Lecture 12 Variation of Parameters

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Variation of parameters.

What about all the equations that are either not constant-coefficient or with $g(t)$ not of any of the above types? Do we have a method that works for all equations? Yes.

- Method of variation of parameters.

  The general solution of

$$y'' + p(t)\, y' + q(t)\, y = g(t) \tag{1}$$

is

$$y = C_1\, y_1 + C_2\, y_2 + y_p \tag{2}$$

where

  - y_1, y_2 form a fundamental set[1] of the homogeneous equation

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{3}$$

    - If p, q are constants, we get y_1, y_2 directly.
    - For general p, q, have to
      1. Guess a solution y_1;
      2. Obtain y_2 from the "reduction of order" formula:[2]

$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{5}$$

    Note. It is a good idea to check that y_2 indeed solves the homogeneous equation.

  - y_p is given by the following formula:

$$y_p = u_1\, y_1 + u_2\, y_2,\ \ u_1 = \int \frac{-y_2\, g}{W[y_1, y_2]},\ \ u_2 = \int \frac{y_1\, g}{W[y_1, y_2]}. \tag{6}$$

    $W[y_1, y_2] = y_1\, y_2' - y_1'\, y_2$ is the Wronskian of y_1, y_2.

    Note. If is a good idea to check that y_p indeed solves the original equation.

Remark. It is a good idea to write the equation given into "standard form"

$$y'' + p(t)\, y' + q(t)\, y = g(t) \tag{7}$$

before applying any formulas.

Examples.

Example 1. Solve

$$y'' + 4\, y' + 4\, y = t^{-2}\, e^{-2t}. \tag{8}$$

Solution. Note that $t^{-2}\, e^{-2t}$ does not belong to any type that can be attacked by undetermined coefficients.

- Write the equation in standard form. Done. We have $p = 4, q = 4, g = t^{-2}\, e^{-2t}$.

1. are solutions and are linearly independent.
2. Later we will see that for a class of equations called "Euler equations", looking like

$$a\, t^2\, y'' + b\, t\, y' + c\, y = 0 \tag{4}$$

we also have formulas for y_1, y_2.

- Find y_1, y_2 by solving
$$y'' + 4\,y' + 4\,y = 0. \tag{9}$$
The characteristic equation is
$$r^2 + 4\,r + 4 = 0 \Longrightarrow r_1 = r_2 = -2. \tag{10}$$
So we have
$$y_1 = e^{-2t}, \qquad y_2 = t\,e^{-2t}. \tag{11}$$
- Find y_p.
    - First compute the Wronskian (as it will be used twice below!)
$$W[y_1, y_2] = (e^{-2t})\,(t\,e^{-2t})' - (t\,e^{-2t})\,(e^{-2t})' = e^{-4t}. \tag{12}$$
    - Now
$$u_1 = \int \frac{-y_2\,g}{W[y_1, y_2]} = \int \frac{-(t\,e^{-2t})\,(t^{-2}\,e^{-2t})}{e^{-4t}} = -\ln|t|. \tag{13}$$
$$u_2 = \int \frac{y_1\,g}{W[y_1, y_2]} = \int \frac{e^{-2t}\,(t^{-2}\,e^{-2t})}{e^{-4t}} = -\frac{1}{t}. \tag{14}$$
    - Write
$$y_p = u_1\,y_1 + u_2\,y_2 = -\ln|t|\,e^{-2t} - \frac{1}{t}\,t\,e^{-2t} = -e^{-2t}\ln|t| - e^{-2t}. \tag{15}$$
    - Check solution: As e^{-2t} solves the homogeneous equation, we only need to put $-e^{-2t}\ln|t|$ into the equation:
$$(-e^{-2t}\ln|t|)' = -\frac{1}{t}\,e^{-2t} + 2\,e^{-2t}\ln|t|; \tag{16}$$
$$(-e^{-2t}\ln|t|)'' = \left(-\frac{1}{t}\,e^{-2t} + 2\,e^{-2t}\ln|t|\right)' = \frac{1}{t^2}\,e^{-2t} + \frac{4}{t}\,e^{-2t} - 4\,e^{-2t}\ln|t|. \tag{17}$$
Now
$$(-e^{-2t}\ln|t|)'' + 4\,(-e^{-2t}\ln|t|)' + 4\,(-e^{-2t}\ln|t|) = \frac{1}{t^2}\,e^{-2t}. \tag{18}$$
- Write down solution.
$$y = C_1\,y_1 + C_2\,y_2 + y_p = C_1\,e^{-2t} + C_2\,t\,e^{-2t} - e^{-2t}\ln|t| - e^{-2t}. \tag{19}$$
Simplify
$$y = C_1\,e^{-2t} + C_2\,t\,e^{-2t} - e^{-2t}\ln|t| \tag{20}$$
(Because $C_1\,e^{-2t}$ and $(C_1 - 1)\,e^{-2t}$ represent the same functions – those that are constant multiples of e^{-2t})

Example 2. Solve
$$t\,y'' - (t+1)\,y' + y = t^2, \qquad t > 0. \tag{21}$$

Solution.

- Write the equation in standard form:
$$y'' - \frac{t+1}{t}\,y' + \frac{1}{t}\,y = t. \tag{22}$$
So
$$p(t) = -\frac{t+1}{t}, \quad q(t) = \frac{1}{t}, \quad g(t) = t. \tag{23}$$
- Find y_1, y_2.
    - Guess y_1. The usual order is exponential then polynomials. For our problem here fortunately e^t solves
$$y'' - \frac{t+1}{t}\,y' + \frac{1}{t}\,y = 0 \tag{24}$$
so $y_1 = e^t$.

- Obtain y_2 through

$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{25}$$

We have

$$\begin{aligned} y_2(t) &= y_1(t) \int \frac{e^{-\int p(t)\,\mathrm{d}t}}{y_1(t)^2}\,\mathrm{d}t \\ &= e^t \int \frac{e^{-\int -(1+1/t)}}{e^{2t}} \\ &= e^t \int \frac{e^t\, e^{\ln t}}{e^{2t}} \quad \text{(Note that as } t>0, |t| = t) \\ &= e^t \int t\, e^{-t}\,\mathrm{d}t \\ &= e^t \left[-\int t\,\mathrm{d}e^{-t} \right] \\ &= e^t \left[-t\, e^{-t} + \int e^{-t}\,\mathrm{d}t \right] \\ &= -(t+1). \end{aligned} \tag{26}$$

Check

$$y_2'' - \frac{t+1}{t}\, y_2' + \frac{1}{t}\, y_2 = 0 + \frac{t+1}{t} + \frac{1}{t}(-t-1) = 0. \tag{27}$$

- Find y_p.

  First compute the Wronskian

$$W[y_1, y_2](t) = y_1\, y_2' - y_1'\, y_2 = -e^t + e^t\,(t+1) = t\, e^t. \tag{28}$$

Then

$$\begin{aligned} u_1(t) &= \int \frac{-g\, y_2}{W} \\ &= \int \frac{-t\,(-(t+1))}{t\, e^t} \\ &= \int (t+1)\, e^{-t}\,\mathrm{d}t \\ &= -\int (t+1)\,\mathrm{d}e^{-t} \\ &= -(t+1)\, e^{-t} + \int e^{-t}\,\mathrm{d}(t+1) \\ &= -(t+1)\, e^{-t} + \int e^{-t}\,\mathrm{d}t \\ &= -(t+2)\, e^{-t}. \end{aligned} \tag{29}$$

$$\begin{aligned} u_2(t) &= \int \frac{g\, y_1}{W} \\ &= \int \frac{t\, e^t}{t\, e^t} \\ &= t. \end{aligned} \tag{30}$$

So

$$y_p = u_1\, y_1 + u_2\, y_2 = -(t+2) - t\,(t+1) = -(t^2 + 2\,t + 2). \tag{31}$$

Check:

$$y_p'' - \frac{t+1}{t}\, y_p' + \frac{1}{t}\, y_p = -2 + \frac{t+1}{t}\,(2\,t+2) - \frac{t^2 + 2\,t + 2}{t} = t. \tag{32}$$

- Write down solution:

$$y = C_1\, y_1 + C_2\, y_2 + y_p = C_1\, e^t - C_2\,(t+1) - -(t^2 + 2\,t + 2). \tag{33}$$

Explanations and remarks.

- The reduction of order formula
$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{34}$$
Consider the equation in standard form
$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{35}$$
Let $y_2 = v\, y_1$. Then
$$y_2' = v'\, y_1 + v\, y_1'; \qquad y_2'' = v''\, y_1 + 2\, v'\, y_1' + v\, y_1''. \tag{36}$$
So y_2 is a solution is the same as
$$[v''\, y_1 + 2\, v'\, y_1' + v\, y_1''] + p(t)\, [v'\, y_1 + v\, y_1'] + q(t)\, v\, y_1 = 0. \tag{37}$$
Grouping v'', v', v terms we get
$$y_1\, v'' + [2\, y_1' + p(t)\, y_1]\, v' + [y_1'' + p(t)\, y_1' + q(t)\, y_1] v = 0. \tag{38}$$
But y_1 is a solution so
$$y_1'' + p(t)\, y_1' + q(t)\, y_1 = 0 \Longrightarrow [y_1'' + p(t)\, y_1' + q(t)\, y_1] v = 0. \tag{39}$$
So v only needs to satisfy
$$y_1\, v'' + [2\, y_1' + p(t)\, y_1]\, v' = 0. \tag{40}$$
Let $w = v'$. The above becomes a 1st order linear equation for w
$$y_1\, w' + [2\, y_1' + p(t)\, y_1]\, w = 0 \Longrightarrow w' + \left(\frac{2\, y_1'}{y_1} + p(t)\right) w = 0. \tag{41}$$
The integrating factor is
$$\exp\left[\int \left(\frac{2\, y_1'}{y_1} + p(t)\right)\right] = \exp\left[(\ln y_1^2) + \int p\right] = y_1^2\, e^{\int p}. \tag{42}$$
So we have
$$\left(y_1^2 e^{\int p}\, w\right)' = 0. \tag{43}$$
Note that we only need one solution, so we take
$$v' = w = y_1^{-2}\, e^{-\int p} \Longrightarrow v = \int \frac{e^{-\int p}}{y_1^2}. \tag{44}$$

- The y_p formula
$$y_p = \left(\int \frac{-y_2\, g}{W[y_1, y_2]}\right) y_1 + \left(\int \frac{y_1\, g}{W[y_1, y_2]}\right) y_2. \tag{45}$$
We look for a particular solution of the form
$$y_p = u_1\, y_1 + u_2\, y_2. \tag{46}$$
This gives
$$y_p' = u_1'\, y_1 + u_2'\, y_2 + u_1\, y_1' + u_2\, y_2'. \tag{47}$$
Then y_p'' would involve 8 terms and the equation will get very complicated. However, things are much simplified if we require
$$u_1'\, y_1 + u_2'\, y_2 = 0. \tag{48}$$
In this case
$$y_p' = u_1\, y_1' + u_2\, y_2' \tag{49}$$
and
$$y_p'' = u_1'\, y_1' + u_2'\, y_2' + u_1\, y_1'' + u_2\, y_2''. \tag{50}$$

Substitute into the equation we get

$$[u_1'\,y_1' + u_2'\,y_2' + u_1\,y_1'' + u_2\,y_2] + p(t)\,[u_1\,y_1' + u_2\,y_2] + q(t)\,[u_1\,y_1 + u_2\,y_2] = g(t). \tag{51}$$

Grouping similar terms we get

$$(u_1'\,y_1' + u_2'\,y_2') + u_1\,[y_1'' + p\,y_1' + q\,y_1] + u_2\,[y_2'' + p\,y_2' + q\,y_2] = g(t). \tag{52}$$

As y_1, y_2 solves the homogeneous equation, this is just

$$u_1'\,y_1' + u_2'\,y_2' = g(t). \tag{53}$$

Summarizing, to get y_p, all we need are u_1, u_2 satisfying

$$u_1'\,y_1 + u_2'\,y_2 = 0 \tag{54}$$

$$u_1'\,y_1' + u_2'\,y_2' = g(t). \tag{55}$$

Multiply the first equation by $-y_2'$, and add to the second multiplied by y_2, we get

$$(-y_1\,y_2' + y_1'\,y_2)\,u_1' = g\,y_2 \Longrightarrow u_1 = \int \frac{-g\,y_2}{y_1\,y_2' - y_1'\,y_2} = \int \frac{-g\,y_2}{W}. \tag{56}$$

Substitute into $u_1'\,y_1 + u_2'\,y_2 = 0$ we get

$$u_2 = \int \frac{g\,y_1}{W}. \tag{57}$$