# Lecture 02 Introduction (cont.), The Simplest DE

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**Introduction (cont.).**

| | Analytic | Geometric |
|---|---|---|
| Differential Equations | Something like $y' = f(x,y)$ | A Slope field: Short line segment at *every* point of the $x$-$y$ plane with slope $f(x,y)$ at point $(x,y)$. |
| A solution: | A function $y(x)$ such that when replacing every $y$ in the equation, makes the equation *identity*. | A curve that, at every point it passes, is tengent to the above short line segment. |

**Table 1.** Analytic and Geometric Points of View for Differential Equations

- General solution of a differential equation: A formula involving constants, such that
  1. Whatever values we assign to these constants, we get a solution to the equation.
  2. The number of constants is the same as the order of the equation.
- "Solve the DE" = Find general solution.

**Example 1.** Which of the following are correct?
Solve the differential equation

$$y'' + 5\,y' + 4\,y = 0. \tag{1}$$

- Answer 1: $y = e^{-4x}$;
- Answer 2: $y = C_1\,e^{-4x} + C_2\,e^{-x}$;
- Answer 3: $y = C_1\,e^{-4x} + C_2\,(3\,e^{-4x})$.

**Solution.** Answer 2 is correct. Answer 1 does not have the correct number of arbitrary constants. Answer 3 seems to have two constants, but in fact has only one:

$$C_1\,e^{-4x} + C_2\,(3\,e^{-4x}) = (C_1 + 3\,C_2)\,e^{-4x}. \tag{2}$$

**Remark 2.** The reason that Answer 3 is not correct is that $3\,e^{-4x}$ is a multiple of $e^{-4x}$, so they are effectively the same thing when considering linear combinations with arbitrary coefficients.

The mathematical jargon for this situation is: $e^{-4x}$ and $3\,e^{-4x}$ are **linearly dependent**. The real mathematical definition is the following:

> Two functions $f, g$ are linearly dependent if there are constants $a, b$, **not both zero**, such that $a\,f(x) + b\,g(x) = 0$ for all $x$ of interest.[1]

It is clear that this definition can be generalized to the case of three, four, or more functions. When writing general solutions of the form

$$y(x) = C_1\,y_1(x) + C_2\,y_2(x) + \cdots + C_n\,y_n(x). \tag{3}$$

We need to make sure that $y_1, y_2, \ldots, y_n$ are **linearly independent** (= not linearly dependent).

More on this issue later.

1. This enables us to talk meaningfully about things like "$f, g$ are linearly dependent over the interval $[0,1]$".

- Boundary value and initial value problems.
  - Differential equations model real world phenomena. The solution tells us how a particle moves, how the price changes, etc. In reality there are no "arbitrary constants". The particle moves along a certain path, the price change gives one specific graph, etc.
  - There are extra conditions in real world phenomena. For example, to determine the path of a particle, we not only require that the particle is subject to Newton's law, we also specify "initial conditions": Where is this particle at the initial time? What is its velocity then?
  - In the abstract mathematical model this is equivalent to require:

$$\ddot{x} = f, \qquad x(0) = x_0, \quad \dot{x}(0) = v_0. \tag{4}$$

  - Technical definition:
    - Initial value problem (IVP): DE + extra conditions at one single point;
    - Boundary value problem (BVP): DE + extra conditions at at least two different points.
  - Examples:

$$y'' + 9\,y' + 8\,y = 0; \quad y(0) = y'(0) = 0: \qquad \text{(IVP)} \tag{5}$$

$$y''' + 8\,y'' + 3\,y = 0; \qquad y(0) = 1, y'(1) = 2, y''(2) = 3: \qquad \text{(BVP)} \tag{6}$$

**From the simplest to the mother of all DEs.**

- The simplest DE that can be solved by anyone after Calculus I:

$$y' = f(x). \tag{7}$$

  Just integrate to get general solution (don't for get the constant!)

$$y = F(x) + C. \tag{8}$$

  Reason: The equation is no other than

$$(y - F(x))' = 0. \tag{9}$$

  So the solution is given by

$$y - F(x) = C. \tag{10}$$

- After Calculus II or III, we can write $(y - F(x))' = 0$ as $\mathrm{d}(y - F(x)) = 0$. As we know that

$$\mathrm{d}u(x, y) = 0 \Longleftrightarrow u(x, y) = C \tag{11}$$

  we can solve the following equations:

$$\mathrm{d}(x\,y) = 0 \Longrightarrow x\,y = C \Longrightarrow y = \frac{C}{x}. \tag{12}$$

$$\mathrm{d}(x^2\,y^3) = 0 \Longrightarrow x^2\,y^3 = C. \tag{13}$$

- Explicit and implicit formulas of the solution.
  - $y = \frac{C}{x}$ is an explicit formula of the solution $y$ ;
  - $x^2\,y^3 = C$ is an implicit formula of the solution $y$.

  Rule of thumb: When it's impossible or not worthwhile to get explicitly $y = \cdots$, we just write the solution in its implicit form. (Warning: whether it's "worthwhile" is different for different people!)

  **Example 3.**

  - $x\,y = C$: Easy. Should write $y = \frac{C}{x}$;
  - $\tan y - y - x^2 = C$: Impossible. Just leave like that.
  - $y^4 + 3\,x\,y^2 - 2\,y + 5\,x = C$: Not worthwhile. Just leave like that.

- $x^2\, y^3 = C$: Borderline, but on the "easy" side.

- In this class, for borderline cases, I will treat both explicit and implicit formulas a OK.
- The problem is that no equation is given in the form $\mathrm{d}(u(x, y)) = 0$. Usually they are given either like $y' = f(x, y)$ or $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$.
  - The two forms are roughly equivalent. As

$$y' = f(x, y) \Longleftrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \Longleftrightarrow -f(x, y)\,\mathrm{d}x + \mathrm{d}y = 0. \tag{14}$$

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \Longleftrightarrow N(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = -M(x, y) \approx y' = -\frac{M(x, y)}{N(x, y)}. \tag{15}$$

    The last $\approx$ is because dividing by $N(x, y)$ in fact changes the equation. We will discuss this issue more in the following lectures.

- **How do we know whether a given $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ is the same as $\mathrm{d}u = 0$?** (When this happens, the equation is called "exact".)
  - This is not always the case. (We will discuss a criterion to check this).
  - Turns out, all solution strategies for first order ODEs can be summarized as: Transform the equation to be exact.
  - Thus we have "jumped" from the simplest DE to the mother of all DEs.

**5-min Quiz.**

**Problem 1.** Solve $\dot{y} = t\,e^t$.

**Solution.** All we need to do is to find the primitive of $t\,e^t$. Calculate:

$$\begin{aligned}
\int t\,e^t\,\mathrm{d}t &= \int u\,\mathrm{d}v \qquad (u = t, v = e^t) \\
&= u\,v - \int v\,\mathrm{d}u \\
&= t\,e^t - \int e^t\,\mathrm{d}t \qquad \text{(Integration by parts)} \\
&= t\,e^t - e^t \\
&= e^t\,(t-1).
\end{aligned} \tag{16}$$

For those who are familiar with the integration by parts process, the step $u = t, v = e^t$ can be omitted. Just write

$$\int t\,e^t\,\mathrm{d}t = \int t\,\mathrm{d}e^t = t\,e^t - \int e^t\,\mathrm{d}t = t\,e^t - e^t = (t-1)\,e^t. \tag{17}$$

Now write down the solution (**don't forget!**)

$$y = (t-1)\,e^t + C. \tag{18}$$

The ability to evaluate integrals like $\int\ t^k\,e^t\,\mathrm{d}t$, $\int\ t^k \sin t\,\mathrm{d}t$, $\int\ e^t \sin t\,\mathrm{d}t$ is important in solving differential equations.