Lecture 06 Integrating Factors (Cont.)

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Review.

- How to solve
$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1}$$
  1. Check whether it is of any "simple" type: Linear? Separable? Homogeneous?
  2. Check whether it is exact.
  3. Try to find an integrating factor $\mu(x,y)$ such that
  $$(\mu(x,y)\,M(x,y))\,\mathrm{d}x+(\mu(x,y)\,N(x,y))\,\mathrm{d}y=0 \tag{2}$$
  is exact. Or equivalently, such that
  $$\frac{\partial}{\partial y}(\mu(x,y)\,M(x,y))=\frac{\partial}{\partial x}(\mu(x,y)\,N(x,y)) \tag{3}$$
  which simplifies to
  $$M(x,y)\,\frac{\partial\mu}{\partial y}-N(x,y)\,\frac{\partial\mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu. \tag{4}$$
- Facts:
  - Such μ always exists;
  - There is no general method to find μ.

How to find integrating factors?.

- Guess (trial and error):
  - Is there a μ of the form $\mu(x)$?
  - If not, is there a μ of the form $\mu(y)$?
  - If not, is there a μ of the form $\mu(x\,y)$ or $\mu(x^2+y^2)$ or $\mu(y/x)$?
  - If you are working on an exercise in a textbook and all the above fails (and the problem gives no hint on what μ should look like), very likely there is some typo in the problem (aside from the possibility that you made mistakes in calculation of course).

Example 1. Solve
$$\left(3\,x+\frac{6}{y}\right)\mathrm{d}x+\left(\frac{x^2}{y}+\frac{3\,y}{x}\right)\mathrm{d}y=0. \tag{5}$$

Solution.

- Check if it is of any simple type.
  - Linear?

    The equation can be written as
    $$y'=-\frac{3\,x+6/y}{x^2/y+3\,y/x} \tag{6}$$
    No.[1]

1. Note that it **may happen** that the equation is indeed of some simple type, but we are not experienced/smart enough to realize this. There are some surer ways to tell:

- Linear: If $y'=f(x,y)$ is linear, then $f(x,y)=-p(x)\,y+g(x)$ which means $\frac{\partial f}{\partial y}=-p(x)$ is independent of y or equivalently $\frac{\partial^2 f}{\partial^2 y}=0$.
- Homogeneous: If $y'=f(x,\,y)$ is homogeneous, that is $f(x,\,y)=H(y/x)$ for some function H, then necessarily $f(t\,x,\,t\,y)=f(x,y)$ for any number t.

- Separable?
  No.
- Homogeneous?
  No.

- Check whether it is exact.

$$M(x,y)=\left(3\,x+\frac{6}{y}\right),\qquad N(x,y)=\left(\frac{x^2}{y}+\frac{3\,y}{x}\right). \tag{7}$$

Taking derivatives

$$\frac{\partial M}{\partial y}=-\frac{6}{y^2};\qquad \frac{\partial N}{\partial x}=\frac{2\,x}{y}-\frac{3\,y}{x^2}. \tag{8}$$

Not exact.

- We have to try to find an integrating factor, that is a function $\mu(x,y)$ which satisfies

$$M(x,y)\,\frac{\partial \mu}{\partial y}-N(x,y)\,\frac{\partial \mu}{\partial x}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu. \tag{9}$$

which in our problem becomes

$$\left(3\,x+\frac{6}{y}\right)\frac{\partial \mu}{\partial y}-\left(\frac{x^2}{y}+\frac{3\,y}{x}\right)\frac{\partial \mu}{\partial x}=\left(\frac{2\,x}{y}-\frac{3\,y}{x^2}+\frac{6}{y^2}\right)\mu. \tag{10}$$

- We guess.
  - 1st guess: $\mu=\mu(x)$.
    In this case we have $\frac{\partial \mu}{\partial y}=0$ and $\frac{\partial \mu}{\partial x}$ becomes just $\mu'(x)$. The equation μ must satisfy becomes

$$-\left(\frac{x^2}{y}+\frac{3\,y}{x}\right)\mu'(x)=\left(\frac{2\,x}{y}-\frac{3\,y}{x^2}+\frac{6}{y^2}\right)\mu(x) \tag{11}$$

which simplifies to

$$\frac{\mu'(x)}{\mu(x)}=-\frac{2\,x^3\,y-3\,y^2+6\,x^2}{x^4\,y+3\,x\,y^3}. \tag{12}$$

Now note that the left-hand-side is a function of x only, while the right-hand-side is not. So the logical conclusion is that there is no integrating factor that is independent of y.

  - 2nd guess: $\mu=\mu(y)$.
    In this case we have $\frac{\partial \mu}{\partial x}=0$ and $\frac{\partial \mu}{\partial y}$ becomes just $\mu'(y)$. The equation becomes

$$\left(3\,x+\frac{6}{y}\right)\mu'(y)=\left(\frac{2\,x}{y}-\frac{3\,y}{x^2}+\frac{6}{y^2}\right)\mu(y) \tag{13}$$

which simplifies to

$$\frac{\mu'(y)}{\mu(y)}=\frac{2\,x^3\,y-3\,y^2+6\,x^2}{3\,x^3\,y^2+6\,x^2\,y}=\frac{2\,x^2\,(x\,y+3)-3\,y^2}{3\,x^2\,y\,(x\,y+2)} \tag{14}$$

Clearly the right-hand-side cannot be made independent of x. So $\mu=\mu(y)$ also doesn't work.

  - 3rd guess: $\mu=\mu(x\,y)$.
    In this case we should be careful what $\frac{\partial \mu}{\partial x}$ and $\frac{\partial \mu}{\partial y}$ become. We have to use the chain rule: Consider the composite function $f(x,y)=g(h(x,y))$. Then we have

$$\frac{\partial f}{\partial x}=g'(h(x,y))\,\frac{\partial h}{\partial x},\qquad \frac{\partial f}{\partial y}=g'(h(x,y))\,\frac{\partial h}{\partial y}. \tag{15}$$

Note that here $g'(h(x,y))$ must be interpreted as follows:
g is a function of one variable, say z. We can compute its derivative: $g'(z)$. Now in the latter set $z=h(x,y)$. For example, if $g(z)=z^2$, $h(x,y)=x\,y$, then $g'(z)=2\,z$, setting $z=x\,y$ we reach $g'(x\,y)=2\,(x\,y)=2\,x\,y$.

I am not aware of any rigorous test to spot separable equations. I would be grateful if someone who found such a test could share his/her insight with me.

Using the chain rule ($g = \mu, h(x, y) = x\,y$) we have

$$\frac{\partial \mu}{\partial x} = \mu'(x\,y)\,\frac{\partial (x\,y)}{\partial x} = y\,\mu'(x\,y); \qquad \frac{\partial \mu}{\partial y} = x\,\mu'(x\,y). \tag{16}$$

Substitute into the equation we obtain

$$\left(3\,x + \frac{6}{y}\right) x\,\mu'(x\,y) - \left(\frac{x^2}{y} + \frac{3\,y}{x}\right) y\,\mu'(x\,y) = \left(\frac{2\,x}{y} - \frac{3\,y}{x^2} + \frac{6}{y^2}\right)\mu(x\,y) \tag{17}$$

which is just

$$\left(3\,x^2 + \frac{6\,x}{y} - x^2 + \frac{3\,y^2}{x}\right)\mu'(x\,y) = \left(\frac{2\,x}{y} - \frac{3\,y}{x^2} + \frac{6}{y^2}\right)\mu(x\,y) \tag{18}$$

and finally simplifies to

$$\frac{\mu'(x\,y)}{\mu(x\,y)} = \frac{1}{x\,y} \tag{19}$$

Letting $z = x\,y$, we have

$$\frac{\mu'(z)}{\mu(z)} = \frac{1}{z} \tag{20}$$

and can take the simplest solution $\mu(z) = z$.

So we have found one integrating factor: $\mu(x, y) = x\,y$.

- Multiply the equation by the integrating factor:

$$\left(3\,x + \frac{6}{y}\right)\mathrm{d}x + \left(\frac{x^2}{y} + \frac{3\,y}{x}\right)\mathrm{d}y = 0. \tag{21}$$

becomes

$$(3\,x^2\,y + 6\,x)\,\mathrm{d}x + (x^3 + 3\,y^2)\,\mathrm{d}y = 0. \tag{22}$$

Note that the two equations have the same general solution.[2]

- Now we solve

$$(3\,x^2\,y + 6\,x)\,\mathrm{d}x + (x^3 + 3\,y^2)\,\mathrm{d}y = 0. \tag{23}$$

First we check that it is indeed exact (otherwise we must have made some mistake in finding μ and we should go back checking the steps, or just re-do everything.)

$$\frac{\partial (3\,x^2\,y + 6\,x)}{\partial y} = 3\,x^2; \qquad \frac{\partial (x^3 + 3\,y^2)}{\partial x} = 3\,x^2. \tag{24}$$

So it is exact.

Write

$$u(x, y) = \int (3\,x^2\,y + 6\,x)\,\mathrm{d}x + g(y) = x^3\,y + 3\,x^2 + g(y). \tag{25}$$

Then compare

$$\frac{\partial (x^3\,y + 3\,x^2 + g(y))}{\partial y} = x^3 + g'(y) \tag{26}$$

and the coefficient for $\mathrm{d}y$:

$$(x^3 + 3\,y^2) \tag{27}$$

we see that $g'(y) = 3\,y^2$ and can take $g(y) = y^3$.

Thus

$$u(x, y) = x^3\,y + 3\,x^2 + y^3 \tag{28}$$

and the solution is given by

$$x^3\,y + 3\,x^2 + y^3 = C. \tag{29}$$

2. One may worry about the zeros of $x\,y$. This is a good point – after all what we did for separable equations is a special case of what we do here. But since solving an equation through general integrating factors is already complicated enough, we just be sloppy and do not discuss this.

Existence and Uniqueness.

Recall that differential equations are mathematical models of real world phenomena. Therefore, given a DE,

- If we can solve it, that is we can somehow get a solution, we ask:

  > Is this really the solution that we are looking for? Is it possible that there are some other solutions which are actually more relevant to the real world phenomena we try to understand?

  In mathematics, the above question becomes

  > Is the solution unique?

- If we cannot solve it, we ask:

  > Is there really a solution to this problem? That is, did we correctly model the real world phenomena?

  In mathematics, this question becomes

  > Does the solution exist?