Lecture 08 2nd Order Equations

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General 2nd order equations.

$$y'' = f(x, y, y'). \tag{1}$$

- General solution involves **two** arbitrary constants.
- Initial value problem

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0, y'(x_0) = y_1. \tag{2}$$

- Boundary value problem

$$y'' = f(x, y, y'), \qquad y(a) = y_1, \;\; y(b) = y_2 \tag{3}$$

or

$$y'' = f(x, y, y'), \qquad y(a) = y_1, \;\; y'(b) = y_2 \tag{4}$$

or

$$y'' = f(x, y, y'), \qquad y'(a) = y_1, \;\; y(b) = y_2 \tag{5}$$

or

$$y'' = f(x, y, y'), \qquad y'(a) = y_1, \;\; y'(b) = y_2 \tag{6}$$

- General 2nd order equations are very difficult. Not only is it highly unlikely that solution formulas can be found, for boundary value problems even the theory is still an active research field.
- We will focus on the following linear equation:

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = g(x) \tag{7}$$

or the standard formed one:

$$y'' + p(x)\, y' + q(x)\, y = g(x). \tag{8}$$

For which we have a complete understanding from the theoretical point of view.

Homogeneous linear 2nd order equations.

- Our starting point is the homogeneous case (that is $g = 0$)

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{9}$$

- Notice: If y_1, y_2 are solutions, so are $C_1\, y_1 + C_2\, y_2$ for any constants C_1, C_2.

Proof. If y_1, y_2 are solutions, then we have

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 = 0 \tag{10}$$

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0. \tag{11}$$

Multiply the first one by C_1 and the second by C_2, add them up and collect same orders of derivatives together:

$$C_1\, y_1'' + C_2\, y_2'' + p(x)\, [C_1\, y_1' + C_2\, y_2'] + q(x)\, [C_1\, y_1 + C_2\, y_2] = 0. \tag{12}$$

Now as

$$C_1\, y_1'' + C_2\, y_2'' = (C_1\, y_1 + C_2\, y_2)'', \qquad C_1\, y_1' + C_2\, y_2 = (C_1\, y_1 + C_2\, y_2)' \tag{13}$$

we reach

$$(C_1\, y_1 + C_2\, y_2)'' + p(x)\, (C_1\, y_1 + C_2\, y_2)' + q(x)\, (C_1\, y_1 + C_2\, y_2) = 0 \tag{14}$$

and end the proof. □

- Recalling that general solution involves two arbitrary constants, we suspect: The general solution to
$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{15}$$
is
$$y = C_1\, y_1 + C_2\, y_2. \tag{16}$$

- This is almost the case. Except that we require

  y_1, y_2 are linearly independent.

  When y_1, y_2 are linearly independent, they form a "fundamental set" of solutions. In other words, if y_1, y_2 form a fundamental set of solutions, then the general solution is given by $C_1\, y_1 + C_2\, y_2$.

- How to check this? Turns out
$$y_1, y_2 \text{ are linearly independent} \Longleftrightarrow W[y_1, y_2](x) \neq 0 \tag{17}$$
where
$$W[y_1, y_2](x) = y_1\, y_2' - y_1'\, y_2 \tag{18}$$
and is called the **Wronskian**.

- Properties of the Wronskian

  - $W[y_1,\, y_2](x) \neq 0 \Longleftrightarrow y_1,\, y_2$ are linearly independent **if y_1, y_2 solve the same equation $y'' + p(x)\, y' + q(x)\, y = 0$**;
  - $W[y_1, y_2](x) = W[y_1, y_2](x_0) \exp\left[-\int_{x_0}^{x} p(s)\, \mathrm{d}s\right]$ (Abel's Theorem).
  - As a consequence Abel's Theorem, if $p(x)$ is bounded in an interval $a < x < b$, and x_0 is in it too, then $W[y_1, y_2](x_0) \neq 0 \Longrightarrow W[y_1, y_2](x) \neq 0$ for all $a < x < b$.
  - In particular, consider the initial value problem
$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{19}$$
with $p(x)$ bounded for $a < x < b$. Let x_0 be inside this interval too.

    Now consider y_1, y_2 solving the equation with the following two sets of initial values respectively
$$y(x_0) = a;\ y'(x_0) = b; \qquad y(x_0) = c;\ y'(x_0) = d. \tag{20}$$
Then y_1, y_2 form a fundamental set if and only if $a\, d - b\, c \neq 0$.