}^{\infty} \left\{\sum_{m=0}^{n} p_{n-m}\, (m+r)\, a_m\right\} x^{n+r-2}. \end{aligned} \tag{8}$$

$$\begin{aligned} q(x)\, x^r \sum_{n=0}^{\infty} a_n x^n &= x^{r-2}\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} q_{n-m}\, a_m\right] x^{n+r-2}. \end{aligned} \tag{9}$$

Now the equation becomes

$$\sum_{n=0}^{\infty} \left\{(n+r)(n+r-1)\, a_n + \sum_{m=0}^{n} [(m+r)\, p_{n-m} + q_{n-m}]\, a_m\right\} x^{n+r-2} = 0. \tag{10}$$

Or equivalently

$$\sum_{n=0}^{\infty} \left\{[(n+r)(n+r-1) + (n+r)\, p_0 + q_0]\, a_n + \sum_{m=0}^{n-1} [(m+r)\, p_{n-m} + q_{n-m}]\, a_m\right\} x^{n+r-2} = 0. \tag{11}$$

This leads to the following equations:

$$(n=0): \qquad [r(r-1) + p_0\, r + q_0]\, a_0 = 0, \tag{12}$$

$$(n \geqslant 1): \qquad [(n+r)(n+r-1) + (n+r)\, p_0 + q_0]\, a_n + \sum_{m=0}^{n-1} [(m+r)\, p_{n-m} + q_{n-m}]\, a_m = 0. \tag{13}$$

The $n=0$ equation is singled out because if we require $a_0 \neq 0$ (which is natural as when $a_0 = 0$, we have $y = x^{r+1} \sum_{m=0}^{\infty} b_m x^m$ where $b_m = a_{m+1}$.), then it becomes a condition on r:

$$r(r-1) + p_0\, r + q_0 = 0. \tag{14}$$

This is called the **indicial** equation and will provide us with two roots r_1, r_2 (Some complicated situation may arise, we will discuss them later). These two roots are called **exponents** of the regular singular point $x=0$. After deciding r, the $n \geqslant 1$ relations provide us with a way to determine a_n one by one.

It turns out that there are three cases: $r_1 \neq r_2$ with $r_1 - r_2$ not an integer; $r_1 = r_2$; $r_1 - r_2$ is an integer. Before we discuss these cases in a bit more detail, let's state the following theorem which summarizes the method of Frobenius in its full glory.

Theorem 2. *Consider the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{15}$$

at an regular singular point x_0. Let ρ be no bigger than the radius of convergence of either $(x - x_0)\, p$ or $(x - x_0)^2\, q$. Let r_1, r_2 solve the indicial equation

$$r\,(r-1) + p_0\, r + q_0 = 0. \tag{16}$$

Then

1. *If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then the two linearly independent solutions are given by*

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \quad y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} \bar{a}_n\,(x - x_0)^n. \tag{17}$$

   The coefficients a_n and $\bar{a}_n$ should be determined through the recursive relation

$$[(n+r)\,(n+r-1) + (n+r)\,p_0 + q_0]\,a_n + \sum_{m=0}^{n-1} [(m+r)\,p_{n-m} + q_{n-m}]\,a_m = 0. \tag{18}$$

2. *If $r_1 = r_2$, then y_1 is given by the same formula as above, and y_2 is of the form*

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{r_1} \sum_{n=1}^{\infty} d_n\,(x - x_0)^n. \tag{19}$$

3. *If $r_1 - r_2$ is an integer, then take r_1 to be the larger root (More precisely, when r_1, r_2 are both complex, take r_1 to be the one with larger real part, that is $\mathrm{Re}(r_1) \geqslant \mathrm{Re}(r_2)$). Then y_1 is still the same, while*

$$y_2(x) = c\, y_1(x) \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=0}^{\infty} e_n\,(x - x_0)^n. \tag{20}$$

   Note that c may be 0.

All the solutions constructed above converge at least for $0 < |x - x_0| < \rho$ (Remember that x_0 is a singular point, so we cannot expect convergence there).

Remark 3. Note that, although ρ is given by radii of convergence of $(x - x_0)\, p$ and $(x - x_0)^2\, q$, in practice, it is the same as the distance from x_0 to the nearest singular point of p and q – no $(x - x_0)$ factor needed.

Remark 4. The proof of this theorem is through careful estimate of the size of a_n using the recurrence relation. See R. P. Agarwal & D. O'Regan "Ordinary and Partial Differential Equations: With Special Functions, Fourier Series, and Boundary Value Problems" Lecture 5.

Remark 5. In fact the converse of this theorem is also true. That is if all solutions of the equation satisfies

$$\lim_{x \to x_0} |x - x_0|^r\, y(x) = 0 \tag{21}$$

for some r, then $(x - x_0)\, p$ and $(x - x_0)^2\, q$ are analytic at x_0. This is called Fuchs' Theorem. Its proof is a tour de force of complex analysis and can be found in K. Yosida "Lectures on Differential and Integral Equations", pp. 37 – 40.

Examples.

In this class we will only require solving equations with $r_1 - r_2$ not an integer.

Example 6. Solve

$$x^2\, y'' + x\left(x - \frac{1}{2}\right) y' + \frac{1}{2}\, y = 0 \tag{22}$$

at $x_0 = 0$.

Solution. We first write it into the standard form

$$y'' + \frac{(x-1/2)}{x}\, y' + \frac{1}{2\,x^2}\, y = 0. \tag{23}$$

Thus $p(x) = \frac{x-1/2}{x}$ and $q(x) = \frac{1}{2\,x^2}$. It is clear that $x\,p(x)$ and $x^2\,q(x)$ are analytic so 0 is a regular singular point, and the method of Frobenius applies.

Now we write

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+r}. \tag{24}$$

Substitute into the equation, we have

$$\left(\sum_{n=0}^{\infty} a_n\, x^{n+r}\right)'' + \frac{x-1/2}{x}\left(\sum_{n=0}^{\infty} a_n\, x^{n+r}\right)' + \frac{1}{2\,x^2}\sum_{n=0}^{\infty} a_n\, x^{n+r} = 0. \tag{25}$$

As p and q are particularly simple, we write the equation as

$$\left(\sum_{n=0}^{\infty} a_n\, x^{n+r}\right)'' + \left(\sum_{n=0}^{\infty} a_n\, x^{n+r}\right)' - \frac{1}{2\,x}\left(\sum_{n=0}^{\infty} a_n\, x^{n+r}\right)' + \frac{1}{2\,x^2}\sum_{n=0}^{\infty} a_n\, x^{n+r} = 0. \tag{26}$$

Carrying out the differentiation, we reach

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n\, x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1} - \frac{1}{2}\sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-2} + \sum_{n=0}^{\infty} \frac{a_n}{2}\, x^{n+r-2} = 0. \tag{27}$$

Shifting index:

$$\sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1} = \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1}\, x^{n+r-2}. \tag{28}$$

Now the equation becomes

$$\left[r\,(r-1) - \frac{r}{2} + \frac{1}{2}\right] a_0\, x^{r-2} + \sum_{n=1}^{\infty}\left\{\left[(n+r)\,(n+r-1) - \frac{1}{2}\,(n+r) + \frac{1}{2}\right] a_n + (n+r-1)\, a_{n-1}\right\} x^{n+r-2} = 0. \tag{29}$$

The indicial equation is

$$r\,(r-1) - \frac{r}{2} + \frac{1}{2} = 0 \Longrightarrow r_1 = 1, r_2 = \frac{1}{2}. \tag{30}$$

Their difference is not an integer.

To find y_1 we set $r = r_1 = 1$. The recurrence relation

$$\left[(n+r)\,(n+r-1) - \frac{1}{2}\,(n+r) + \frac{1}{2}\right] a_n + (n+r-1)\, a_{n-1} = 0 \tag{31}$$

becomes

$$\left[n\,(n+1) - \frac{1}{2}\,(n+1) + \frac{1}{2}\right] a_n + n\, a_{n-1} = 0 \tag{32}$$

which simplifies to

$$a_n = -\frac{2}{2\,n+1}\, a_{n-1}. \tag{33}$$

This gives

$$a_n = (-1)^n \frac{2^n}{(2\,n+1)\,(2\,n-1)\cdots 3}\, a_0. \tag{34}$$

Setting $a_0 = 1$ we obtain

$$y_1(x) = |x| \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2\,n+1)\,(2\,n-1)\cdots 3}\, x^n. \tag{35}$$

To find y_2 we set $r = r_2 = 1/2$. The recurrence relation becomes

$$a_n = -\frac{1}{n} a_{n-1} \Longrightarrow a_n = (-1)^n \frac{1}{n!} a_0 \tag{36}$$

so

$$y_2(x) = |x|^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n = |x|^{1/2} e^{-x}. \tag{37}$$

Finall the general solution is

$$y(x) = C_1 |x| \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n+1)(2n-1)\cdots 3} x^n + C_2 |x|^{1/2} e^{-x}. \tag{38}$$

Remark 7. Of course, for anyone who can remember the formulas, there is no need to do all these differentiation and index-shifting.

Remark 8. After getting $r_{1,2}$, one can also write y_1, y_2 explicitly and solve a_n by substituting them into the equation. See homework 8 solution.

Remark 9. For detailed discussion of the why and how of the other two cases, see my Math 334 2010 note at http://www.math.ualberta.ca/~xinweiyu/334.1.10f/DE_series_sol.pdf. They are **not required** for Math 334 2011.