Lecture 17 Power Series Method: Introduction (Cont.)

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Radius of convergence.

For every power series

$$a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \tag{1}$$

(or in compact notation $\sum_{n=0}^{\infty} a_n (x - x_0)^n$) there is a number $0 \leqslant \rho \leqslant \infty$, calld "radius of convergence", such that the power series converges – that is indeed represent something – for all $|x - x_0| < \rho$ while diverges – that is does not have any obvious meaning – for all $|x - x_0| > \rho$.

Remark 1. For those points on the boundary $|x - x_0| = \rho$, we have to analyze case by case. For example

$$\sum_{n=0}^{\infty} \frac{x^n}{n} \tag{2}$$

has $\rho = 1$. But the series converges at $x = -1$ while diverges at $x = 1$.

There are two ways to compute this important number.

- Method 1.

$$\rho^{-1} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{3}$$

  when the limit exists.

- Method 2.

$$\rho^{-1} = \limsup_{n \to \infty} |a_n|^{1/n}. \tag{4}$$

Remark 2. The first method only applies to some power series while the second method can be used for all power series. On the other hand, when the first method applies, it usually is easier to use than the second one. In this class, we will not meet any power series that the first method (together with a few ad hoc tricks) does not work.

Example 3. Let's see a few examples of determining ρ and then the interval of convergence.

1.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n\, n}. \tag{5}$$

   First identify

$$a_n = \frac{1}{2^n\, n}; \qquad x_0 = 1. \tag{6}$$

   Then compute

$$\begin{aligned} \rho^{-1} &= \lim_{n \longrightarrow \infty} \left| \frac{1/2^{n+1} (n+1)}{1/2^n\, n} \right| \\ &= \lim_{n \to \infty} \left| \frac{1}{2\, \frac{n+1}{n}} \right| = \frac{1}{2}. \end{aligned} \tag{7}$$

   So $\rho = 2$ and the power series converges for $|x - 1| < 2$ (or $-1 < x < 3$) while diverges for $|x - 1| > 2$.

2.

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3\, n)!} x^n. \tag{8}$$

First identify

$$a_n = \frac{(n!)^3}{(3\,n)!}; \qquad x_0 = 0. \tag{9}$$

Then compute

$$\rho^{-1} = \lim_{n \to \infty} \left| \frac{((n+1)!)^3/(3\,(n+1))!}{(n!)^3/(3\,n)!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^3}{(3\,n+3)\,(3\,n+2)\,(3\,n+1)} \right| = \frac{1}{27}. \tag{10}$$

So $\rho = 27$ and the series converges for $|x| < 27$, diverges for $|x| > 27$.

Remark 4. The meaning of

- $\rho = 0$: The power series only converges at $x = x_0$ and diverges at every other x; Example: $\sum_{n=0}^{\infty} (n!)\, x^n$.
- $\rho = \infty$: The power series converges everywhere; Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case, if we can find out the limit function, as in

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \tag{11}$$

then we can say that the power series $\sum_{n=0}^{\infty} \frac{x^n}{n1}$ and the explicit formula e^x are the **same thing**. In contrast, when $\rho < \infty$ this is not true even when we can find an explicit formula for the sum. For example,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{12}$$

is only true when $|x| < \rho = 1$.

Operation of power series.

As we are using power series to represent functions, we would like to do all the usual operations: Differentiation, integration, addition/subtraction, multiplication, division, ...

- Operations on one power series. Let $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ has radius of convergence $\rho > 0$.
  - Differentiation:

$$\left(\sum_{n=0}^{\infty} a_n\,(x - x_0)^n \right)' = \sum_{n=1}^{\infty} a_n\, n\,(x - x_0)^{n-1} = \sum_{n=0}^{\infty} [a_{n+1}\,(n+1)]\,(x - x_0)^n. \tag{13}$$

    Note that we have shifted the index $n - 1 \to n$ in the last step.

  - Integration:

$$\int \left(\sum_{n=0}^{\infty} a_n\,(x - x_0)^n \right) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\,(x - x_0)^{n+1} + C. \tag{14}$$

    We can further write it into the standard form

$$\sum_{n=0}^{\infty} b_n\,(x - x_0)^n, \qquad b_n = \begin{cases} C & n = 0 \\ \frac{a_{n-1}}{n} & n \geqslant 1 \end{cases}. \tag{15}$$

  These resulting power series have the same radius of convergence ρ. As a consequence, inside $|x - x_0| < \rho$ we can differentiate/integrate as many times as we need.

- Operations of two (or more) power series. Let $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n\,(x - x_0)^n$ have radii of convergence ρ_1, ρ_2 respectively. Let $\rho := \min\,(\rho_1, \rho_2)$. Then inside $|x - x_0| < \rho$, the following is legal, and the resulting power series has radius of convergence ρ.
  - Addition/Subtraction:

$$\sum_{n=0}^{\infty} a_n\,(x - x_0)^n \pm \sum_{n=0}^{\infty} b_n\,(x - x_0)^n = \sum_{n=0}^{\infty} (a_n \pm b_n)\,(x - x_0)^n. \tag{16}$$

- ○ Multiplication:

$$\left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] = a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty}\left[\sum_{i+j=n} a_i b_j\right](x - x_0)^n. \quad (17)$$

Division

$$\frac{\sum_{n=0}^{\infty} a_n (x - x_0)^n}{\sum_{n=0}^{\infty} b_n (x - x_0)^n} \quad (18)$$

is a bit different, due to the fact that the denominator may be 0. When doing division, the resulting radius of convergence is **at least**[1]

$$\min\left(\rho, \text{the distance between } x_0 \text{ and the closest zero of } \sum_{n=0}^{\infty} b_n (x - x_0)^n\right). \quad (19)$$

- How to compute division: Write

$$\frac{\sum_{n=0}^{\infty} a_n (x - x_0)^n}{\sum_{n=0}^{\infty} b_n (x - x_0)^n} = \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad (20)$$

into

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \left[\sum_{n=0}^{\infty} c_n (x - x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] = c_0 b_0 + (c_1 b_0 + c_0 b_1)(x - x_0) + \cdots \quad (21)$$

and then determine $c_0, c_1, c_2, \ldots$ one by one through

$$c_0 b_0 = a_0 \quad (22)$$

$$c_1 b_0 = a_1 - c_0 b_1 \quad (23)$$

$$c_2 b_0 = a_2 - c_0 b_2 - c_1 b_1 \quad (24)$$

$$\vdots \quad \vdots \quad \vdots$$

1. See this week's homework for more on this point.