Nov. 18, 2011

Math 334 Fall 2011 Homework 9 Solutions

Basic

Problem 1. Use definition to compute $\mathcal{L}\{\sin b\,t\}$.
Solution.

$$\begin{aligned}\mathcal{L}\{\sin b\,t\}(s) &= \int_0^\infty \sin b\,t\, e^{-st}\,\mathrm{d}t \\ &= -\frac{1}{s}\int_0^\infty \sin b\,t\,\mathrm{d}e^{-st} \\ &= -\frac{1}{s}\sin b\,t\,e^{-st}|_0^\infty + \frac{1}{s}\int e^{-st}\,\mathrm{d}\sin b\,t \\ &= 0 + \frac{b}{s}\int e^{-st}\cos b\,t\,\mathrm{d}t \\ &= -\frac{b}{s^2}\int \cos b\,t\,\mathrm{d}e^{-st} \\ &= -\frac{b}{s^2}\left[\cos b\,t\,e^{-st}|_0^\infty - \int e^{-st}\,\mathrm{d}\cos b\,t\right] \\ &= -\frac{b}{s^2}\left[-1 + b\int e^{-st}\sin b\,t\right] \\ &= \frac{b}{s^2} - \frac{b^2}{s^2}\mathcal{L}\{\sin b\,t\}(s). \qquad (1)\end{aligned}$$

This gives

$$\mathcal{L}\{\sin b\,t\}(s) = \frac{b}{s^2+b^2}, \qquad s>0. \qquad (2)$$

Problem 2. Compute

$$\mathcal{L}\{e^{-t}\,t\sin 2\,t\}(s). \qquad (3)$$

Solution. We have

$$\begin{aligned}\mathcal{L}\{e^{-t}\,t\sin 2\,t\}(s) &= \mathcal{L}\{t\sin 2\,t\}(s+1) \\ &= (-1)\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin 2\,t\}\right](s+1) \\ &= -\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right)\right](s+1) \\ &= F(s+1) \qquad (4)\end{aligned}$$

where

$$F(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right) = \frac{4\,s}{(s^2+4)^2}. \qquad (5)$$

So

$$\mathcal{L}\{e^{-t}\,t\sin 2\,t\}(s) = \frac{4\,(s+1)}{[(s+1)^2+4]^2}. \qquad (6)$$

Problem 3. Compute the following inverse transforms.

a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2\,s+10}\right\}$.

b) $\mathcal{L}^{-1}\left\{\frac{3}{(2\,s+5)^3}\right\}$.

c) $\mathcal{L}^{-1}\left\{\frac{s-1}{2\,s^2+s+6}\right\}$.

Solution.

a) We notice

$$\frac{s+1}{s^2+2\,s+10} = \frac{s+1}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+3^2}. \qquad (7)$$

Thus recalling the transform formulas for $e^{at} f$ and for $\cos b\,t$, we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2\,s+10}\right\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = e^{-t}\cos 3\,t. \tag{8}$$

b) We have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{(2\,s+5)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{2^3\left(s+\frac{5}{2}\right)^3}\right\}\\ &= \frac{3}{8}\,\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^3}\right\}\\ &= \frac{3}{8}\,e^{-\frac{5}{2}t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}\\ &= \frac{3}{16}\,e^{-\frac{5}{2}t}\,t^2. \end{aligned} \tag{9}$$

c) Note that the denominator is not of the form $(s-1)^2+b^2$. But we can write

$$\begin{aligned}\frac{s-1}{2\,s^2+s+6} &= \frac{1}{2}\left[\frac{s-1}{s^2+s/2+3}\right]\\ &= \frac{1}{2}\left[\frac{s-1}{(s+1/4)^2+47/16}\right]\\ &= \frac{1}{2}\,\frac{s+1/4}{(s+1/4)^2+47/16}-\frac{1}{2}\,\frac{5/4}{(s+1/4)^2+47/16}\\ &= \frac{1}{2}\,\frac{s+1/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}-\frac{5}{2\sqrt{47}}\,\frac{\sqrt{47}/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}. \end{aligned} \tag{10}$$

Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-1}{2\,s^2+s+6}\right\} &= \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{s+1/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}\right\}-\frac{5}{2\sqrt{47}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{47}/4}{(s+1/4)^2+\left(\sqrt{47}/4\right)^2}\right\}\\ &= \frac{1}{2}\,e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{47}}{4}\,t\right)-\frac{5}{2\sqrt{47}}\,e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{47}}{4}\,t\right). \end{aligned} \tag{11}$$

Problem 4. Use Laplace transform to solve the following problem:

a) $y''+3\,y'+2\,y=0; \quad y(0)=1,\ y'(0)=0;$

b) $y^{(4)}-4\,y'''+6\,y''-4\,y'+y=0; \quad y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y'''(0)=1$

c) $y''-2\,y'+2\,y=\cos t; \quad y(0)=1,\ y'(0)=0.$

Solution.

a) We transform

$$\mathcal{L}\{y''\}=s^2\,Y-s\,y(0)-y'(0)=s^2\,Y-s;\ \ \mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y-1 \tag{12}$$

so the transformed equation is

$$(s^2+3\,s+2)\,Y-s-3=0 \Longrightarrow Y=\frac{s+3}{s^2+3\,s+2}. \tag{13}$$

To inverse transform, factorize

$$s^2+3\,s+2=(s+1)\,(s+2) \tag{14}$$

and write

$$\frac{s+3}{s^2+3\,s+2}=\frac{A}{s+1}+\frac{B}{s+2}. \tag{15}$$

Therefore

$$A(s+2)+B\,(s+1)=s+3 \tag{16}$$

which gives

$$A+B=1; \qquad 2\,A+B=3 \Longrightarrow A=2, B=-1. \tag{17}$$

So

$$y=\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3\,s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{2}{s+1}-\frac{1}{s+2}\right\}=2\,e^{-t}-e^{-2t}. \tag{18}$$

b) Compute

$$\begin{aligned}
\mathcal{L}\{y^{(4)}\} &= s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = s^4 Y - s^2 - 1; \qquad (19)\\
\mathcal{L}\{y'''\} &= s^3 Y - s^2 y(0) - s y'(0) - y''(0) = s^3 Y - s; \qquad (20)\\
\mathcal{L}\{y''\} &= s^2 Y - s y(0) - y'(0) = s^2 Y - 1; \qquad (21)\\
\mathcal{L}\{y'\} &= s Y - y(0) = s Y.
\end{aligned}$$

So the transformed equation is

$$[s^4 Y - s^2 - 1] - 4[s^3 Y - s] + 6[s^2 Y - 1] - 4[s Y] + Y = 0 \qquad (22)$$

which simplifies to

$$(s^4 - 4s^3 + 6s^2 - 4s + 1) Y - s^2 - 1 + 4s - 6 = 0 \qquad (23)$$

and gives

$$Y = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}. \qquad (24)$$

To get y, we use partial fraction. First factorize

$$s^4 - 4s^3 + 6s^2 - 4s + 1 = (s-1)^4. \qquad (25)$$

So

$$\frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1} = \frac{A}{(s-1)^4} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{s-1}. \qquad (26)$$

A — D are determined by setting

$$s^2 - 4s + 7 = A + B(s-1) + C(s-1)^2 + D(s-1)^3. \qquad (27)$$

One can always expand the right hand side into a polynomial of s and compare the coefficients, but for this problem the easiest way is to write the left hand side into a polynomial of $s-1$:

$$s^2 - 4s + 7 = (s-1)^2 - 2s + 6 = (s-1)^2 - 2(s-1) + 4 \qquad (28)$$

So

$$A = 4, B = -2, c = 1, D = 0. \qquad (29)$$

We have

$$y = \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^4} + \frac{-2}{(s-1)^3} + \frac{1}{(s-1)^2}\right\}. \qquad (30)$$

Recall

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a). \qquad (31)$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{4}{(s-1)^4}\right\} = e^t f(t) \qquad (32)$$

with $f(t)$ satisfying

$$\mathcal{L}\{f(t)\} = \frac{4}{s^4}. \qquad (33)$$

Recalling $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ we have

$$f(t) = \frac{2}{3} t^3. \qquad (34)$$

So

$$\mathcal{L}^{-1}\left\{\frac{4}{(s-1)^4}\right\} = \frac{2}{3} t^3 e^t. \qquad (35)$$

Similarly we get

$$\mathcal{L}^{-1}\left\{\frac{-2}{(s-1)^3}\right\} = -t^2 e^t; \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = t e^t. \qquad (36)$$

So finally

$$y = t e^t - t^2 e^t + \frac{2}{3} t^3 e^t. \qquad (37)$$

c) We have

$$\mathcal{L}\{y''\} = s^2 Y - s y(0) - y'(0) = s^2 Y - s; \qquad \mathcal{L}\{y'\} = s Y - y(0) = s Y - 1. \qquad (38)$$

and

$$\mathcal{L}\{\cos t\} = \frac{s}{1 + s^2}. \qquad (39)$$

The transformed equation is

$$s^2 Y - s - 2\,(s\,Y - 1) + 2\,Y = \frac{s}{1+s^2} \tag{40}$$

which simplifies to

$$(s^2 - 2\,s + 2)\,Y = \frac{s}{1+s^2} + s - 2. \tag{41}$$

So

$$Y = \frac{s}{(1+s^2)\,(s^2-2\,s+2)} + \frac{s-2}{s^2-2\,s+2}. \tag{42}$$

Notice that $s^2 - 2\,s + 2 = (s-1)^2 + 1$ cannot be factorized anymore. We now compute $\mathcal{L}^{-1}\{Y\}$.

- $\mathcal{L}^{-1}\left\{\frac{s}{(1+s^2)\,((s-1)^2+1)}\right\}$: Use partial fraction.

$$\frac{s}{(1+s^2)\,((s-1)^2+1)} = \frac{A\,s+B}{s^2+1} + \frac{C\,(s-1)+D}{(s-1)^2+1} = \frac{(A\,s+B)\,[(s-1)^2+1] + [C\,(s-1)+D]\,(s^2+1)}{(1+s^2)\,((s-1)^2+1)}. \tag{43}$$

Thus A — D satisfy

$$s = (A\,s+B)\,[(s-1)^2+1] + [C\,(s-1)+D]\,(s^2+1) \tag{44}$$

Write everything in s:

$$\begin{aligned}(A\,s+B)\,[(s-1)^2+1] + [C\,(s-1)+D]\,(s^2+1) &= (A\,s+B)\,(s^2-2\,s+2) \\ &\quad + (C\,s + (D-C))\,(s^2+1) \\ &= A\,s^3 + (B-2\,A)\,s^2 + 2\,(A-B)\,s + 2\,B \\ &\quad + C\,s^3 + (D-C)\,s^2 + C\,s + D - C \\ &= (A+C)\,s^3 + (B-2\,A+D-C)\,s^2 \\ &\quad + (2\,A-2\,B+C)\,s + (2\,B+D-C)\end{aligned}$$

So

$$\begin{aligned} A + C &= 0 && (45)\\ B - 2\,A + D - C &= 0 && (46)\\ 2\,A - 2\,B + C &= 1 && (47)\\ 2\,B + D - C &= 0 && (48)\end{aligned}$$

The first equation gives $C = -A$. Substitute into the other three, we get

$$\begin{aligned} B - A + D &= 0 && (49)\\ A - 2\,B &= 1 && (50)\\ 2\,B + D + A &= 0 && (51)\end{aligned}$$

Adding the 1st and the 3rd we get $3\,B + 2\,D = 0$. Substitute into the 1st. The first two equations now become

$$\begin{aligned} -A - \frac{B}{2} &= 0 && (52)\\ A - 2\,B &= 1. && (53)\end{aligned}$$

So we have

$$B = -\frac{2}{5}; \qquad A = \frac{1}{5}. \tag{54}$$

Consequently

$$C = -\frac{1}{5}, \qquad D = \frac{3}{5}. \tag{55}$$

Therefore

$$\frac{s}{(1+s^2)\,((s-1)^2+1)} = \frac{\frac{1}{5}\,s - \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}\,(s-1) + \frac{3}{5}}{(s-1)^2+1}. \tag{56}$$

Taking inverse transform

$$\mathcal{L}^{-1}\left\{\frac{s}{(1+s^2)\,((s-1)^2+1)}\right\} = \frac{1}{5}\cos t - \frac{2}{5}\sin t - \frac{1}{5}\,e^t\cos t + \frac{3}{5}\,e^t\sin t. \tag{57}$$

- $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^2+1}\right\}$:

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}\right\} = e^t \cos t - e^t \sin t. \tag{58}$$

So finally we have

$$y = \mathcal{L}^{-1}\left\{\frac{s}{(1+s^2)\,(s^2-2\,s+2)} + \frac{s-2}{s^2-2\,s+2}\right\} = \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5}e^t \cos t - \frac{2}{5}e^t \sin t. \tag{59}$$

Intermediate

Advanced

Problem 5. Compute

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\}. \tag{60}$$

Solution. We notice that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\left(\frac{s+2}{s-5}\right)\right] = \frac{1}{s+2} - \frac{1}{s-5} \tag{61}$$

is of the form we just discussed.

Thus we have

$$\begin{aligned} -t\,\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\} &= \mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\left(\frac{s+2}{s-5}\right)\right]\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} \\ &= e^{-2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{5t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= e^{-2t} - e^{5t}. \end{aligned} \tag{62}$$

Dividing both sides by $-t$ we reach

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+2}{s-5}\right)\right\} = (e^{5t} - e^{-2t})/t. \tag{63}$$

Challenge

Problem 6. Solve

$$y'' + 3\,t\,y' - 6\,y = 1, \qquad y(0) = 0, \quad y'(0) = 0. \tag{64}$$

Solution. The three steps are still the same. However the details inside each step is different from the constant-coefficient case.

1. Transform the equation. As before, we denote $Y = \mathcal{L}\{y\}$.

   a. Transform the LHS;

      We compute

$$\mathcal{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0) = s^2\,Y. \tag{65}$$

   To compute $\mathcal{L}\{t\,y'\}$, recall that

$$\mathcal{L}\{t^n\,f\} = (-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f\}. \tag{66}$$

   Thus

$$\mathcal{L}\{t\,y'\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{y'\} = -\frac{\mathrm{d}}{\mathrm{d}s}[s\,Y - y(0)] = -\frac{\mathrm{d}}{\mathrm{d}s}(s\,Y) = -Y - s\,\frac{\mathrm{d}Y}{\mathrm{d}s}. \tag{67}$$

   Thus

$$\mathcal{L}\{y'' + 3\,t\,y' - 6\,y\} = s^2\,Y - 3\,Y - 3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s} - 6\,Y = (s^2-9)\,Y - 3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s}. \tag{68}$$

   b. Transform the RHS. We have

$$\mathcal{L}\{1\} = \frac{1}{s}. \tag{69}$$

2. Solve the transformed equation.

   The transformed equation reads

$$-3\,s\,\frac{\mathrm{d}Y}{\mathrm{d}s}+(s^2-9)\,Y=\frac{1}{s}. \tag{70}$$

   Divide both sides by $-3\,s$, we reach

$$\frac{\mathrm{d}Y}{\mathrm{d}s}+\left(\frac{3}{s}-\frac{s}{3}\right)Y=-\frac{1}{3\,s^2}. \tag{71}$$

   Recall how to solve first order linear equations:

$$y'+Py=Q\Longrightarrow\left(e^{\int P}y\right)'=e^{\int P}Q\Longrightarrow y=e^{-\int P}\int e^{\int P}Q. \tag{72}$$

   The integration factor is

$$e^{\int\frac{3}{s}-\frac{s}{3}}=e^{-\frac{s^2}{6}}\,s^3. \tag{73}$$

   Note that in general the domain of Y is contained in $s>0$. Therefore $\ln|s|=\ln s$. We have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[s^3\,e^{-\frac{s^2}{6}}\,Y\right]=e^{-\frac{s^2}{6}}\,s^3\left(-\frac{1}{3\,s^2}\right)=-\frac{s}{3}\,e^{-\frac{s^2}{6}}. \tag{74}$$

   Integrating, we obtain

$$s^3\,e^{-\frac{s^2}{6}}\,Y=e^{-\frac{s^2}{6}}+C\Longrightarrow Y=\frac{1}{s^3}+C\,s^{-3}\,e^{\frac{s^2}{6}}. \tag{75}$$

   To determine the constant C, we use the following property:

$$\lim_{s\nearrow+\infty}\mathcal{L}\{f\}(s)=0 \tag{76}$$

   for all reasonable f.[1] Thus $Y(s)\longrightarrow 0$ as $s\longrightarrow+\infty$ which immediately leads to $C=0$. Thus

$$Y=\frac{1}{s^3}. \tag{77}$$

3. Compute $y=\mathcal{L}^{-1}\{Y\}$.

   We compute

$$y=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{1}{2}\,t^2. \tag{78}$$

Problem 7. Solve

$$x' = 3\,x-2\,y,\qquad x(0)=1; \tag{79}$$
$$y' = 3\,y-2\,x;\qquad y(0)=1. \tag{80}$$

Solution. Still the same old three steps. But first we write the system into

$$x'-3\,x+2\,y = 0,\qquad x(0)=1 \tag{81}$$
$$y'-3\,y+2\,x = 0,\qquad y(0)=1. \tag{82}$$

We use X,Y to denote the Laplace transforms of x,y.

1. Transform the equations.

   a. Transform the LHS;

      We have

$$\mathcal{L}\{x'\}=s\,X-x(0)=s\,X-1 \tag{83}$$

$$\mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y-1. \tag{84}$$

      Thus the LHS are

$$(s-3)\,X+2\,Y-1 \tag{85}$$

      and

$$(s-3)\,Y+2\,X-1. \tag{86}$$

   b. Transform the RHS;

1. More specifically, for f piecewise continuous and of exponential order.

The transformed RHS are simply $0, 0$.

2. Solve the transformed system.

   The transformed system is

$$(s-3)\,X + 2\,Y = 1 \tag{87}$$

$$2\,X + (s-3)\,Y = 1. \tag{88}$$

   Solving it[2] we obtain

$$X = \frac{s-5}{(s-3)^2-4} = \frac{1}{s-1}; \qquad Y = \frac{1}{s-1}. \tag{89}$$

3. Invert $x = \mathcal{L}^{-1}(X)$, $y = \mathcal{L}^{-1}(Y)$.

   Clearly we have

$$x = y = e^t. \tag{90}$$

2. Either using Cramer's rule, or multiply the first equation by 2 and the second by $(s-3)$, then take the difference to obtain Y, etc.