Nov. 04, 2011

Math 334 Fall 2011 Homework 8 Solutions

Basic

Problem 1. Find the general solution for the following:

a) $x^2\,y''+4\,x\,y'+2\,y=0$.

b) $x^2\,y''+5\,x\,y'+4\,y=0$.

c) $2\,x^2\,y''+3\,x\,y'+4\,y=0$.

Solution.

a) Set $y=x^r$ we reach

$$r\,(r-1)+4\,r+2=0\Longrightarrow r_{1,2}=-2,-1. \tag{1}$$

So the general solution is

$$y=C_1x^{-2}+C_2\,x^{-1}. \tag{2}$$

b) Set $y=x^r$ we get the indicial equation

$$r\,(r-1)+5\,r+4=0\Longrightarrow r_{1,2}=-2. \tag{3}$$

So the general solution is

$$y=C_1\,x^{-2}+C_2\,x^{-2}\ln x. \tag{4}$$

c) Set $y=x^r$ we get the indicial equation

$$2\,r\,(r-1)+3\,r+4=0\Longrightarrow r_{1,2}=\frac{-1\pm\sqrt{-31}}{4}=-\frac{1}{4}\pm\frac{\sqrt{31}}{4}\,i. \tag{5}$$

So the general solution is

$$y=C_1\,x^{-1/4}\cos\left(\frac{\sqrt{31}}{4}\ln x\right)+C_2\,x^{-1/4}\sin\left(\frac{\sqrt{31}}{4}\ln x\right). \tag{6}$$

Problem 2. Find all singular points of

$$x^2\,(1-x)\,y''+(x-2)\,y'-3\,x\,y=0, \tag{7}$$

and determine whether each one is regular or irregular.

Solution. Write the equation into standard form:

$$y''+\frac{x-2}{x^2\,(1-x)}\,y'-\frac{3}{x\,(1-x)}\,y=0. \tag{8}$$

We see that there are two singular points $x=0, x=1$.

- At $x=0$, we have

$$x\,p=\frac{x-2}{x\,(1-x)},\quad x^2\,q=-\frac{3\,x}{1-x}. \tag{9}$$

  We see that $x\,p$ is not analytic (still has singularity at 0). So $x=0$ is an irregular singuar point.

- At $x=1$, we have

$$(x-1)\,p=\frac{x-2}{x^2},\quad (x-1)^2\,q=\frac{3\,(1-x)}{x} \tag{10}$$

  both are analytic at $x=1$. So $x=1$ is a regular singular point.

Intermediate

Problem 3. Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the differential equation

$$(1+x^3)\,y''+4\,x\,y'+4\,y=0;\qquad x_0=0,\ x_0=2. \tag{11}$$

Solution. Write the equation into standard form

$$y'' + \frac{4\,x}{1+x^3}\,y' + \frac{4}{1+x^3}\,y = 0. \tag{12}$$

We see that the singular points are solutions to

$$x^3 + 1 = 0. \tag{13}$$

or equivalently

$$x^3 = -1. \tag{14}$$

To find all such x, we need to write $-1 = R\,e^{i\theta}$. Clearly $R = 1$. To determine θ we solve

$$\cos\theta = -1, \qquad \sin\theta = 0 \tag{15}$$

which gives $\theta = \pi + 2\,k\,\pi$. Thus the solutions are given by

$$x = e^{i\frac{2k+1}{3}\pi}. \tag{16}$$

Notice that k and $k+3$ gives the same x. Therefore the three roots are given by setting $k = 0, 1, 2$.

$$k = 0 \Longrightarrow x = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i;\ k = 1 \Longrightarrow x = -1;\ k = 2 \Longrightarrow x = \frac{1}{2} - \frac{\sqrt{3}}{2}i. \tag{17}$$

Now we discuss

- $x_0 = 0$. The distance from 4 to the three roots are:

$$\left|0 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)\right| = 1 \tag{18}$$

$$|0 - (-1)| = 1; \tag{19}$$

$$\left|0 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)\right| = 1; \tag{20}$$

  The smallest distance is 1. So the radius of convergence is at least 1.

- $x_0 = 2$. The distances are

$$\left|2 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)\right| = \left|\frac{3}{2} - \frac{\sqrt{3}}{2}\,i\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}; \tag{21}$$

$$|2 - (-1)| = 3; \tag{22}$$

$$\left|2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)\right| = \sqrt{3}. \tag{23}$$

  The smallest distance is $\sqrt{3}$. So the radius of convergence is $\sqrt{3}$.

ADVANCED

Problem 4. Find the first five terms of the power series solution for

$$x\,y'' + y\ln(1-x) = 0 \tag{24}$$

and determine a lower bound for its radius of convergence.
Solution.

First we determine the radius of convergence. Write the equation in standard form

$$y'' + \frac{\ln(1-x)}{x}\,y = 0. \tag{25}$$

We need to find all singular points for $\frac{\ln(1-x)}{x}$. As this is a ratio, we first determine:

- Singular point for $\ln(1-x)$: $x = 1$;[1]

1. The issue is in fact a bit more complicated than this.

- Singular point for x: None.

Now as this is a ratio, one more possible singular point is $x = 0$, which makes the denominator vanish. However, we realize that $\ln(1-x) = 0$ at this point too. Expanding $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ we see that the x is in fact cancelled. So $x = 0$ is in fact a regular point. The distance from 0 to 1 is 1, so the radius of convergence for the solution is at least 1.

Or we can proceed more directly as follows:

$$\frac{\ln(1-x)}{x} = \frac{-\sum_{n=1}^{\infty} \frac{x^n}{n}}{x} = -\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1} \tag{26}$$

when $|x| < 1$. As the resulting power series indeed has radius of convergence 1, we have

$$\frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1} \tag{27}$$

for $|x| < 1$. The radius of convergence for the solution is therefore at least 1.

Next we solve the equation. Write

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{28}$$

We can substitute this into either the original equation or the standard form. As we have already obtained the expansion for $\frac{\ln(1-x)}{x}$, we use the standard form:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' - \left(\sum_{n=0}^{\infty} \frac{x^n}{n+1}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0. \tag{29}$$

This simplifies to

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \left(\sum_{n=0}^{\infty} \frac{x^n}{n+1}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0. \tag{30}$$

We need to find the first 5 nonzero terms so we have to compute at least up to a_4. The lowest order term containing a_4 is the x^2 term so we try first to calculate up to x^2:

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) - \left(1 + \frac{x}{2} + \frac{x^2}{3} + \cdots\right)(a_0 + a_1 x + a_2 x^2 + \cdots) = 0. \tag{31}$$

The left hand side is

$$(2a_2 - a_0) + \left(6a_3 - \frac{a_0}{2} - a_1\right) x + \left(12a_4 - a_2 - \frac{a_1}{2} - \frac{a_0}{3}\right) x^2 + \cdots = 0. \tag{32}$$

Thus we have

$$2a_2 - a_0 = 0; \tag{33}$$

$$6a_3 - \frac{a_0}{2} - a_1 = 0; \tag{34}$$

$$12a_4 - a_2 - \frac{a_1}{2} - \frac{a_0}{3} = 0; \tag{35}$$

They lead to

$$a_2 = \frac{a_0}{2}; \qquad a_3 = \frac{a_0}{12} + \frac{a_1}{6}; \qquad a_4 = \frac{5a_0}{72} + \frac{a_1}{24}; \tag{36}$$

None of them is 0 so we already have 5 nonzero terms:

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 + \left(\frac{a_0}{12} + \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{72} + \frac{a_1}{24}\right) x^4 + \cdots \tag{37}$$

Problem 5. Consider the equation

$$2x^2 y'' + x(2x+1) y' - y = 0. \tag{38}$$

a) Is 0 a regular(ordinary) point, a regular singular point, or an irregular singular point?

b) Write down and solve the indicial equation.

c) Write down the correct forms of y_1, y_2.

d) If the two roots of the indicial equation does not differ by an integer, find y_1, y_2.

Solution.

a) First write the equation into standard form:

$$y''+\frac{2\,x+1}{2\,x}\,y'-\frac{1}{2\,x^2}\,y=0. \tag{39}$$

Thus $p=\frac{2\,x+1}{2\,x}$, $q=-\frac{1}{2\,x^2}$. Obviously they both are not analytic at 0. Therefore 0 is not a regular point. Next consider $x\,p=\frac{2\,x+1}{2}$, $x^2\,q=-1$. Both are analytic at 0. Therefore 0 is a regular singular point.

b) The indicial equation is $r\,(r-1)+p_0\,r+q_0=0$. Here p_0, q_0 are the constant terms in the Taylor expansions of $x\,p$ and $x^2\,q$. So $p_0=\frac{1}{2}$, $q_0=-\frac{1}{2}$. The indicial equation is then

$$r\,(r-1)+\frac{r}{2}-\frac{1}{2}=0\Longleftrightarrow 2\,r^2-r-1=0\Longrightarrow r_{1,2}=1,-\frac{1}{2}. \tag{40}$$

c) As the two roots are distinct and their difference is not an integer, we have

$$y_1=x\sum_{n=0}^{\infty}a_n\,x^n,\qquad y_2=x^{-1/2}\sum_{n=0}^{\infty}b_n\,x^n. \tag{41}$$

d)

- Finding y_1: Substitute

$$y_1=x\sum_{n=0}^{\infty}a_n\,x^n=\sum_{n=0}^{\infty}a_n\,x^{n+1} \tag{42}$$

into the equation we have

$$y_1' = \sum_{n=0}^{\infty}(n+1)\,a_n\,x^n; \tag{43}$$

$$y_1'' = \sum_{n=1}^{\infty}(n+1)\,n\,a_n\,x^{n-1}; \tag{44}$$

Note that the summation starts from $n=0$ and 1, not 1 and 2!!

Now we have

$$\sum_{n=1}^{\infty}(n+1)\,n\,a_n\,x^{n-1}+\frac{2\,x+1}{2\,x}\sum_{n=0}^{\infty}(n+1)\,a_n\,x^n-\frac{1}{2\,x^2}\sum_{n=0}^{\infty}a_n\,x^{n+1}=0. \tag{45}$$

Simplify:

$$\sum_{n=1}^{\infty}(n+1)\,n\,a_n\,x^{n-1}+\sum_{n=0}^{\infty}(n+1)\,a_n\,x^n+\frac{1}{2}\sum_{n=0}^{\infty}(n+1)\,a_n\,x^{n-1}-\frac{1}{2}\sum_{n=0}^{\infty}a_n\,x^{n-1}=0. \tag{46}$$

Shift indices (for this problem it is easier to shift everything to x^{n-1} since thus only one term needs to be shifted)

$$\sum_{n=1}^{\infty}(n+1)\,n\,a_n\,x^{n-1}+\sum_{n=1}^{\infty}n\,a_{n-1}\,x^{n-1}+\frac{1}{2}\sum_{n=0}^{\infty}(n+1)\,a_n\,x^{n-1}-\frac{1}{2}\sum_{n=0}^{\infty}a_n\,x^{n-1}=0. \tag{47}$$

Simplify:

$$\frac{1}{2}\,a_0\,x^{-1}-\frac{1}{2}\,a_0\,x^{-1}+\sum_{n=1}^{\infty}\left[(n+1)\,n\,a_n+n\,a_{n-1}+\frac{1}{2}\,(n+1)\,a_n-\frac{1}{2}\,a_n\right]x^{n-1}=0 \tag{48}$$

Note that the blue term must be 0 if we have solved the indicial equation correctly.

So the recurrence relation is (for $n\geqslant 1$; Remember that a_0 is arbitrary)

$$(n+1)\,n\,a_n+n\,a_{n-1}+\frac{1}{2}\,(n+1)\,a_n-\frac{1}{2}\,a_n=0 \tag{49}$$

which simplifies to

$$a_n=-\frac{a_{n-1}}{(n+3/2)}. \tag{50}$$

Thus we have

$$a_n=-\frac{a_{n-1}}{(n+3/2)}=(-1)^2\,\frac{a_{n-2}}{\left(n+\frac{3}{2}\right)\left(n-1+\frac{3}{2}\right)}=\cdots=(-1)^n\frac{a_0}{\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\cdots\frac{5}{2}}. \tag{51}$$

So

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\cdots\frac{5}{2}} x^{n+1}. \tag{52}$$

- y_2. We have

$$y_2(x) = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}. \tag{53}$$

So

$$y_2' = \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right) x^{n-\frac{3}{2}}; \tag{54}$$

$$y_2'' = \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) x^{n-\frac{5}{2}}. \tag{55}$$

Substitute into equation we get

$$\sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) x^{n-\frac{5}{2}} + \frac{2x+1}{2x} \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right) x^{n-\frac{3}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}} = 0. \tag{56}$$

Simplify:

$$\sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) x^{n-\frac{5}{2}} + \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right) x^{n-\frac{3}{2}} + \frac{1}{2} \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right) x^{n-\frac{5}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n-\frac{5}{2}} = 0 \tag{57}$$

Shift index for the 2nd term to make all generic terms the same (all $x^{n-\frac{5}{2}}$):

$$\sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) x^{n-\frac{5}{2}} + \sum_{n=1}^{\infty} a_{n-1} \left(n-\frac{3}{2}\right) x^{n-\frac{5}{2}} + \frac{1}{2} \sum_{n=0}^{\infty} a_n \left(n-\frac{1}{2}\right) x^{n-\frac{5}{2}} - \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n-\frac{5}{2}} = 0. \tag{58}$$

Simplify

$$a_0 \left[\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\right] x^{-\frac{5}{2}} + \sum_{n=1}^{\infty} \left[a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + a_{n-1}\left(n-\frac{3}{2}\right) + \frac{1}{2}\left(n-\frac{1}{2}\right) a_n - \frac{1}{2} a_n\right] x^{n-\frac{5}{2}} = 0. \tag{59}$$

Again, the blue term vanishes because we have solved the indicial equation correctly.

The recurrence relation is then

$$a_n \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + a_{n-1}\left(n-\frac{3}{2}\right) + \frac{1}{2}\left(n-\frac{1}{2}\right) a_n - \frac{1}{2} a_n \tag{60}$$

which simplifies to

$$a_n = -\frac{a_{n-1}}{n} \Longrightarrow a_n = \frac{(-1)^n}{n!} a_0. \tag{61}$$

So

$$y_2 = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = x^{-\frac{1}{2}} e^{-x}. \tag{62}$$

Challenge

Problem 6. Construct an example of an equation that does not have a solution of the form $x^{\alpha} \sum_{n=0}^{\infty} a_n x^n$. (Hint: What equation does e^{1/x^2} solve?)

Problem 7. Prove the following: If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for $|x-x_0| < R$ for some $R > 0$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{63}$$

In other words, if f is analytic at some point x_0, then the corresponding power series is necessarily the Taylor expansion of f.

Proof. As $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for all $|x - x_0| < R$, then necessarily $f(x_0) = a_0$.

Now show $a_1 = f'(x_0)$. As $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, we know that the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges for all $|x - x_0| < R$ which means its radius of convergence $\rho \geqslant R$. Therefore we have $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is differentiable and

$$\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n \right)' = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n. \tag{64}$$

Consequently f is differentiable at x_0 and

$$f'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n \tag{65}$$

for all $|x - x_0| < R$. Setting $x = x_0$ we get

$$a_1 = f'(x_0). \tag{66}$$

Differentiate again we reach

$$f''(x) = \sum_{n=0}^{\infty} (n+2)\,(n+1)\, a_{n+2} (x - x_0)^n \tag{67}$$

for all $|x - x_0| < R$. Setting $x = x_0$ we have

$$f''(x_0) = 2\, a_2. \tag{68}$$

In general, taking k derivatives we reach

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n+k) \cdots (n+1)\, a_{n+k} (x - x_0)^n \tag{69}$$

which gives

$$f^{(k)}(x_0) = k!\, a_k. \tag{70}$$

□

Problem 8. Show that Euler equations

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0 \tag{71}$$

can be transformed to 2nd order constant-coefficient linear equations through the change of variable: $t = \ln x$. Write down that equation.

Solution.

$t = \ln x$ so $x = e^t$. The chain rule then gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}t} = e^t\, y' = x\, y'; \tag{72}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t} \right) = \frac{\mathrm{d}}{\mathrm{d}t}(x\, y') = \frac{\mathrm{d}x}{\mathrm{d}t}\, y' + x\, \frac{\mathrm{d}}{\mathrm{d}t}(y') = x\, y' + x\, \frac{\mathrm{d}(y')}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}t} = x\, y' + x^2\, y''. \tag{73}$$

Thus

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0 \tag{74}$$

becomes

$$a\, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (b - a)\, \frac{\mathrm{d}y}{\mathrm{d}t} + c\, y = 0. \tag{75}$$

Problem 9. Find a function $p(x)$ for which $\lim_{x \to x_0} (x - x_0)\, p(x)$ is finite, but $(x - x_0)\, p(x)$ is **not** analytic at x_0. Then prove that if $p(x)$ is rational, that is $p(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials, then the finiteness of the above limit indeed implies the analyticity of $(x - x_0)\, p$.

Proof. The example can be, say, $x_0 = 0$ and $p(x) = \frac{1}{x}\, e^{-\frac{1}{x^2}}$.

To prove the claim, use the following fact: Any polynomial of degree k can be factorized:

$$a_0 + \cdots + a_k\, x^k = a_k\, (x - r_1) \cdots (x - r_k) \tag{76}$$

where $r_1, \ldots, r_k$ are the k complex roots of the polynomial.

Therefore if $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$ are the roots of P, Q, we have

$$(x - x_0)\, p(x) = (x - x_0)\, \frac{p_n}{q_n} \frac{(x - r_1) \cdots (x - r_n)}{(x - s_1) \cdots (x - s_m)} = \frac{\tilde{P}(x)}{\tilde{Q}(x)}. \tag{77}$$

Here $\tilde{P}, \tilde{Q}$ are polynomials such that no further cancellation can be done. In particular, $\tilde{P}(x)$ and $\tilde{Q}(x)$ do not both have the factor $(x - x_0)$.

The finiteness of the limit $\lim_{x \to x_0} (x - x_0)\, p(x)$ then means that $\tilde{Q}(x_0) \neq 0$. Therefore (according to our rules) $\frac{\tilde{P}}{\tilde{Q}}$ is analytic at x_0. □

Problem 10. Prove the following. If all solutions to $y'' + p(x)\, y' + q(x) y = 0$ are analytic at $x_0 = 0$, then p, q are analytic there too.

Proof. Since all solutions are analytic, in particular y_1, y_2 are, with $y_1(0) = y_2'(0) = 1$, $y_1'(0) = y_2(0) = 0$. Now we have

$$y_1'' + p(x)\, y_1' + q(x) y_1 = 0, \qquad y_2'' + p(x)\, y_2' + q(x) y_2 = 0. \tag{78}$$

Treating this as a system with unknown p, q we reach

$$y_1'\, p + y_1\, q = -y_1'' \tag{79}$$

$$y_2'\, p + y_2\, q = -y_2'' \tag{80}$$

Solving it we get

$$p = \frac{-y_1''\, y_2 + y_2''\, y_1}{y_2\, y_1' - y_1\, y_2'}; \qquad q = \frac{-y_1''\, y_2' + y_2''\, y_1'}{y_1\, y_2' - y_2\, y_1'}. \tag{81}$$

Both are of the form f/g with f, g analytic at 0. So all we need to check is the the denominator is not 0 at $x_0 = 0$, which is obvious. □