Oct. 21, 2011

Math 334 Fall 2011 Homework 6 Solutions

Basic

Intermediate

Advanced

Problem 1. Solve the following equations:

a) $y''' + 2\,y'' + 9\,y' + 18\,y = 0$;

b) $y^{(4)} - 2\,y'' + 4\,y = 0$;

Solution.

a) Characteristic equation:

$$r^3 + 2\,r^2 + 9\,r + 18 = 0. \tag{1}$$

Clearly there is no positive roots. For negative numbers, first guess -1. Doesn't work. Next guess -2:

$$(-2)^3 + 2\,(-2)^2 + 9\,(-2) + 18 = -8 + 8 - 18 + 18 = 0. \tag{2}$$

So $r_1 = -2$ is a root.

Factorize:

$$r^3 + 2\,r^2 + 9\,r + 18 = (r - (-2))\,(\cdots) = (r+2)\,(r^2+9). \tag{3}$$

Therefore the other two roots are those of $r^2 + 9 = 0$ which are $r_{2,3} = \pm 3\,i$. Summarize: List of roots: -2, $\pm 3\,i$ – one real root, and one pair of complex roots.

Now

$$-2 \longrightarrow e^{-2t}; \qquad \pm 3\,i = 0 \pm 3\,i \longrightarrow e^{0t}\cos 3\,t, e^{0t}\sin 3\,t = \cos 3\,t, \sin 3\,t. \tag{4}$$

Therefore the general solution is given by

$$y = C_1\,e^{-2t} + C_2\cos 3\,t + C_3\sin 3\,t. \tag{5}$$

b) Characteristic equation:

$$r^4 - 2\,r^2 + 4 = 0. \tag{6}$$

If we let $q = r^2$, we have a quadratic equation for q:

$$q^2 - 2\,q + 4 = 0 \Longrightarrow q_{1,2} = 1 \pm \sqrt{3}\,i. \tag{7}$$

Therefore the 4 roots r_1 — r_4 are obtained through computing $\left(1 \pm \sqrt{3}\,i\right)^{1/2}$.

- $\left(1 + \sqrt{3}\,i\right)^{1/2}$:

  First write $1 + \sqrt{3}\,i = R\,e^{i(\theta_0 + 2k\pi)}$:

$$R = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2; \qquad \cos\theta_0 = \frac{1}{2},\ \sin\theta_0 = \frac{\sqrt{3}}{2} \Longrightarrow \theta_0 = \frac{\pi}{3}. \tag{8}$$

  So

$$1 + \sqrt{3}\,i = 2\,e^{i\left(\frac{\pi}{3} + 2k\pi\right)}. \tag{9}$$

  Now

$$\left(1 + \sqrt{3}\,i\right)^{1/2} = \sqrt{2}\,e^{i\left(\frac{\pi}{6} + k\pi\right)}. \tag{10}$$

  Take two consecutive values of k, say $0, 1$.

  - $k = 0$ gives

$$\sqrt{2}\,e^{i\frac{\pi}{6}} = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = \frac{\sqrt{6}}{2} + i\,\frac{\sqrt{2}}{2}; \tag{11}$$

- $k=1$ gives

$$\sqrt{2}\, e^{i\frac{7\pi}{6}} = \sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\frac{\sqrt{6}}{2} - i\,\frac{\sqrt{2}}{2}; \tag{12}$$

Note that since we are taking square root, we actually know that the second root has to be "−first root", so we should be able to save some time here.

To feel save, check:

$$\left(\frac{\sqrt{6}}{2} + i\,\frac{\sqrt{2}}{2}\right)^2 = \frac{6}{4} + 2\,\frac{\sqrt{6}\,\sqrt{2}}{2\cdot 2}\,i - \frac{2}{4} = 1 + \sqrt{3}\,i. \tag{13}$$

Note. The two roots we obtained here are not conjugates (or, not a "pair")!

- $(1-\sqrt{3}\,i)^{1/2}$:
  First write $1-\sqrt{3}\,i = R^{i(\theta_0+2k\pi)}$:

$$R = \sqrt{1^2 + (-\sqrt{3})^2} = 2; \qquad \cos\theta_0 = \frac{1}{2},\ \sin\theta_0 = -\frac{\sqrt{3}}{2} \Longrightarrow \theta_0 = -\frac{\pi}{3}. \tag{14}$$

So

$$1-\sqrt{3}\,i = 2\,e^{i\left(-\frac{\pi}{3}+2k\pi\right)} \tag{15}$$

and

$$(1-\sqrt{3}\,i)^{1/2} = \sqrt{2}\,e^{i\left(-\frac{\pi}{6}+k\pi\right)}. \tag{16}$$

Taking $k=0,1$ we obtain

$$\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \frac{\sqrt{6}}{2} - i\,\frac{\sqrt{2}}{2}; \tag{17}$$

$$\sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = -\frac{\sqrt{6}}{2} + i\,\frac{\sqrt{2}}{2} \tag{18}$$

Notice how these two roots pair up with those obtained in $(1+\sqrt{3}\,i)^{1/2}$!

So we finally obtained the four roots in two pairs:

$$\frac{\sqrt{6}}{2} \pm i\,\frac{\sqrt{2}}{2}; \qquad -\frac{\sqrt{6}}{2} \pm i\,\frac{\sqrt{2}}{2}. \tag{19}$$

and the fundamental set is obtained as:

$$\frac{\sqrt{6}}{2} \pm i\,\frac{\sqrt{2}}{2} \Longrightarrow e^{\frac{\sqrt{6}}{2}t}\cos\frac{\sqrt{2}}{2}t,\, e^{\frac{\sqrt{6}}{2}t}\sin\frac{\sqrt{2}}{2}t \tag{20}$$

and

$$-\frac{\sqrt{6}}{2} \pm i\,\frac{\sqrt{2}}{2} \Longrightarrow e^{-\frac{\sqrt{6}}{2}t}\cos\frac{\sqrt{2}}{2}t,\, e^{-\frac{\sqrt{6}}{2}t}\sin\frac{\sqrt{2}}{2}t. \tag{21}$$

The final answer is then

$$y = C_1\, e^{\frac{\sqrt{6}}{2}t}\cos\frac{\sqrt{2}}{2}t + C_2\, e^{\frac{\sqrt{6}}{2}t}\sin\frac{\sqrt{2}}{2}t + C_3\, e^{-\frac{\sqrt{6}}{2}t}\cos\frac{\sqrt{2}}{2}t + C_4\, e^{-\frac{\sqrt{6}}{2}t}\sin\frac{\sqrt{2}}{2}t. \tag{22}$$

Problem 2. Solve the following equations:

a) $y^{(4)} - 4\,y'' = 2\,t^2$;

b) $y^{(4)} + 2\,y'' + y = 3\,t + 4;\qquad y(0) = y'(0) = 0,\quad y''(0) = y'''(0) = 1.$

Solution.

a) We use undetermined coefficients.
First solve the homogeneous equation:

$$y^{(4)} - 4\,y'' = 0. \tag{23}$$

The characteristic equation is

$$r^4 - 4\,r^2 = 0 \Longrightarrow r_{1,2} = 0, r_3 = 2, r_4 = -2. \tag{24}$$

The general solution (to the homogeneous equation) is

$$y = C_1 + C_2\, t + C_3\, e^{2t} + C_4\, e^{-2t}. \tag{25}$$

Next guess the form of y_p. As

$$2\, t^2 = e^{0t}\,(0 + 0\, t + 2\, t^2) \tag{26}$$

we have

$$y_p = t^s\, e^{0t}\,(A_0 + A_1\, t + A_2\, t^2). \tag{27}$$

To determine s we check the multiplicity of 0 as a root to the characteristic equation (that is how many time 0 appears in the list of roots): 2. So $s = 2$.

$$y_p = t^2\,(A_0 + A_1\, t + A_2\, t^2). \tag{28}$$

Compute

$$y_p^{(4)} = 24\, A_2; \qquad y_p'' = 2\, A_0 + 6\, A_1\, t + 12\, A_2\, t^2. \tag{29}$$

Substitute into the equation:

$$24\, A_2 - 4\,(2\, A_0 + 6\, A_1\, t + 12\, A_2\, t^2) = 2\, t^2 \tag{30}$$

So

$$24\, A_2 - 8\, A_0 = 0; \qquad -24\, A_1 = 0; \qquad -48\, A_2 = 2 \tag{31}$$

which gives

$$A_2 = -\frac{1}{24}, \qquad A_0 = -\frac{1}{8}. \tag{32}$$

Therefore

$$y_p = -\frac{t^2}{8} - \frac{t^4}{24}. \tag{33}$$

(Check y_p is time allows!)

Finally the general solution is

$$y = C_1 + C_2\, t + C_3\, e^{2t} + C_4\, e^{-2t} - \frac{t^2}{8} - \frac{t^4}{24}. \tag{34}$$

b) This is initial value problem. So we should

1. Find the general solution: First get solution to homogeneous equation, then get y_p, then put everything together;
2. Use initial conditions to determine the four constants.

To find the general solution we first solve the homogeneous equation

$$y^{(4)} + 2\, y'' + y = 0 \tag{35}$$

whose characteristic equation is

$$r^4 + 2\, r^2 + 1 = 0 \Longrightarrow r^2 = -1 \text{ (repeated)} \tag{36}$$

therefore

$$r_{1,2} = \pm i, \quad r_{3,4} = \pm i. \tag{37}$$

The fundamental set is then

$$\pm i \text{ (repeated 2 times)} \Longrightarrow \cos t, \sin t, t\cos t, t\sin t. \tag{38}$$

The general solution to the homogeneous equation is

$$y = C_1 \cos t + C_2 \sin t + C_3\, t \cos t + C_4\, t \sin t. \tag{39}$$

Next we try to fix the form of y_p. As the right hand side is

$$3\, t + 4 = e^{0t}\,(4 + 3\, t) \tag{40}$$

we guess

$$y_p = t^s\, e^{0t}\,(A_0 + A_1\, t). \tag{41}$$

As 0 does not appear in the list of roots (appears 0 times; has multiplicity 0), $s = 0$. So

$$y_p = A_0 + A_1\, t. \tag{42}$$

Now compute

$$y_p'' = 0, y_p^{(4)} = 0. \tag{43}$$

So

$$A_0 + A_1\, t = 3\, t + 4 \Longrightarrow A_0 = 4, A_1 = 3. \tag{44}$$

Thus

$$y_p = 3\, t + 4. \tag{45}$$

The general solution for the nonhomogeneous equation is

$$y = C_1 \cos t + C_2 \sin t + C_3\, t \cos t + C_4\, t \sin t + 3\, t + 4. \tag{46}$$

To apply initial conditions, prepare:

$$y' = (C_2 + C_3) \cos t + (-C_1 + C_4) \sin t + C_4\, t \cos t - C_3\, t \sin t + 3 \tag{47}$$

$$y'' = (-C_1 + 2\, C_4) \cos t + (-C_2 - 2\, C_3) \sin t - C_3\, t \cos t - C_4\, t \sin t \tag{48}$$

$$y''' = (-C_2 - 3\, C_3) \cos t + (C_1 - 3\, C_4) \sin t - C_4\, t \cos t + C_3\, t \sin t. \tag{49}$$

Apply initial conditions:

$$y(0) = 0 \implies C_1 + 4 = 0; \tag{50}$$

$$y'(0) = 0 \implies C_2 + C_3 + 3 = 0; \tag{51}$$

$$y''(0) = 1 \implies -C_1 + 2\, C_4 = 1; \tag{52}$$

$$y'''(0) = 1 \implies -C_2 - 3\, C_3 = 1. \tag{53}$$

Instead of using the general procedure (Gaussian elimination), we observe that this system is special: equations 1, 3 only involve C_1, C_4 while equations 2,4 only involve C_2, C_3. So it's more efficient to solve the system in the following ad hoc manner:

Getting C_1, C_4:

$$\begin{array}{rcl} C_1 + 4 & = & 0 \\ -C_1 + 2\, C_4 & = & 1 \end{array} \Longrightarrow C_1 = -4, C_4 = -\frac{3}{2}; \tag{54}$$

Getting C_2, C_3:

$$\begin{array}{rcl} C_2 + C_3 + 3 & = & 0 \\ -C_2 - 3\, C_3 & = & 1 \end{array} \Longrightarrow C_3 = 1, C_2 = -4. \tag{55}$$

Therefore

$$y = -4 \cos t - 4 \sin t + t \cos t - \frac{3}{2}\, t \sin t + 3\, t + 4. \tag{56}$$

CHALLENGE