Dec. 5, 2011

MATH 334 FALL 2011 HOMEWORK 12

BASIC

INTERMEDIATE

Problem 1. Solve the following system

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{x} \tag{1}$$

Problem 2. Solve the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

Problem 3. Solve the following initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3}$$

ADVANCED

Problem 4. Find the fundamental matrix satisfying $\Phi(0) = I$ (In other words, compute e^{At})

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \boldsymbol{x}. \tag{4}$$

CHALLENGE

Problem 5. Find the solution of the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{5}$$

Problem 6. Let A be an $n \times n$ matrix with all a_{ij}'s real. Let $\lambda = \alpha + \beta i$ be an eigenvalue, with $\boldsymbol{z} = \boldsymbol{x} + i\,\boldsymbol{y}$ as one of its corresponding eigenvectors. Show the following:

a) $\bar{\lambda} = \alpha - \beta i$ is also an eigenvalue, and $\boldsymbol{x} - i\,\boldsymbol{y}$ is one of its corresponding eigenvector.

b) The real vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent.

Problem 7. Consider the linear system with constant coefficients:

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}. \tag{6}$$

Assume that A has n distinct eigenvalues.

Try solve it using Laplace transform and reach the conclusion: The general solution takes the form

$$C_1\, e^{\lambda_1 t}\, \boldsymbol{x}_0^{(1)} + \cdots + C_n\, e^{\lambda_n t}\, \boldsymbol{x}_0^{(n)} \tag{7}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues and $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ are eigenvectors corresponding (respectively) to these eigenvalues.

Problem 8. Consider the linear system with variable coefficients

$$\dot{\boldsymbol{x}} = A(t)\,\boldsymbol{x}. \tag{8}$$

Explain why in general the solution is not given by $e^{\int_0^t A(s)\,\mathrm{d}s}\,\boldsymbol{x}_0$. In other words, if we let $X = e^{\int_0^t A(s)\,\mathrm{d}s}$, in general $\dot{X} \neq A\,X$. (Notice that this is in sharp contrast to the constant-coefficient case: $\dot{\boldsymbol{x}} = A\,\boldsymbol{x} \Longrightarrow \boldsymbol{x} = e^{\int_0^t A}\,\boldsymbol{x}_0$ and also the first order linear equation case: $\dot{x} = a(t)\,x \Longrightarrow x = e^{\int_0^t a(s)\,\mathrm{d}s}\,x_0$)

- Problem 1: $C_1\,e^t\begin{pmatrix}1\\-2\\1\end{pmatrix}+C_2\,e^{4t}\begin{pmatrix}1\\1\\1\end{pmatrix}+C_3\,e^{-t}\begin{pmatrix}1\\0\\-1\end{pmatrix}$.
- Problem 2: $e^t\begin{pmatrix}0\\-2\\1\end{pmatrix}+2e^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix}$.
- Problem 3: $e^{-t}\begin{pmatrix}\cos t-3\sin t\\ \cos t-\sin t\end{pmatrix}$.
- Problem 4: $\begin{pmatrix}-3\,e^{-t}+2\,e^{-2t} & e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -4\,e^{-t}+\frac{3}{2}\,e^{-t}+\frac{5}{2}\,e^{-2t} & -\frac{4}{3}\,e^{-t}+\frac{13}{12}\,e^{2t}+\frac{5}{4}\,e^{-2t} & -\frac{4}{3}\,e^{-t}+\frac{1}{12}\,e^{2t}+\frac{5}{4}\,e^{-2t}\\ -2\,e^{-t}-\frac{3}{2}\,e^{2t}+\frac{7}{2}\,e^{-2t} & -\frac{2}{3}\,e^{-t}-\frac{13}{12}\,e^{2t}+\frac{7}{4}\,e^{-2t} & -\frac{2}{3}\,e^{-t}-\frac{1}{12}\,e^{2t}+\frac{7}{4}\,e^{-2t}\end{pmatrix}$
- Problem 5: $e^{-3t}\begin{pmatrix}3+4\,t\\2+4\,t\end{pmatrix}$.