Lecture 01 Introduction

Sep. 7, 2011

Notation.

- Standard notation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}, \frac{\mathrm{d}^4y}{\mathrm{d}x^4}, \ldots \tag{1}$$

- Shorthand:

$$y', \; y'', \; y''', \; y^{(4)}(\text{or } y^{\text{iv}} \text{ in some literature}), \ldots \tag{2}$$

- Mechanics:

$$\dot{x}, \; \ddot{x}, \; \dddot{x}, \; \ddddot{x}, \; \ldots \tag{3}$$

  When this notation is used, t is often the independent variable. In other words, these are shorthands of

$$\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^3x}{\mathrm{d}t^3}, \frac{\mathrm{d}^4x}{\mathrm{d}t^4}, \ldots \tag{4}$$

- When there are more than one independent variables, partial derivatives are involved. For example, if $u = u(x, y)$, we have
  - First order derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \tag{5}$$

  - Second order derivatives

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y^2} \tag{6}$$

  - Third order derivatives

$$\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial x \partial x \partial y}, \frac{\partial^3 u}{\partial x \partial y \partial x}, \ldots \tag{7}$$

  Usually the following shorthands are used:

$$u_x, u_y, u_{xx}, u_{xy}, u_{yx}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyx}, \ldots \tag{8}$$

What is a differential equation?.

- Informal definition: An equation involving derivatives.
- Can write down as many as we want:

$$u_{xy} = 3 \sin x + u_x^2 \tag{9}$$

$$\ddot{x} - 2\,\dot{x} + x = 0 \tag{10}$$

  even very exotic ones:

$$\exp\left[\exp\left(\sin \dot{x}\right)^2\right] = 5 \cos\left(\exp t\right) \tag{11}$$

- As you can guess, we have 0% chance solving an arbitrarily written differential equation (you may know from calculus that most functions cannot be explicitly integrated – the situation is worse here as $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ is, of course, the simplest class of differential equations).
- Fortunately, important differential equations – those arising from science and technology – are much simpler than arbitrary ones.

  Example 1. (Newton's Second Law) This is the earliest and most significant source of differential equations.

$$\text{Force} = \text{Mass} \times \text{Acceleration} = \text{Mass} \times \frac{\mathrm{d}(\text{Velocity})}{\mathrm{d}t} = \text{Mass} \times \frac{\mathrm{d}^2(\text{Location})}{\mathrm{d}t^2}. \tag{12}$$

In general, force is a function of velocity and/or location. Different mechanical systems give different force functions and thus different DEs.

For example, in the mass-spring system, force is given by Hooke's law, which leads to

$$-k\,x = m\,\ddot{x}. \tag{13}$$

How to solve differential equations?.

- What is a solution?
  - Analytic point of view: A function which, when substituted into the equation, makes it an identity.

    Example 2. Check whether $\sin t$ is a solution to $\ddot{x} + x = 0$.

    We substitute $x(t) = \sin t$ into the equation:

    $$\ddot{x} + x = \frac{\mathrm{d}^2(\sin t)}{\mathrm{d}t^2} + \sin t = -\sin t + \sin t = 0 \tag{14}$$

    So yes.

  - Geometric point of view: A curve in the t-x plane (or x-y plane) (which is the graph of the solution function) such that at every point along the curve, it is tangent to the slope vector there.
    - To understand this we need the geometric interpretation of differential equations.

      Example 3. Consider the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 3\,x$. Let $x(t)$ be a solution passing a point (t_0, x_0). Then at that particular point, we have $\frac{\mathrm{d}x}{\mathrm{d}t} = 3\,x_0$. Thus the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 3\,x$, at the point (t_0, x_0), is equivalent to the requirement that any solution curve passing that point has to have slope $3\,x_0$.

      If we draw a short line passing $(t_0,\ x_0)$, with slope $3\ x_0$, then any solution curve passing this point has to be tangent to this short line.

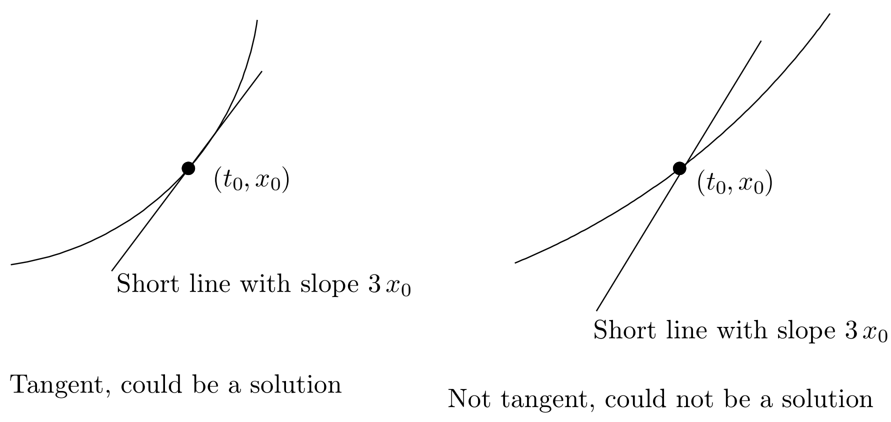


Figure 1. Solution curves have to be tangent to the short line at every point the curve passes

- Geometric interpretation of a differential equation: A specification of slopes at every point in the t-x plane. In other words, a differential equation is equivalent to a slope field.

- A solution is then a curve that is tangent to this slope field. Such a curve is called an **integral curve** of the slope field.
- To get better sense of slope fields, go to http://www.math.rutgers.edu/~sontag/JODE/JOdeApplet.html .

- How to obtain solutions?
  - To obtain analytic formula: Use either brain or computer (Maple, Mathematica, etc.)
  - To obtain geometric graph: Use either hand or computer (Maple, Mathematica, etc.)

  In this course we will focus on the basic brain-hand approach, although in the real world computers dominate. (Analogy: Being able to do basic arithmetics is still valuable despite the existence of calculators)

Jargons.

- What does "general solution" mean?
  - The "general solution" is a solution formula involving the correct number of arbitrary constants.
  - For example, the general solution to the equation $\dot{x} = 3\,x$ is $x(t) = C\,e^{3t}$. Here we have one arbitrary constant C. It is "arbitrary" because whatever value assigned to it will produce a solution to the differential equation. For example, set $C = 2$, $2\,e^{3t}$ is a solution; Set $C = 4$, $4\,e^{3t}$ is a solution...
  - How do we know what is the "correct number"? Answer is "look at the **order** of the equation". The order of a differential equation is the highest derivative involved. Examples:
    - $\ddot{x} + 3\,x = 0$, highest derivative term: $\ddot{x}$, order is 2;
    - $(y')^2 + 3\,x = 0$, highest derivative term y', order is 1.

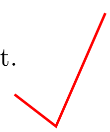
    Rule of thumb:

    $$\text{Number of arbitrary constants} = \text{Order of the equation} \tag{15}$$

  - "Solve the equation" = Find general solution.

    Solve $\dot{x} = 3\,x$

    $x(t) = C\,e^{3t}$, C is an arbitrary constant.

    $x(t) = 2\,e^{3t}$

  - To be continued...