Lecture 20 Power Series Method (Cont.)

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Review.

Power series method for solving

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{1}$$

Write

$$y = \sum_{n=0}^{\infty} a_n\, (x - x_0)^n \tag{2}$$

and substitute into the equation.

The method works (meaning: finds the general solution) if all solutions to the equation are analytic at x_0. This is guaranteed by p, q being analytic at x_0. Therefore the definition:

- x_0 is said to be an ordinary point for the equation $y'' + p(x)\, y' + q(x)\, y = 0$ if both p, q are analytic at x_0. Otherwise x_0 is said to be singular.

So the methods we used so far works at ordinary points. Furthermore we can determine the following lower bound for the power series solution before actually finding out a_n's:

The radius of convergence is at least the distance from x_0 to the nearest singular point **in** $\mathbb{C}$.

Rules for checking analyticity:

1. $e^x, \sin x, \cos x$ and polynomials are analytic for all x;
   $\ln(x)$ is **not** analytic at 0.
2. If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then the composite function $g(f(x))$ is analytic at x_0. For example, e^{x^2} is analytic everywhere.
3. If $f(x)$ and $g(x)$ are both analytic at x_0, then $f \pm g$ and $f g$ are analytic at x_0;
4. If $f(x)$ and $g(x)$ are analytic at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is analytic at x_0.

Remark 1. Even if $g(x_0) = 0$, f/g may still be analytic. See example below.

Remark 2. It's quite hard to prove a function $f(x)$ is **not** analytic at a certain point. However in most situations the following rule is enough:

$f(x)$ is **not** analytic at x_0 if $f^{(n)}(x_0)$ is infinity for some n.

Example 3. e^{x^3} is analytic everywhere because it's a composite function $g(f(x))$ with $g = e^x$, $f = x^3$, both g, f are analytic everywhere.

Example 4. $\frac{1}{1+x^2}$ is of the form f/g with $f = 1, g = 1 + x^2$. Both are polynomials so analytic everywhere. As $1 + x^2 = 0$ only at $\pm i$, we know that $\frac{1}{1+x^2}$ is analytic at all $x \neq \pm i$.

At $\pm i$ we notice that $\frac{1}{1+x^2}$ becomes infinity and therefore is not analytic there.

Example 5. $\frac{x^3}{\sin x - x}$ is analytic at $x = 0$ despite $\sin x - x = 0$ there. To see why, expand $\sin x = x - \frac{x^3}{6} + \cdots$ thus

$$\frac{x^3}{\sin x - x} = \frac{x^3}{-\frac{x^3}{6} + \cdots} = \frac{1}{-\frac{1}{6} + \frac{x^2}{5!} - \cdots}. \tag{3}$$

Now the new ratio, after cancelling the common factor x^3, is still of the form f/g with $f = 1, g = -\frac{1}{6} + \frac{x^2}{5!} - \cdots$. It can be shown that both f, g are analytic everywhere. As $g \neq 0$ at 0, the ratio is analytic at 0.

Remark 6. Remembering Taylor expansions.

- e^x: No other way but just remember it.

- $\cos x, \sin x$: use the fact
$$\sum_{n=0}^{\infty} \frac{(i\,x)^n}{n!} = e^{i\,x} = \cos x + i \sin x \tag{4}$$
- $\ln(1 \pm x)$: Use the fact
$$\ln(1-x) = \int \frac{1}{1-x} = \int \sum_{n=0}^{\infty} x^n. \tag{5}$$
$\ln(1+x) = \ln(1-(-x))$.

Power series method at singular points.

Consider solving
$$x^2\, y'' + x\, y' + y = 0 \tag{6}$$
by writing
$$y = \sum_{n=0}^{\infty} a_n\, x^n. \tag{7}$$
Substitute into the equation we get
$$\begin{aligned} & x^2 \left(\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} \right) + x \left(\sum_{n=1}^{\infty} a_n\, n\, x^{n-1} \right) + \sum_{n=0}^{\infty} a_n\, x^n = 0 \\ \Longrightarrow\ & \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^n + \sum_{n=1}^{\infty} a_n\, n\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0 \\ \Longrightarrow\ & a_0 + 2\, a_1\, x + \sum_{n=2}^{\infty} (n^2+1)\, a_n\, x^n = 0 \\ \Longrightarrow\ & a_0 = a_1 = \cdots = a_n = \cdots = 0. \end{aligned} \tag{8}$$
The reason is that, at a singular point, not all solutions are analytic. So the ansatz $y = \sum_{n=0}^{\infty} a_n\, x^n$ cannot "catch" those solutions.

Fortunately this equation is a special case of the so-called "Euler equations" which can be solved.

Euler equations.

a, b, c constants,
$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0. \tag{9}$$
To solve it, guess $y = x^r$. Substitute into the equation we get the characteristic (indicial) equation
$$a\, r\,(r-1) + b\, r + c = 0. \tag{10}$$
This equation has two roots:

- $r_1 \neq r_2$, both real. Then
$$y = C_1\, x^{r_1} + C_2\, x^{r_2}; \tag{11}$$
- $r_1 = r_2 = r$. Then
$$y = C_1\, x^r + C_2\, x^r \ln x. \tag{12}$$
- $r_{1,2} = \alpha \pm i\,\beta$, then
$$y = C_1\, x^{\alpha} \cos(\beta \ln x) + C_2\, x^{\alpha} \sin(\beta \ln x). \tag{13}$$

Thus for our equation
$$x^2\, y'' + x\, y' + y = 0 \tag{14}$$
we get $r_{1,2} = \pm i$ so $\alpha = 0, \beta = 1$ which gives
$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x). \tag{15}$$
Such solutions are not analytic at 0 unless $C_1 = C_2 = 0$.

Regular singular points.

It turns out that, if the equation $y'' + p(x)\, y' + q(x)\, y = 0$ are no more singular than the Euler equation, then the solutions are not analytic only by a factor x^r (with possible $\ln x$).[1] This leads to the following definition:

- x_0 is "regular singular" point if
  - x_0 is singular;
  - $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at x_0.

Warning. In p.274 of the textbook, there are two formulas (29), (30) which claim that x_0 is regular singular if

$$\lim_{x \to x_0} (x - x_0)\, p(x), \qquad \lim_{x \to x_0} (x - x_0)^2\, q(x) \tag{16}$$

are finite. **This is only true when p, q are ratios of polynomials.** This restriction is only implicitly mentioned in the textbook. So don't take (29),(30) out of context and use them to justify your claims.

1. $\cos(\beta \ln x) + i \sin(\beta \ln x) = e^{i\beta \ln x} = x^{i\beta}$.