Sept. 30, 2011

Math 334 Fall 2011 Homework 3 Solutions

Basic

Problem 1. Solve the following equations

a)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 - 2\,y}{x} \tag{1}$$

b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \cos x}{2 - \sin y} \tag{2}$$

c)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\,x + y}{3 + 3\,y^2 - x}, \qquad y(0) = 0. \tag{3}$$

Solution.

a) First we check whether this equation is linear or separable (because these two are usually easiest to solve!). It is linear. So we write it as

$$y' = x^2 - \frac{2}{x}\,y \Longrightarrow y' + \frac{2}{x}\,y = x^2. \tag{4}$$

Compare with the standard form $y' + p(x)\,y = g(x)$, we see that $p(x) = 2/x$.

Therefore an integrating factor is given by

$$\mu(x) = e^{\int p} = e^{\int \frac{2}{x}\mathrm{d}x} = e^{2\ln x} = x^2. \tag{5}$$

Multiply the equation by $\mu(x)$ we should get

$$(x^2\,y)' = x^4. \tag{6}$$

Check:

$$(x^2\,y)' = (x^2)'\,y + x^2\,y' = x^2\left[y' + \frac{2}{x}\,y\right]. \tag{7}$$

Integrate

$$(x^2\,y)' = x^4 \Longrightarrow x^2\,y = \frac{1}{5}\,x^5 + C \Longrightarrow y = \frac{1}{5}\,x^3 + \frac{C}{x^2}. \tag{8}$$

Finally check solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{5}\,x^2 - 2\,\frac{C}{x^3}; \tag{9}$$

$$\frac{x^3 - 2\,y}{x} = \frac{x^3 - \frac{2}{5}\,x^3 - \frac{2\,C}{x^2}}{x} = \frac{3}{5}\,x^2 - 2\,\frac{C}{x^3}. \tag{10}$$

We see that we have found the correct solution.

b) First notice that this equation is separable. So we move all the y's to the left and all the x's to the right.

$$(2 - \sin y)\,\mathrm{d}y = (1 + \cos x)\,\mathrm{d}x. \tag{11}$$

Integrate:

$$2\,y + \cos y = x + \sin x + C. \tag{12}$$

This is an implicit formula for the solution. Also note that as we divided both sides by $1/(2 - \sin y)$ which is never 0, we didn't lose any solution.

So the final answer is

$$2\,y + \cos y - x - \sin x = C. \tag{13}$$

We check the solution:

$$\mathrm{d}(2\,y + \cos y - x - \sin x) = 0 \Longrightarrow (2 - \sin y)\,\mathrm{d}y - (1 + \cos x)\,\mathrm{d}x = 0 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \cos x}{2 - \sin y}. \tag{14}$$

c) First this equation doesn't look like linear or separable. So we try to see whether it's exact.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\,x + y}{3 + 3\,y^2 - x} \Longrightarrow (2\,x + y)\,\mathrm{d}x - (3 + 3\,y^2 - x)\,\mathrm{d}y = 0. \tag{15}$$

So $M = 2x + y$, $N = -(3 + 3y^2 - x)$ (Note the "−"!!)

We check

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 1 \tag{16}$$

so the equation is exact. We solve it through

$$u(x, y) = \int (2x + y)\,\mathrm{d}x + g(y) = x^2 + xy + g(y). \tag{17}$$

To determine $g(y)$, take the y derivative

$$\frac{\partial u}{\partial y} = x + g'(y). \tag{18}$$

Comparing with $N = -(3 + 3y^2 - x)$, we see that

$$g'(y) = -(3 + 3y^2) \tag{19}$$

which gives

$$g(y) = -3y - y^3. \tag{20}$$

So

$$u(x, y) = x^2 + xy - 3y - y^3 \tag{21}$$

and the general solution is

$$x^2 + xy - 3y - y^3 = C. \tag{22}$$

To check our solution, take "d":

$$\begin{aligned} \mathrm{d}(x^2 + xy - 3y - y^3) &= \frac{\partial(x^2 + xy - 3y - y^3)}{\partial x}\,\mathrm{d}x + \frac{\partial(x^2 + xy - 3y - y^3)}{\partial y}\,\mathrm{d}y \\ &= (2x + y - 0 - 0)\,\mathrm{d}x + (0 + x - 3 - 3y^2)\,\mathrm{d}y \\ &= (2x + y)\,\mathrm{d}x + (x - 3 - 3y^2)\,\mathrm{d}y. \end{aligned} \tag{23}$$

We see that

$$\mathrm{d}(x^2 + xy - 3y - y^3) = 0 \Longrightarrow (2x + y)\,\mathrm{d}x + (x - 3 - 3y^2)\,\mathrm{d}y = 0 \tag{24}$$

which is in turn the same as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y}{x - 3 - 3y^2} = \frac{2x + y}{3 + 3y^2 - x}. \tag{25}$$

So our solution is correct.

Finally as we are solving an initial value problem, we have to find out C. Substitute $x_0 = 0, y_0 = 0$ into the formula

$$x^2 + xy - 3y - y^3 = C \tag{26}$$

we see that $C = 0$.

So the final answer is

$$x^2 + xy - 3y - y^3 = 0. \tag{27}$$

INTERMEDIATE

Problem 2. Compute the Wronskian of the following pairs of functions

a) $e^{3t}, t^2 e^t$

b) $\sin x,\ \cos x^2$

c) $x,\ x \ln x$.

Solution.

a) We have

$$W = (e^{3t})\,(t^2 e^t)' - (t^2 e^t)\,(e^{3t})' = e^{3t}\,[2te^t + t^2 e^t] - (t^2 e^t)\,[3e^{3t}] = 2te^{4t} - 2t^2 e^{4t}. \tag{28}$$

b) We have

$W = (\sin x)\,(\cos x^2)' - (\cos x^2)\,(\sin x)' = (\sin x)\,(-2\,x\sin x^2) - (\cos x^2)\,(\cos x) = -2\,x\,(\sin x)\,(\sin x^2) - (\cos x^2)\,(\cos x).$ (29)

c) We have

$$W = x\,(x\ln x)' - (x\ln x)\,(x)' = x\,[\ln x + 1] - x\ln x = x. \tag{30}$$

Problem 3. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\,x + y^2 + 2\,x\,y^2. \tag{31}$$

Solution. This equation is clearly not linear. To see whether it's separable, we try to simplify the right hand side:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\,x + y^2 + 2\,x\,y^2 = 1 + 2\,x + y^2\,(1 + 2\,x) = (1 + y^2)\,(1 + 2\,x). \tag{32}$$

So it is! To solve it, we divide both sides by $1 + y^2$ and move $\mathrm{d}x$ to the right hand side:

$$\frac{\mathrm{d}y}{1 + y^2} = (1 + 2\,x)\mathrm{d}x. \tag{33}$$

Integrate

$$\int \frac{\mathrm{d}y}{1 + y^2} = \arctan y, \tag{34}$$

so

$$\arctan y = x + x^2 + C. \tag{35}$$

As $1 + y^2 \neq 0$, we didn't lose any solution.

We choose to leave it like this to avoid writing

$$y = \tan(x + x^2 + C) + k\,\pi, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{36}$$

Finally check

$$\mathrm{d}[\arctan y - x - x^2] = \frac{\mathrm{d}y}{1 + y^2} - (1 + 2\,x)\,\mathrm{d}x. \tag{37}$$

So

$$\arctan y = x + x^2 + C \Longleftrightarrow \mathrm{d}[\arctan y - x - x^2] = 0 \Longleftrightarrow \frac{\mathrm{d}y}{1 + y^2} - (1 + 2\,x)\,\mathrm{d}x = 0 \tag{38}$$

which is equivalent to the original equation.

Advanced

Problem 4. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3\,x^2\,y + y^2}{2\,x^3 + 3\,x\,y}, \qquad y(1) = -2. \tag{39}$$

Solution. This is clearly not linear and unlikely to be separable, and our effort of writing it as homogeneous fail:

$$\text{Divide both by } x^3\text{: } -\frac{3\,x^2\,y + y^2}{2\,x^3 + 3\,x\,y} = -\frac{3\,\frac{y}{x} + \left(\frac{y}{x}\right)^2\frac{1}{x}}{2 + 3\,\frac{y}{x}\,\frac{1}{x}}. \tag{40}$$

So we check whether it is exact.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3\,x^2\,y + y^2}{2\,x^3 + 3\,x\,y} \Longleftrightarrow (3\,x^2\,y + y^2)\,\mathrm{d}x + (2\,x^3 + 3\,x\,y)\,\mathrm{d}y = 0 \tag{41}$$

so $M = 3\,x^2\,y + y^2$, $N = 2\,x^3 + 3\,x\,y$. We compute

$$\frac{\partial M}{\partial y} = 3\,x^2 + 2\,y, \qquad \frac{\partial N}{\partial x} = 6\,x^2 + 3\,y \tag{42}$$

They are not equal so the equation is not exact.

As we have excluded linear, separable, and homogeneous, we have to try finding an integrating factor. Recall the equation for integrating factor

$$M\,\frac{\partial\mu}{\partial y} - N\,\frac{\partial\mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu \tag{43}$$

(Check whether we have remembered the correct equation: μ makes the equation exact that is $(\mu\, M)\,\mathrm{d}x + (\mu\, N)\,\mathrm{d}y = 0$ is exact, so $\frac{\partial(\mu\, M)}{\partial y} = \frac{\partial(\mu\, N)}{\partial x}$ which reduces to the above).

Substitute $M = 3\,x^2\,y + y^2$, $N = 2\,x^3 + 3\,x\,y$, $\frac{\partial M}{\partial y} = 3\,x^2 + 2\,y$, $\frac{\partial N}{\partial x} = 6\,x^2 + 3\,y$ into the μ equation:

$$(3\,x^2\,y + y^2)\,\frac{\partial\mu}{\partial y} - (2\,x^3 + 3\,x\,y)\,\frac{\partial\mu}{\partial x} = [(6\,x^2 + 3\,y) - (3\,x^2 + 2\,y)]\,\mu = (3\,x^2 + y)\,\mu. \tag{44}$$

Now guess

- $\mu = \mu(x)$ independent of y. If this is true then $\frac{\partial\mu}{\partial y} = 0$ and $\mu(x)$ satisfies

$$-(2\,x^3 + 3\,x\,y)\,\mu' = (3\,x^2 + y)\,\mu \Longleftrightarrow \frac{\mu'(x)}{\mu(x)} = -\frac{3\,x^2 + y}{2\,x^3 + 3\,x\,y}. \tag{45}$$

  The right hand side depends on y also so such μ cannot exist.

- $\mu = \mu(y)$ independent of x. If this is true then $\frac{\partial\mu}{\partial x} = 0$ and $\mu(y)$ satisfies

$$(3\,x^2\,y + y^2)\,\mu' = (3\,x^2 + y)\,\mu \Longleftrightarrow \frac{\mu'(y)}{\mu(y)} = \frac{3\,x^2 + y}{3\,x^2\,y + y^2} = \frac{1}{y}. \tag{46}$$

  This can be solved! We can take

$$\mu(y) = y. \tag{47}$$

Multiplying our equation by this integrating factor, we get

$$(3\,x^2\,y^2 + y^3)\,\mathrm{d}x + (2\,x^3\,y + 3\,x\,y^2)\,\mathrm{d}y = 0. \tag{48}$$

To make sure we have done everything right so far, check exactness:

$$\frac{\partial(3\,x^2\,y^2 + y^3)}{\partial y} = 6\,x^2\,y + 3\,y^2; \qquad \frac{\partial(2\,x^3\,y + 3\,x\,y^2)}{\partial x} = 6\,x^2\,y + 3\,y^2. \tag{49}$$

Now we solve the equation (already multiplied by the integrating factor):

$$u(x, y) = \int (3\,x^2\,y^2 + y^3)\,\mathrm{d}x + g(y) = x^3\,y^2 + x\,y^3 + g(y). \tag{50}$$

To determine $g(y)$, take y derivative

$$\frac{\partial u}{\partial y} = 2\,x^3\,y + 3\,x\,y^2 + g'(y) \tag{51}$$

and compare with

$$\mu\,N = (2\,x^3\,y + 3\,x\,y^2). \tag{52}$$

We see that we only need $g' = 0$ so can take $g = 0$.

The general solution is then

$$x^3\,y^2 + x\,y^3 = C. \tag{53}$$

Check:

$$\mathrm{d}(x^3\,y^2 + x\,y^3) = (3\,x^2\,y^2 + y^3)\,\mathrm{d}x + (2\,x^3\,y + 3\,x\,y^2)\,\mathrm{d}y = y\,[(3\,x^2\,y + y^2)\,\mathrm{d}x + (2\,x^3 + 3\,x\,y)\,\mathrm{d}y]. \tag{54}$$

Finally we fix C using the initial value $y(1) = -2 \Longrightarrow x_0 = 1, y_0 = -2$. So

$$1^3\,(-2)^2 + 1\,(-2)^3 = C \Longrightarrow C = -4. \tag{55}$$

Therefore the final answer is

$$x^3\,y^2 + x\,y^3 = -4. \tag{56}$$

CHALLENGE

Problem 5. Solve

$$x\,y' + y - y^2\,e^{2x} = 0, \qquad y(1) = 2. \tag{57}$$

Solution.

The key is to observe that if we let $z = x\,y$, the equation becomes

$$z' - \left(\frac{z}{x}\right)^2 e^{2x} = 0 \Longleftrightarrow z' = z^2\,\frac{e^{2x}}{x^2} \tag{58}$$

which is separable.

We write it as

$$\frac{\mathrm{d}z}{z^2} = \frac{e^{2x}}{x^2}\,\mathrm{d}x \tag{59}$$

and integrate:

$$-\frac{1}{z} = \int \frac{e^{2x}}{x^2}\,\mathrm{d}x + C. \tag{60}$$

Now if $y(1) = 2$, we have $z(1) = 2$. To use this information, it's better to write the above formula as a definite integral:

$$-\frac{1}{z(x)} = \int_1^x \frac{e^{2s}}{s^2}\,\mathrm{d}s + C. \tag{61}$$

Now $z(1) = 2$ gives

$$-\frac{1}{2} = C \Longrightarrow C = -\frac{1}{2}. \tag{62}$$

So

$$-\frac{1}{z} = \int_1^x \frac{e^{2s}}{s^2}\,\mathrm{d}s - \frac{1}{2}. \tag{63}$$

Recall that $y = z/x$, we have

$$-\frac{1}{y} = x\int_1^x \frac{e^{2s}}{s^2}\,\mathrm{d}s - \frac{x}{2} \Longrightarrow \frac{1}{y} = -x\int_1^x \frac{e^{2s}}{s^2}\,\mathrm{d}s + \frac{x}{2}. \tag{64}$$

Problem 6. Consider the homogeneous equation

$$y' = H(y/x). \tag{65}$$

Find an integrating factor.

Problem 7. Consider the Bernoulli equation

$$y' + p(x)\,y = g(x)\,y^n. \tag{66}$$

Find an integrating factor of it. (Hint: The procedure of solving it is as follows.

- Divide both sides by y^n:

$$y^{-n}\,y' + p(x)\,y^{1-n} = g(x). \tag{67}$$

- Let $v = y^{1-n}$. Use the fact $v' = (1-n)\,y^{-n}\,y'$ to get

$$\frac{1}{1-n}\,v' + p(x)\,v = g(x). \tag{68}$$

  This is a linear equation and can be solved.)

Problem 8. Show that the general second order equation $y'' = f(x, y, y')$ can be re-written into a system of two equations, each of first order:

$$u' = A(x, u, v) \tag{69}$$

$$v' = B(x, u, v) \tag{70}$$

through appropriate definition of the new unknowns u, v. Then argue, without any rigorous justification, that such can be done for general differential equations of any order. Finally argue without proof that the method we used to prove the existence/uniqueness for $y' = f(x, y)$ also would give existence/uniqueness for general equations.

Problem 9. Give a rigorous proof of the following. If

$$|y(x) - z(x)| \leqslant M\int_{x_0}^x |y(\tau) - z(\tau)|\,\mathrm{d}\tau \tag{71}$$

then $y(x) - z(x) = 0$ for $|x - x_0| < M^{-1}$.

Problem 10. Consider the general first order equation (initial value problem)

$$y' = f(x, y), \qquad y(0) = y_0. \tag{72}$$

Try to prove existence of solution through the following strategy.

- Show that all the derivatives $y^{(n)}(0)$ can be represented using the values of f and its derivatives at the point $(0, y_0)$.

- Now write

$$y(x) = y_0 + y'(0)\,x + \frac{1}{2}\,y''(0)\,x^2 + \cdots \tag{73}$$

  and show convergence of the right hand side. What conditions on f will you need?
- Show that the sum $y(x)$ indeed solve the equation.
- Compare this approach with the Picard iteration proof in the textbook. Which one is better?