Sept. 23, 2011

Math 334 Fall 2011 Homework 3

Basic

Problem 1. Solve the following equations

a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 - 2\,y}{x} \tag{1}$$

b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \cos x}{2 - \sin y} \tag{2}$$

c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\,x + y}{3 + 3\,y^2 - x}, \qquad y(0) = 0. \tag{3}$$

Intermediate

Problem 2. Compute the Wronskian of the following pairs of functions

a) $e^{3t}, t^2\,e^t$

b) $\sin x,\ \cos x^2$

c) $x,\ x \ln x$.

Problem 3. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\,x + y^2 + 2\,x\,y^2. \tag{4}$$

Advanced

Problem 4. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3\,x^2\,y + y^2}{2\,x^3 + 3\,x\,y}, \qquad y(1) = -2. \tag{5}$$

Challenge

Problem 5. Solve

$$x\,y' + y - y^2\,e^{2x} = 0, \qquad y(1) = 2. \tag{6}$$

Problem 6. Consider the homogeneous equation

$$y' = H(y/x). \tag{7}$$

Find an integrating factor.

Problem 7. Consider the Bernoulli equation

$$y' + p(x)\, y = g(x)\, y^n. \tag{8}$$

Find an integrating factor of it. (Hint: The procedure of solving it is as follows.

- Divide both sides by y^n:

$$y^{-n}\, y' + p(x)\, y^{1-n} = g(x). \tag{9}$$

- Let $v = y^{1-n}$. Use the fact $v' = (1-n)\, y^{-n}\, y'$ to get

$$\frac{1}{1-n}\, v' + p(x)\, v = g(x). \tag{10}$$

  This is a linear equation and can be solved.)

Problem 8. Show that the general second order equation $y'' = f(x, y, y')$ can be re-written into a system of two equations, each of first order:

$$\begin{aligned} u' &= A(x, u, v) \quad (11)\\ v' &= B(x, u, v) \quad (12) \end{aligned}$$

through appropriate definition of the new unknowns u, v. Then argue, without any rigorous justification, that such can be done for general differential equations of any order. Finally argue without proof that the method we used to prove the existence/uniqueness for $y' = f(x, y)$ also would give existence/uniqueness for general equations.

Problem 9. Give a rigorous proof of the following. If

$$|y(x) - z(x)| \leqslant M \int_{x_0}^{x} |y(\tau) - z(\tau)|\, \mathrm{d}\tau \tag{13}$$

then $y(x) - z(x) = 0$ for $|x - x_0| < M^{-1}$.

Problem 10. Consider the general first order equation (initial value problem)

$$y' = f(x, y), \qquad y(0) = y_0. \tag{14}$$

Try to prove existence of solution through the following strategy.

- Show that all the derivatives $y^{(n)}(0)$ can be represented using the values of f and its derivatives at the point $(0, y_0)$.

- Now write

$$y(x) = y_0 + y'(0)\, x + \frac{1}{2}\, y''(0)\, x^2 + \cdots \tag{15}$$

  and show convergence of the right hand side. What conditions on f will you need?

- Show that the sum $y(x)$ indeed solve the equation.

- Compare this approach with the Picard iteration proof in the textbook. Which one is better?

See Next Page for Answers

Answers

- Problem 1:

  a) $y = \frac{1}{5}\,x^3 + \frac{C}{x^2}$.

  b) $2\,y + \cos y - x - \sin x = C$.

  c) $x^2 + x\,y - 3\,y - y^3 = 0$.

- Problem 2:

  a) $2\,t\,e^{4t} - 2\,t^2\,e^{4t}$.

  b) $-2\,x\,(\sin x)\,(\sin x^2) - (\cos x^2)\,(\cos x)$.

  c) x.

- Problem 3:
  $\arctan y = x + x^2 + C$.

- Problem 4:
  $x^3\,y^2 + x\,y^3 = -4$.

- Problem 5:
  $\frac{1}{y} = -x \int_1^x \frac{e^{2s}}{s^2}\,\mathrm{d}s + \frac{x}{2}$.

- Problems 6 – 10 are either theoretical or of the “you know you are right when you are right”-type. Come discuss with me if you have questions.