Dec. 7, 2011

Math 334 Fall 2011 Homework 12 Solutions

Basic

Intermediate

Problem 1. Solve the following system

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{x} \tag{1}$$

Solution. We need to find all the eigenvalues and eigenvectors for the matrix $\left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{smallmatrix}\right)$.

First find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(1-\lambda) + 2 + 2 - 4(2-\lambda) - (1-\lambda) - (1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] + 4 - 4(2-\lambda) \\ &= (1-\lambda)[-3\lambda + \lambda^2] - 4(1-\lambda) \\ &= (1-\lambda)(\lambda^2 - 3\lambda - 4) \\ &= (1-\lambda)(\lambda - 4)(\lambda + 1). \end{aligned} \tag{2}$$

Therefore we have 3 eigenvalues: $1, 4, -1$.

- Eigenvectors corresponding to 1: Solve

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-1 & 1 & 2 \\ 1 & 2-1 & 1 \\ 2 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{3}$$

  The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \tag{4}$$

- Eigenvectors corresponding to 4: Solve

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

  The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{6}$$

- Eigenvectors corresponding to -1:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

  The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{8}$$

The general solution is then

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{9}$$

Problem 2. Solve the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

Solution. We first find the general solution, then use the initial values to fix the constants C_1, C_2, C_3.

First find the eigenvalues:

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} &= (1-\lambda)(2-\lambda)(3-\lambda) - 2 + 2(2-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)(2-\lambda)(3-\lambda). \end{aligned} \tag{11}$$

Thus the eigenvalues are $1, 2, 3$.

- Eigenvectors corresponding to 1: Solve

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{12}$$

  Use Gaussian elimination:

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

  So the eigenvectors are characterized by

$$x_2 + 2x_3 = 0, \ -x_1 = 0 \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{14}$$

- Eigenvectors corresponding to 2: Solve

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{15}$$

- Eigenvectors corresponding to 3: Solve

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}. \tag{16}$$

The general solution is then

$$C_1 e^t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}. \tag{17}$$

The initial condition gives

$$C_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \tag{18}$$

that is

$$\begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \tag{19}$$

Solving it using Gaussian elimination, we get

$$C_1 = 1, C_2 = 2, C_3 = 0. \tag{20}$$

So the final answer is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = e^t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{21}$$

Problem 3. Solve the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{22}$$

Solution. First solve

$$\det \begin{pmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda = -1 \pm i. \tag{23}$$

Next find the eigenvectors. As the eigenvalues form a pair of conjugate complex numbers, we only need to find the eigenvectors corresponding to one of them.

We find the eigenvectors for $-1+i$. Solve

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{25}$$

Now expand

$$e^{(-1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{-t} [\cos t + i \sin t] \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = e^{-t} \left[\cos t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + i\, e^{-t} \left[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]. \tag{26}$$

So the general solution is given by

$$C_1 e^{-t} \left[\cos t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + C_2 e^{-t} \left[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]. \tag{27}$$

Now apply the initial conditions:

$$C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow C_1 = 1, \quad C_2 = -1. \tag{28}$$

So finally the answer is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t - 3 \sin t \\ \cos t - \sin t \end{pmatrix} \tag{29}$$

Advanced

Problem 4. Find the fundamental matrix satisfying $\Phi(0) = I$ (In other words, compute e^{At})

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \boldsymbol{x}. \tag{30}$$

Solution. First find all eigenvalues:

$$0 = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{pmatrix} = -\lambda^3 - \lambda^2 + 4\lambda + 4 = (\lambda+1)(2-\lambda)(2+\lambda) \Longrightarrow \lambda_{1,2,3} = -1, 2, -2. \tag{31}$$

As we have three distinct eigenvalues, we know that the existence of 3 linearly independent eigenvectors is guaranteed.

- Eigenvectors for -1: Solve

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

  We obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix} = c \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}. \tag{33}$$

- Eigenvectors for 2: Solve
$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{34}$$
We obtain
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=x_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}. \tag{35}$$
- Eigenvectors for -2: Solve
$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{36}$$
We obtain
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ -\frac{5}{4} \\ -\frac{7}{4} \end{pmatrix}=c\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} \tag{37}$$

We have now
$$X=\begin{pmatrix} -3 & 0 & 4 \\ 4 & -1 & -5 \\ 2 & 1 & -7 \end{pmatrix}. \tag{38}$$

Now use Gaussian elimination to compute X^{-1}:
$$\begin{aligned}\begin{pmatrix} -3 & 0 & 4 & 1 & 0 & 0 \\ 4 & -1 & -5 & 0 & 1 & 0 \\ 2 & 1 & -7 & 0 & 0 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 4 & -1 & -5 & 0 & 1 & 0 \\ 2 & 1 & -7 & 0 & 0 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 2 & 1 & -7 & 0 & 0 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 1 & -\frac{13}{3} & \frac{2}{3} & 0 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{4}{3} & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{13}{12} & -\frac{1}{12} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}.\end{aligned} \tag{39}$$

So
$$X^{-1}=\begin{pmatrix} -1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{3}{2} & -\frac{13}{12} & -\frac{1}{12} \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}=-\frac{1}{12}\begin{pmatrix} 12 & 4 & 4 \\ 18 & 13 & 1 \\ 6 & 3 & 3 \end{pmatrix}. \tag{40}$$
We have
$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}=\begin{pmatrix} -3 & 0 & 4 \\ 4 & -1 & -5 \\ 2 & 1 & -7 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 2 & \\ & & -2 \end{pmatrix}\left[-\frac{1}{12}\begin{pmatrix} 12 & 4 & 4 \\ 18 & 13 & 1 \\ 6 & 3 & 3 \end{pmatrix}\right]. \tag{41}$$

Therefore

$$
\begin{aligned}
e^{At} &= \begin{pmatrix} -3 & 0 & 4 \\ 4 & -1 & -5 \\ 2 & 1 & -7 \end{pmatrix} \begin{pmatrix} e^{-t} & & \\ & e^{2t} & \\ & & e^{-2t} \end{pmatrix} \left[-\frac{1}{12} \begin{pmatrix} 12 & 4 & 4 \\ 18 & 13 & 1 \\ 6 & 3 & 3 \end{pmatrix} \right] \\
&= -\frac{1}{12} \begin{pmatrix} -36\,e^{-t} + 24\,e^{-2t} & -12\,e^{-t} + 12\,e^{-2t} & -12\,e^{-t} + 12\,e^{-2t} \\ 48\,e^{-t} - 18\,e^{2t} - 30\,e^{-2t} & 16\,e^{-t} - 13\,e^{2t} - 15\,e^{-2t} & 16\,e^{-t} - e^{2t} - 15\,e^{-2t} \\ 24\,e^{-t} + 18\,e^{2t} - 42\,e^{-2t} & 8\,e^{-t} + 13\,e^{2t} - 21\,e^{-2t} & 8\,e^{-t} + e^{2t} - 21\,e^{-2t} \end{pmatrix} \\
&= \begin{pmatrix} -3\,e^{-t} + 2\,e^{-2t} & e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -4\,e^{-t} + \frac{3}{2}\,e^{-t} + \frac{5}{2}\,e^{-2t} & -\frac{4}{3}\,e^{-t} + \frac{13}{12}\,e^{2t} + \frac{5}{4}\,e^{-2t} & -\frac{4}{3}\,e^{-t} + \frac{1}{12}\,e^{2t} + \frac{5}{4}\,e^{-2t} \\ -2\,e^{-t} - \frac{3}{2}\,e^{2t} + \frac{7}{2}\,e^{-2t} & -\frac{2}{3}\,e^{-t} - \frac{13}{12}\,e^{2t} + \frac{7}{4}\,e^{-2t} & -\frac{2}{3}\,e^{-t} - \frac{1}{12}\,e^{2t} + \frac{7}{4}\,e^{-2t} \end{pmatrix}
\end{aligned}
$$

Challenge

Problem 5. Find the solution of the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{42}$$

Solution. Solve

$$0 = \det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = \lambda^2 + 6\,\lambda + 9 \Longrightarrow \lambda_1 = \lambda_2 = -3. \tag{43}$$

So there is only one eigenvalue -3.

Next we find the eigenvector corresponding to -3. Solving

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{45}$$

As we have only one eigenvector, we have to go on solving (remember that we only need one solution here)

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix}. \tag{46}$$

Thus the second solution to the system is

$$e^{-3t} \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} + t\,e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{47}$$

Finally the general solution to the problem is given by

$$C_1\,e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left[e^{-3t} \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} + t\,e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \tag{48}$$

which can be simplified to

$$e^{-3t} \left[C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} + t\,C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]. \tag{49}$$

Now apply the initial conditions.

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} \Longrightarrow C_1 = -1, \quad C_2 = 4; \tag{50}$$

Substitute back into the formula for general solutions, we get the final answer:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-3t} \begin{pmatrix} 3 + 4\,t \\ 2 + 4\,t \end{pmatrix}. \tag{51}$$

Problem 6. Let A be an $n \times n$ matrix with all a_{ij}'s real. Let $\lambda = \alpha + \beta\,i$ be an eigenvalue, with $\boldsymbol{z} = \boldsymbol{x} + i\,\boldsymbol{y}$ as one of its corresponding eigenvectors. Show the following:

a) $\bar{\lambda} = \alpha - \beta\,i$ is also an eigenvalue, and $\boldsymbol{x} - i\,\boldsymbol{y}$ is one of its corresponding eigenvector.

b) The real vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent.

Proof.

a) Short proof:

$$A\,\boldsymbol{z} = \lambda\,\boldsymbol{z} \Longrightarrow \overline{A\,\boldsymbol{z}} = \overline{\lambda\,\boldsymbol{z}} \Longrightarrow \bar{A}\,\bar{z} = \bar{\lambda}\,\bar{z} \Longrightarrow A\,\bar{z} = \bar{\lambda}\,\bar{z}. \tag{52}$$

Long proof:

$$\begin{aligned} A\,(\boldsymbol{x} + i\,\boldsymbol{y}) = (\alpha + \beta\,i)\,(\boldsymbol{x} + i\,\boldsymbol{y}) &\Longrightarrow A\,\boldsymbol{x} + i\,A\,\boldsymbol{y} = (\alpha\,\boldsymbol{x} - \beta\,\boldsymbol{y}) + i\,(\alpha\,\boldsymbol{y} + \beta\,\boldsymbol{x}) \\ &\Longrightarrow A\,\boldsymbol{x} - i\,A\,\boldsymbol{y} = (\alpha\,\boldsymbol{x} - \beta\,\boldsymbol{y}) - i\,(\alpha\,\boldsymbol{y} + \beta\,\boldsymbol{x}) \\ &\Longrightarrow A\,(\boldsymbol{x} - i\,\boldsymbol{y}) = (\alpha - \beta\,i)\,(\boldsymbol{x} - i\,\boldsymbol{y}). \end{aligned} \tag{53}$$

b) Assume the contrary: there are constants a, b, not both zero, such that $a\,\boldsymbol{x} + b\,\boldsymbol{y} = \mathbf{0}$. Without loss of generality we can assume $\boldsymbol{x} = c\,\boldsymbol{y}$ for some constant c. This leads to

$$\boldsymbol{x} + i\,\boldsymbol{y} = (c + i)\,\boldsymbol{y}, \qquad (\boldsymbol{x} - i\,y) = (c - i)\,\boldsymbol{y} \tag{54}$$

and consequently

$$(c - i)\,(\boldsymbol{x} + i\,\boldsymbol{y}) - (c + i)\,(\boldsymbol{x} - i\,\boldsymbol{y}) = \mathbf{0} \tag{55}$$

that is $\boldsymbol{x} + i\,\boldsymbol{y}$ and $\boldsymbol{x} - i\,\boldsymbol{y}$ are linearly dependent.

However, as $\alpha + \beta\,i$ and $\alpha - \beta\,i$ are different eigenvalues, we know that their eigenvectors must be linearly independent.

Thus we reach a contradiction, which means the assumption we make at the beginning: $\boldsymbol{x}$, $\boldsymbol{y}$ linearly dependent, must be false. □

Problem 7. Consider the linear system with constant coefficients:

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}. \tag{56}$$

Assume that A has n distinct eigenvalues.

Try solve it using Laplace transform and reach the conclusion: The general solution takes the form

$$C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)} + \cdots + C_n\,e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)} \tag{57}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues and $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ are eigenvectors corresponding (respectively) to these eigenvalues.

Proof. Taking Laplace transform we obtain:

$$s\,X - \boldsymbol{x}(0) = A\,X \tag{58}$$

Here

$$X(s) := \begin{pmatrix} \mathcal{L}\{x_1\} \\ \vdots \\ \mathcal{L}\{x_n\} \end{pmatrix}. \tag{59}$$

This gives

$$(s\,I - A)\,X = \boldsymbol{x}(0) \Longrightarrow X(s) = (s\,I - A)^{-1}\,\boldsymbol{x}(0). \tag{60}$$

Now using Cramer's rule we have

$$(s\,I - A)^{-1} = \frac{B(s)}{\det\,(s\,I - A)} \tag{61}$$

where $B(s)$ is a certain matrix satisfying

$$(s\,I - A)\,B(s) = (\det\,(s\,I - A))\,I. \tag{62}$$

Substituting into the formula for $X(s)$ we get

$$X(s) = \frac{\boldsymbol{p}(s)}{\det\,(s\,I - A)} \tag{63}$$

where $\boldsymbol{p}(s) = B(s)\,\boldsymbol{x}(0)$ is an n-vector with each entry a polynomial of s of degree at most $n - 1$.

By assumption A has n distinct eigenvalues, therefore we can factorize

$$\det\,(s\,I - A) = (s - \lambda_1) \cdots (s - \lambda_n). \tag{64}$$

Applying the method of partial fraction, we have

$$X(s) = \frac{\boldsymbol{c}_1}{s - \lambda_1} + \cdots + \frac{\boldsymbol{c}_n}{s - \lambda_n}. \tag{65}$$

To decide $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ we return to

$$(s\,I - A)\,\boldsymbol{X}(s) = \boldsymbol{x}(0). \tag{66}$$

Multiply both sides by $(s - \lambda_1)$ we have

$$(s\,I - A)\left[\boldsymbol{c}_1 + (s - \lambda_1)\left(\frac{\boldsymbol{c}_2}{s - \lambda_2} + \cdots + \frac{\boldsymbol{c}_n}{s - \lambda_n}\right)\right] = (s - \lambda_1)\,\boldsymbol{x}(0). \tag{67}$$

Now set $s = \lambda_1$ we reach

$$(\lambda_1\,I - A)\,\boldsymbol{c}_1 = \mathbf{0}. \tag{68}$$

Therefore $\boldsymbol{c}_1$ is an eigenvector corresponding to λ_1 and can be written as

$$\boldsymbol{c}_1 = C_1\,\boldsymbol{x}_0^{(1)}. \tag{69}$$

By multiplying the equation by $s - \lambda_i$ for other i's we reach similar conclusions for $\boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$.

Now that we have established

$$X(s) = \frac{C_1}{s - \lambda_1}\,\boldsymbol{x}_0^{(1)} + \cdots + \frac{C_n}{s - \lambda_n}\,\boldsymbol{x}_0^{(n)}, \tag{70}$$

it is immediately that

$$\boldsymbol{x}(t) = C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)} + \cdots + C_n\,e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)}. \tag{71}$$

Thus ends the proof. □

Problem 8. Consider the linear system with variable coefficients

$$\dot{\boldsymbol{x}} = A(t)\,\boldsymbol{x}. \tag{72}$$

Explain why in general the solution is not given by $e^{\int_0^t A(s)\,\mathrm{d}s}\,\boldsymbol{x}_0$. In other words, if we let $X = e^{\int_0^t A(s)\,\mathrm{d}s}$, in general $\dot{X} \neq A\,X$. (Notice that this is in sharp contrast to the constant-coefficient case: $\dot{\boldsymbol{x}} = A\,\boldsymbol{x} \Longrightarrow \boldsymbol{x} = e^{\int_0^t A}\,\boldsymbol{x}_0$ and also the first order linear equation case: $\dot{x} = a(t)\,x \Longrightarrow x = e^{\int_0^t a(s)\,\mathrm{d}s}\,x_0$)

Solution. By definition we have

$$X := e^{\int_0^t A(s)\,\mathrm{d}s} = I + B + \frac{B^2}{2} + \cdots \tag{73}$$

where $B = \int_0^t A(s)\,\mathrm{d}s$. All we need to show is that in general $\dot{X} \neq A\,X$.

Compute

$$\dot{B} = A; \qquad \frac{\mathrm{d}}{\mathrm{d}t}(B^2) = \dot{B}\,B + B\,\dot{B} = A\,B + B\,A, \ldots \tag{74}$$

Note that $A\,B + B\,A \neq 2\,B\,A$ or $2\,A\,B$ because in general $\int_0^t A(s)\,\mathrm{d}s$ and $A(t)$ do not commute – For a fixed $A(t)$ we can make $A(s), s < t$ be any matrices totally unrelated to $A(t)$ so $\int_0^t A(s)\,\mathrm{d}s$, which is a "sum" of these matrices, clearly does not need to commute with $A(t)$.

Now it's clear that

$$\dot{X} = \dot{B} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B^2}{2}\right) + \cdots = A + \frac{A\,B + B\,A}{2} + \cdots \neq A + A\,B + \cdots = A\,X. \tag{75}$$