Lecture 16 Power Series Method

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Taylor Expansion.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \tag{1}$$

In essence, Taylor expansion is the following relation

$$f(x) = \text{a power series} = \text{a polynomial of degree infinity.} \tag{2}$$

A power series is an infinite sum:

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{3}$$

which is often denoted in the compact form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{4}$$

Such a sum is often called "formal" to emphasize the fact that adding up infinity many numbers may not be meaningful. For example, we know

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{5}$$

is meaningful because this sum converges to the number e. However the formal sum

$$\sum_{n=0}^{\infty} (-1)^n n \tag{6}$$

is just "formal" since there is no reasonable way to equate it with a number.

Remark. It is crucial to understand that the index n in the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is only a "place holder". It's whole purpose is to indicate that the subscript of the coefficient and the power of $x - x_0$ are the same, and that the sum starts from the zeroth term. Therefore we can replace n by any other symbol:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \sum_{m=0}^{\infty} a_m (x - x_0)^m, \quad \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{7}$$

all denote the **same** power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{8}$$

However, they are not the same as

$$\sum_{n=2}^{\infty} a_n (x - x_0)^n \text{ or } \sum_{k=0}^{\infty} a_k (x - x_0)^{k+1} \tag{9}$$

as the former starts from a different term, and the latter has a different relation between the subscript and the power.

Power Series Method – An Example.

We solve the Airy's equation

$$y'' = x\, y. \tag{10}$$

This equation looks very simple but its general solution cannot be written in "closed form". On the other hand, these solutions are very useful in practice that they were given a name and became one class of the so-called "Special functions".

1. Write power series expansion for y:

$$y = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n. \tag{11}$$

   Note that here we are taking $x_0 = 0$ as it is not specified.

2. Substitute into the equation:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)'' = x \left(\sum_{n=0}^{\infty} a_n x^n \right). \tag{12}$$

3. Simplify both sides (Red $=$ means "the operation needs justification". However they will be justified very soon)

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)'' = \sum_{n=0}^{\infty} (a_n x^n)'' = \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} \tag{13}$$

$$x \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{14}$$

   Note that the $n=0$ term a_0 and the $n=1$ terms $a_1 x$ disappear when taking two derivatives.

   So the equation becomes

$$\sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{15}$$

   Remark. Keep in mind that $\sum \cdots$ is just a **notation**, a **short hand**. So $\sum_{n=2}^{\infty} a_n n (n-1) x^{n-2}$ is just a notation for the "real" power series

$$2 a_2 + 6 a_3 x + \cdots \tag{16}$$

   and $\sum_{n=0}^{\infty} a_n x^{n+1}$ is just a notation for $a_0 x + a_1 x^2 + \cdots$. In particular, we are free to change either or both n to other symbols:

$$\sum_{m=2}^{\infty} a_m m (m-1) x^{m-2} = \sum_{n=0}^{\infty} a_n x^{n+1}, \tag{17}$$

$$\sum_{m=2}^{\infty} a_m m (m-1) x^{m-2} = \sum_{l=0}^{\infty} a_l x^{l+1}, \tag{18}$$

   whatever... They all mean the **same** equation.

4. Shift indices. Note that if we do not use the short hands and just write what the equation really means:

$$2 a_2 + 6 a_3 x + \cdots = a_0 x + a_1 x^2 + \cdots \tag{19}$$

   we can immediately conclude

$$2 a_2 = 0, \qquad 6 a_3 = a_1, \tag{20}$$

   and so on. This is not satisfactory though[1] as we won't be able to get a "universal" relation between a_n's. To get such relation we have to analyze

$$\sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{21}$$

1. For a certain type of problems this is satisfactory. We will see.

We try to do the same thing – Equate the terms on both sides with the same power of x. To do this efficiently we need to "shift indices", that is introducing one or more new indices so that the generic terms (currently $a_n\, n\,(n-1)\, x^{n-2}$ and $a_n\, x^{n+1}$) has the same power of x.

- Shift left hand side. We try to re-write

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} = \sum \cdots x^k. \tag{22}$$

It is clear that we should let the new index, k, be related to the old one, n, through

$$k = n-2. \tag{23}$$

Now replace every n by $k+2$:

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} = \sum_{k+2=2}^{\infty} a_{k+2}\,(k+2)\,(k+1)\, x^k \tag{24}$$

and simplify to

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\, x^k; \tag{25}$$

- Similarly for the right hand side,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{26}$$

5. Balance the equation. The equation now becomes

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\, x^k = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{27}$$

Equating terms with same x^k on both sides, we reach

$$(k=0)\colon 2\, a_2 \;=\; 0; \tag{28}$$

$$(k \geqslant 1)\colon a_{k+2}\,(k+2)\,(k+1) \;=\; a_{k-1}; \tag{29}$$

This is called the "recurrence relation".

Remark. Note that when shifting indices, we use "k" as the new index for both sides. This is just for convenience since at the end of the day we would like to pick terms with the same x^k from both sides. We can also use difference symbols and reach:

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\, x^k = \sum_{l=1}^{\infty} a_{l-1}\, x^l. \tag{30}$$

This is the same equation as

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\, x^k = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{31}$$

The only difference is that we cannot proceed before renaming l by k!

Remark. Also, usually the index-shifting is done somewhat implicitly, that is we do not explicitly introduce any new symbols. In the future we will just replace every n by $n+2$ and obtain

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\, x^n. \tag{32}$$

6. Write down the solution.

There are several ways to write down the final solution. Check with your problem for specific requirement.

- General formula.

  From the recurrence relation we can get:

$$a_{k+2} = \frac{a_{k-1}}{(k+2)\,(k+1)} = \frac{a_{k-4}}{(k+2)\,(k+1)\,(k-1)\,(k-2)} = \cdots \tag{33}$$

  which finally can be written as

$$a_k = \begin{cases} \dfrac{a_0}{\Pi_{i=1}^{l}(3\,i)\,(3\,i-1)} & k = 3\,l \\ \dfrac{a_1}{\Pi_{i=1}^{l}(3\,i+1)\,(3\,i)} & k = 3\,l+1 \\ 0 & k = 3\,l+2 \end{cases}. \tag{34}$$

  So the final answer is

$$y = \sum_{k=0}^{\infty} a_k\,x^k \tag{35}$$

  with a_k given by the above.

- First few terms/First few nonzero terms.

  Say we are asked to write down the first 4 nonzero terms of the solution. We compute: a_0, a_1 are free, so two nonzero terms $a_0 + a_1\,x$ now. $a_2 = 0$ so $a_2\,x^2 = 0$ does not give us the 3rd term. We compute $a_3 = \frac{a_0}{6}$ so $a_3\,x^3 = \frac{a_0}{6}\,x^3$ is the 3rd nonzero term. Then $a_4\,x^4 = \frac{a_1}{12}\,x^4$ gives the 4th term. So

$$y = a_0 + a_1\,x + \frac{a_0}{6}\,x^3 + \frac{a_1}{12}\,x^4 + \cdots \tag{36}$$

  is the answer.

- y_1, y_2. Recall that as $y'' = x\,y$ is a linear second order equation, its general solution should look like

$$y = C_1\,y_1 + C_2\,y_2. \tag{37}$$

  We can put our power series solution into this form as follows:

  Note that a_0, a_1 are "free", so they are in fact the C_1, C_2. Now group all a_0 terms together, all a_1 terms together we get:

$$y = a_0\left(1 + \frac{x^3}{6} + \cdots\right) + a_1\left(x + \frac{x^4}{12} + \cdots\right) \tag{38}$$

  which gives

$$y_1 = 1 + \frac{x^3}{6} + \cdots; \qquad y_2 = x + \frac{x^4}{12} + \cdots. \tag{39}$$

Theoretical Issues.

In the above solution we have done many dubious operations and thus left many theoretical gaps open. For example

$$\left(\sum_{n=0}^{\infty} a_n\,x^n\right)'' = \sum_{n=0}^{\infty} (a_n\,x^n)''. \tag{40}$$

Why can we do this? Does this hold for all x? If not, how do we tell for which x the above holds?

Turns out, all these gaps are filled as long as we restrict ourselves to $|x| < \rho$, where $0 \leqslant \rho \leqslant \infty$[2] is a certain number, determined by the coefficients $a_0, a_1, \ldots$, called "radius of convergence".

In short, given a power series

$$\sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \tag{41}$$

let ρ be its radius of convergence. Then in $|x - x_0| < \rho$ we can treat it as if it's a polynomial: Termwise differentiation, termwise integration, re-arrangement of terms, etc. All OK.

2. Notice it's $\leqslant$, not $<$!

On the other hand, for those x such that $|x - x_0| > \rho$, the infinite sum

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{42}$$

diverges. In other words, for those x this sum does not represent a function at all.