Lecture 18 Power Series Method

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The Problem.

Equation (+ initial value);

- Possible question 1 (Q1): Solve the equation (or find general formula for a_n);
- Possible question 2 (Q2): Find the first 5 nonzero terms;
- Possible question 3 (Q3): Find the first 4 nonzero terms for y_1, y_2.

The Method.

1. Identify x_0;
2. Write $y =$ power series;
3. Substitute into the equation;
4. Simplify; Shift indices where necessary;
5. Get recurrence relation;
6. Depending on the question asked, find a general formula for a_n, or compute a_n one by one until satisfactory.

Example.

$$y'' - x\,y' - y = 0, \qquad x_0 = 1. \tag{1}$$

Remark 1. When $x_0 \neq 0$, there are two ways to proceed.

1. (Recommended) Change of variable $t = x - x_0$.
   - Then the equation becomes
   $$y'' - (t+1)\,y' - y = 0, \qquad t_0 = 0. \tag{2}$$
   - Solve it. Get $y = \sum \cdots t^n$.
   - Write down $y = \sum \cdots (x-1)^n$;
2. Write
   $$y = \sum a_n\,(x - x_0)^n \tag{3}$$
   and substitute into equation.

   Caution: Do not forget to write every x into $(x - x_0) + x_0$!

Solution 1. We use the first method to try Q1 and Q2.

We solve

$$y'' - (t+1)\,y' - y = 0, \qquad t_0 = 0. \tag{4}$$

Write

$$y = \sum_{n=0}^{\infty} a_n\,t^n. \tag{5}$$

Substitute into equation:

$$y'' = \sum_{n=2}^{\infty} a_n\,n\,(n-1)\,t^{n-2} = \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,t^n. \tag{6}$$

$$y' = \sum_{n=1}^{\infty} a_n\,n\,t^{n-1} = \sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n. \tag{7}$$

So equation becomes

$$\sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,t^n-(t+1)\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n-\sum_{n=0}^{\infty} a_n\,t^n=0. \tag{8}$$

We can get recurrence relation only if all sums are with generic term $\cdots t^n$ so we have to try to write the 2nd term into this form.

$$\begin{aligned}(t+1)\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n &= t\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n+\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n\\ &= \sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^{n+1}+\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n\\ &= \sum_{n=1}^{\infty} a_n\,n\,t^n+\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n. \end{aligned}\tag{9}$$

Now the equation is

$$\sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,t^n-\sum_{n=1}^{\infty} a_n\,n\,t^n-\sum_{n=0}^{\infty} a_{n+1}\,(n+1)\,t^n-\sum_{n=0}^{\infty} a_n\,t^n=0. \tag{10}$$

Note that we have to be careful here as the 2nd sum starts from $n=1$ while all others starts from $n=0$. So the recurrence relation for $n=0$ is different from those for $n\geqslant 1$.

- $n=0$:

$$2\,a_2-a_1-a_0=0. \tag{11}$$

- $n\geqslant 1$:

$$(n+2)\,a_{n+2}-a_{n+1}-a_n=0. \tag{12}$$

Now we deal with Q1 and Q2.

- Q1. A correct answer to Q1 should be a formula for a_n involving only a_0, a_1 and n. Sometimes this can be done, other times it cannot. The steps are the following.

  1. Start from the general recurrence relation:

  $$(n+2)\,a_{n+2}-a_{n+1}-a_n=0. \tag{13}$$

  Rewrite it as $a_n=\cdots$ by shifting index:

  $$a_{n+2}=\frac{a_{n+1}+a_n}{n+2}\Longrightarrow a_n=\frac{a_{n-1}+a_{n-2}}{n}. \tag{14}$$

  2. Observe that the above represent a_n by a_{n-1} and a_{n-2}, while we need a representation of a_n by a_0, a_1. So we proceed as follows: Represent a_n by a_{n-2}, a_{n-3} then by a_{n-3}, a_{n-4} then by $a_{n-4}, a_{n-5}\cdots$

  $$\begin{aligned} a_n &= \frac{a_{n-1}+a_{n-2}}{n}\\ &= \frac{\frac{a_{n-2}+a_{n-3}}{n-1}+a_{n-2}}{n}\\ &= \frac{n\,a_{n-2}+a_{n-3}}{n\,(n-1)}\\ &= \frac{n\,\frac{a_{n-3}+a_{n-4}}{n-2}+a_{n-3}}{n\,(n-1)}\\ &= \frac{(2\,n-2)\,a_{n-3}+n\,a_{n-4}}{n\,(n-1)\,(n-2)}\\ &= \frac{(2\,n-2)\,\frac{a_{n-4}+a_{n-5}}{n-3}+n\,a_{n-4}}{n\,(n-1)\,(n-2)}\\ &= \frac{(n^2-n-2)\,a_{n-4}+(2\,n-2)\,a_{n-5}}{n\,(n-1)\,(n-2)\,(n-3)} \end{aligned}\tag{15}$$

3. The idea now is to look at the above several steps and try to make a clever guess of what the general formula for a_n is. It doesn't seem to lead to any reasonable general formula so we give up.

- Q2. To answer Q2 we only need to start from a_0, a_1 and compute the a_n's one by one, until we have 5 nonzero a_n's (including a_0, a_1).

$$a_2 = \frac{a_1 + a_0}{2} \text{ nonzero;} \tag{16}$$

$$a_3 = \frac{a_2 + a_1}{3} = \frac{\frac{a_1 + a_0}{2} + a_1}{3} = \frac{3\,a_1 + a_0}{6} \text{ nonzero;} \tag{17}$$

$$a_4 = \frac{a_3 + a_2}{4} = \frac{\frac{3\,a_1 + a_0}{6} + \frac{a_1 + a_0}{2}}{4} = \frac{3\,a_1 + 2\,a_0}{12} \text{ nonzero;} \tag{18}$$

As we already have 5 nonzero terms (a_0 — a_4), we stop here and write

$$\begin{aligned} y &= a_0 + a_1\,t + \frac{a_1 + a_0}{2}\,t^2 + \frac{3\,a_1 + a_0}{6}\,t^3 + \frac{3\,a_1 + 2\,a_0}{12}\,t^4 + \cdots \\ &= a_0 + a_1\,(x-1) + \frac{a_1 + a_0}{2}\,(x-1)^2 + \frac{3\,a_1 + a_0}{6}\,(x-1)^3 + \frac{3\,a_1 + 2\,a_0}{12}\,(x-1)^4 + \cdots \end{aligned} \tag{19}$$

Solution 2. We illustrate the 2nd method by trying Q3. Write

$$y = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n = \sum_{n=0}^{\infty} a_n\,(x-1)^n. \tag{20}$$

Substitute into the equation, we have

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' - [(x-1)+1]\left(\sum_{n=0}^{\infty} a_n(x-1)^n\right)' - \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0. \tag{21}$$

First compute the first term:

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' = \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,(x-1)^{n-2}. \tag{22}$$

Shifting index, we reach

$$\left(\sum_{n=0}^{\infty} a_n\,(x-1)^n\right)'' = \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,(x-1)^n. \tag{23}$$

Now compute the second term

$$-[(x-1)+1]\left(\sum_{n=0}^{\infty} a_n(x-1)^n\right)' = -(x-1)\sum_{n=1}^{\infty} n\,a_n\,(x-1)^{n-1} - \sum_{n=1}^{\infty} n\,a_n\,(x-1)^{n-1} \tag{24}$$

$$= -\sum_{n=1}^{\infty} n\,a_n\,(x-1)^n - \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,(x-1)^n. \tag{25}$$

Now the equation becomes

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,(x-1)^n - \sum_{n=1}^{\infty} n\,a_n\,(x-1)^n - \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,(x-1)^n - \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0. \tag{26}$$

Note that in the above, three sums start from 0 while one starts from 1. Thus we need to write the $n=0$ term separately:

$$2\,a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2} - n\,a_n - (n+1)\,a_{n+1} - a_n] = 0. \tag{27}$$

The recurrence relations are

$$2\,a_2 - a_1 - a_0 = 0 \tag{28}$$

$$(n+2)\,(n+1)\,a_{n+2} - (n+1)\,a_n - (n+1)\,a_{n+1} = 0 \qquad n \geqslant 1 \tag{29}$$

The second relation can be simplified to

$$(n+2)\,a_{n+2} = a_n + a_{n+1}. \qquad n \geqslant 1 \tag{30}$$

Solving them one by one, we have

$$(n=0) \qquad a_2 = \frac{1}{2}\,a_0 + \frac{1}{2}\,a_1 \tag{31}$$

$$(n=1) \qquad a_3 = \frac{1}{3}\,(a_1 + a_2) = \frac{1}{6}\,a_0 + \frac{1}{2}\,a_1 \tag{32}$$

$$(n=2) \qquad a_4 = \frac{1}{4}\,(a_2 + a_3) = \frac{1}{4}\left(\frac{2}{3}\,a_0 + a_1\right) = \frac{1}{6}\,a_0 + \frac{1}{4}\,a_1 \tag{33}$$

The general solution is

$$y(x) = a_0 + a_1\,(x-1) + \left(\frac{1}{2}\,a_0 + \frac{1}{2}\,a_1\right)(x-1)^2 + \left(\frac{1}{6}\,a_0 + \frac{1}{2}\,a_1\right)(x-1)^3 + \left(\frac{1}{6}\,a_0 + \frac{1}{4}\,a_1\right)(x-1)^4 + \cdots \tag{34}$$

Collecting all the a_0's and the a_1's together we have

$$y(x) = a_0\left[1 + \frac{1}{2}\,(x-1)^2 + \frac{1}{6}\,(x-1)^3 + \frac{1}{6}\,(x-1)^4 + \cdots\right] + a_1\left[x - 1 + \frac{1}{2}\,(x-1)^2 + \frac{1}{2}\,(x-1)^3 + \frac{1}{4}\,(x-1)^4 + \cdots\right]. \tag{35}$$

So

$$y_1(x) = 1 + \frac{1}{2}\,(x-1)^2 + \frac{1}{6}\,(x-1)^3 + \frac{1}{6}\,(x-1)^4 + \cdots \tag{36}$$

$$y_2(x) = x - 1 + \frac{1}{2}\,(x-1)^2 + \frac{1}{2}\,(x-1)^3 + \frac{1}{4}\,(x-1)^4 + \cdots \tag{37}$$