Lecture 14 Higher Order Linear Equations (Cont.)

10/12/2011

Review: Solving constant-coefficient, homogeneous, linear equations.

$$a_0\, y^{(n)} + \cdots + a_n\, y = 0. \tag{1}$$

1. Solve the characteristic equation

   $$a_0\, r^n + \cdots + a_n = 0 \tag{2}$$

   and get a list of roots.

2. Order the roots: real roots first, then complex conjugate pairs. Write down the fundamental set $y_1, \ldots, y_n$ according to the following rules:

   - A real root r, repeated k times, yields k solutions in the fundamental set: $e^{rt}, t\, e^{rt}, \ldots, t^{k-1}\, e^{rt}$ (Note that when r is a single root, this automatically give only e^{rt}.
   - A pair of complex roots $\alpha \pm i\, \beta$, repeated k times, yields $2\, k$ solutions in the fundamental set: $e^{\alpha t} \cos \beta\, t, e^{\alpha t} \sin \beta\, t, \ldots, t^{k-1}\, e^{\alpha t} \cos \beta\, t, t^{k-1}\, e^{\alpha t} \sin \beta\, t$.

3. $y = C_1\, y_1 + \cdots + C_n\, y_n$.

- In general, solving the characteristic equation is not possible for humans. However there are two cases that are solvable:

  - When the equation is simple. Meaning: the characteristic equation can be solve by repeating:

    Find a root -> Factorize -> Find another root -> Factorize -> ...

  - When the equation is special. Meaning: It is of the form $y^{(n)} - a\, y = 0$ where a is a constant, or it can be reduced to such form. Example of the latter case:

    $$y^{(6)} + y'' = 0 \Longrightarrow r^2 (r^4 + 1) = 0 \tag{3}$$

    so the roots are $0, 0$ and the four roots of $(-1)^{1/4}$.

- Example of simple equation.

  $$y^{(4)} - 4\, y''' + 4\, y'' = 0, \qquad y(1) = -1, y'(1) = 2, y''(1) = 0, y'''(1) = 0. \tag{4}$$

  Solution. We first find the general solution, then use initial conditions to determine the four constants.

  - Characteristic equation:

    $$r^4 - 4\, r^3 + 4\, r^2 = 0 \Longrightarrow r^2\, (r^2 - 4\, r + 4) = 0 \Longrightarrow r^2\, (r - 2)^2 = 0. \tag{5}$$

    So the roots are $0, 0, 2, 2$.

  - Write down $y_1, \ldots, y_4$.

    We have

    $$0\text{: Real, repeated 2 times} \Longrightarrow e^{0t}, t\, e^{0t} \tag{6}$$

    $$2\text{: Real, repeated 2 times} \Longrightarrow e^{2t}, t\, e^{2t}. \tag{7}$$

- Write down y.

$$y = C_1 + C_2\, t + C_3\, e^{2t} + C_4\, t\, e^{2t}. \tag{8}$$

- Use initial conditions.
  - Preparation.

$$y' = C_2 + (2\, C_3 + C_4)\, e^{2t} + 2\, C_4\, t\, e^{2t} \tag{9}$$

$$y'' = [2\,(2\, C_3 + C_4) + 2\, C_4]\, e^{2t} + 4\, C_4\, t\, e^{2t} = (4\, C_3 + 4\, C_4)\, e^{2t} + 4\, C_4\, t\, e^{2t} \tag{10}$$

$$y''' = [2\,(4\, C_3 + 4\, C_4) + 4\, C_4]\, e^{2t} + 8\, C_4\, t\, e^{2t} = (8\, C_3 + 12\, C_4)\, e^{2t} + 8\, C_4\, t\, e^{2t}. \tag{11}$$

  - Use initial conditions.

$$y(1) = -1 \implies C_1 + C_2 + C_3\, e^2 + C_4\, e^2 = -1 \tag{12}$$
$$y'(1) = 2 \implies C_2 + (2\, C_3 + C_4)\, e^2 + 2\, C_4\, e^2 = 2 \tag{13}$$
$$y''(1) = 0 \implies (4\, C_3 + 4\, C_4)\, e^2 + 4\, C_4\, e^2 = 0 \tag{14}$$
$$y'''(1) = 0 \implies (8\, C_3 + 12\, C_4)\, e^2 + 8\, C_4\, e^2 = 0 \tag{15}$$

Simplify to get a 4×4 system for $C_1, \ldots, C_4$:

$$C_1 + C_2 + e^2\, C_3 + e^2\, C_4 = -1 \tag{16}$$
$$C_2 + 2\, e^2\, C_3 + 3\, e^2\, C_4 = 2 \tag{17}$$
$$4\, C_3 + 8\, C_4 = 0 \tag{18}$$
$$8\, C_3 + 20\, C_4 = 0 \tag{19}$$

  - Solve the 4×4 system.
    - Ad hoc method.

      For this problem we notice that we can solve C_3, C_4 first and then C_1, C_2. The C_3, C_4 equations yields $C_3 = C_4 = 0$. Substitute back into the C_1, C_2 equation we get

$$C_1 + C_2 = -1 \tag{20}$$
$$C_2 = 2 \tag{21}$$

      We immediately get $C_1 = -3, C_2 = 2$.

    - General method.

      In general we have to use the method of Gaussian elimination. See your linear algebra textbook or the following links for explanations:

      - http://mathworld.wolfram.com/GaussianElimination.html
      - http://www.youtube.com/watch?v=woqq3Sls1d8

      First write down the matrix: Coefficient matrix with an extra column of the right hand side:

$$\begin{pmatrix} 1 & 1 & e^2 & e^2 & -1 \\ 0 & 1 & 2\, e^2 & 3\, e^2 & 2 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 8 & 20 & 0 \end{pmatrix} \tag{22}$$

Now transform.

$$\begin{pmatrix} 1 & 1 & e^2 & e^2 & -1 \\ 0 & 1 & 2\,e^2 & 3\,e^2 & 2 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 8 & 20 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & e^2 & e^2 & -1 \\ 0 & 1 & 2\,e^2 & 3\,e^2 & 2 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

(Row 3 times -2 add to row 4)

$$\Longrightarrow \begin{pmatrix} 1 & 1 & e^2 & e^2 & -1 \\ 0 & 1 & 2\,e^2 & 3\,e^2 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

(Row 3 divided by 4 to make the leading number 1)

$$\Longrightarrow \begin{pmatrix} 1 & 1 & e^2 & e^2 & -1 \\ 0 & 1 & 2\,e^2 & 3\,e^2 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(Row 4 divided by 4 to make the leading number 1) (23)

Now we are ready the solve the system bottom up – from C_4 to C_1. The corresponding transformed system is

$$C_1 + C_2 + e^2\,C_3 + e^2\,C_4 = -1 \quad (24)$$

$$C_2 + 2\,e^2\,C_3 + 3\,e^2\,C_4 = 2 \quad (25)$$

$$C_3 + 2\,C_4 = 0 \quad (26)$$

$$C_4 = 0 \quad (27)$$

- Finally the solution is

$$y = -3 + 2\,t. \quad (28)$$

It is easy to check that this indeed the solution.

Complex numbers.

To be able to solve "special" equations like $y^{(4)} + y = 0$, we have to be familiar with complex numbers.

- Two ways to represent a complex number.
  - $a + b\,i$: Good for addition, subtraction;
  - $R\,e^{i\theta}$: Good for multiplication, especially powers and roots.
- Transforming back and forth.
  - Write $R\,e^{i\theta}$ into $a + b\,i$.

$$R\,e^{i\theta} = R\,(\cos\theta + i\sin\theta) \Longrightarrow a = R\cos\theta, b = R\sin\theta. \quad (29)$$

  - Write $a + b\,i$ into $R\,e^{i\theta}$.

$$R = \sqrt{a^2 + b^2}; \quad (30)$$

and θ is determined through requiring

$$\cos\theta = \frac{a}{R}; \qquad \sin\theta = \frac{b}{R}. \quad (31)$$

For example, consider $1 + \sqrt{3}\,i$. We have

$$R = \sqrt{1^2 + (\sqrt{3})^2} = 2. \quad (32)$$

and θ must satisfy

$$\cos\theta = \frac{1}{2}, \qquad \sin\theta = \frac{\sqrt{3}}{2} \tag{33}$$

and we conclude $\theta = \frac{\pi}{3}$... **wait a minute**... $+2\,k\,\pi$... Now it's right. So

$$\theta = \frac{\pi}{3} + 2\,k\,\pi, \qquad k \text{ arbitrary integer} \tag{34}$$

and

$$1 + \sqrt{3}\,i = 2\,e^{i\left(\frac{\pi}{3} + 2k\pi\right)}. \tag{35}$$

In general,

$$a + b\,i = R\,e^{i(\theta_0 + 2k\pi)} \tag{36}$$

with k taking any integer.

- Taking roots.
  - To compute $(a + b\,i)^{1/n}$,
    1. Write $a + b\,i = R\,e^{i(\theta_0 + 2k\pi)}$;
    2. Write
    $$(a + b\,i)^{1/n} = R^{1/n} \exp\left[i\,\frac{\theta_0 + 2\,k\,\pi}{n}\right]. \tag{37}$$
    3. Set $k =$ n consecutive numbers (for example $0, 1, \ldots, n-1$, or $-\frac{n}{2} + 1, \ldots, 0, \ldots, \frac{n}{2}$ when n is even and similarly when n is odd. Each value of k gives one root.
    4. Simplify if possible.
  - Example. Compute $\left(1 + \sqrt{3}\,i\right)^{1/4}$.
    - First step is already done:
    $$1 + \sqrt{3}\,i = 2\,e^{i\left(\frac{\pi}{3} + 2k\pi\right)} \tag{38}$$
    - Now we need to evaluate
    $$2^{1/4} \exp\left[i\,\frac{\frac{\pi}{3} + 2\,k\,\pi}{4}\right] \tag{39}$$
    for 4 consecutive values of k.
    - Take $-1, 0, 1, 2$.
    $$\begin{aligned} k = -1 &\Longrightarrow 2^{1/4}\,e^{-i\frac{5}{12}\pi}; \\ k = 0 &\Longrightarrow 2^{1/4}\,e^{i\frac{\pi}{12}}; \\ k = 1 &\Longrightarrow 2^{1/4}\,e^{i\frac{7}{12}\pi}; \\ k = 2 &\Longrightarrow 2^{1/4}\,e^{i\frac{13}{12}\pi}. \end{aligned}$$
    - Not really possible to further simplify.
- Solving $y^{(n)} - a\,y = 0$.

Example 1. Solve

$$y^{(4)} + y = 0 \tag{40}$$

Solution.

Characteristic equation

$$r^4 + 1 = 0 \Longrightarrow r^4 = -1. \tag{41}$$

We need to find all 4 roots of $(-1)^{1/4}$.

Write -1 into $R\,e^{i\theta}$. We have

$$R = 1, \quad \cos\theta = -1, \quad \sin\theta = 0. \tag{42}$$

So can take $\theta_0 = \pi$. Now

$$-1 = e^{i(\pi + 2k\pi)}. \tag{43}$$

The four roots are given by

$$e^{i\frac{(2k+1)\pi}{4}}. \tag{44}$$

We take $k = -1, 0, 1, 2$.

$$\begin{aligned}
k = -1 &\implies e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i \\
k = 0 &\implies e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i \\
k = 1 &\implies e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i \\
k = 2 &\implies e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i
\end{aligned}$$

We end up with two pairs of roots:

$$\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}\,i, \quad -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}\,i. \tag{45}$$

The general solution is then

$$y = C_1\,e^{\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}\,t + C_2\,e^{\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}\,t + C_3\,e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}\,t + C_4\,e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}\,t. \tag{46}$$