Nov. 18, 2011

Math 334 Fall 2011 Homework 10

Basic

Problem 1. Express the following function using the unit step function. And sketch their graphs.

a) $g(t) = \begin{cases} 1 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$

b) $g(t) = \begin{cases} t & t < 1 \\ t^2 & 1 < t < 2 \\ t^3 & t > 2 \end{cases}$

Problem 2. Compute the following Laplace transforms:

a) $\mathcal{L}\{t\, u(t-2)\}$

b) $\mathcal{L}\left\{\cos 2t\, u\left(t - \frac{\pi}{8}\right) + (9\, t^2 + 2\, t - 1)\, u(t-2)\right\}$.

Problem 3. Compute $\mathcal{L}\{\cos(e^{t^2-1})\, \delta(t-1)\}$.

Intermediate

Problem 4. Find the inverse Laplace transform for the following functions.

a) $F(s) = \frac{2\,(s-1)\,e^{-2s}}{s^2 - 2\,s + 2}$.

b) $F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$.

Problem 5. Solve

$$y'' + y = g(t) = \begin{cases} t/2 & 0 \leqslant t < 6 \\ 3 & t \geqslant 6 \end{cases}, \qquad y(0) = 0, \quad y'(0) = 1. \tag{1}$$

Problem 6. Solve

$$y'' + y = \delta(t - 2\,\pi) \cos t, \qquad y(0) = 0, y'(0) = 1. \tag{2}$$

Advanced

Problem 7. Let f satisfy $f(t+T) = f(t)$ for all $t \geqslant 0$ and some fixed positive number T. Show that

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, \mathrm{d}t}{1 - e^{-sT}}. \tag{3}$$

Challenge

Problem 8. Let $f(t)$ be a bounded function (not necessarily continuous). Prove that its Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \tag{4}$$

is continuous at all $s > 0$.[1] Therefore usually there is no need to consider the inverse transform of functions with jumps.

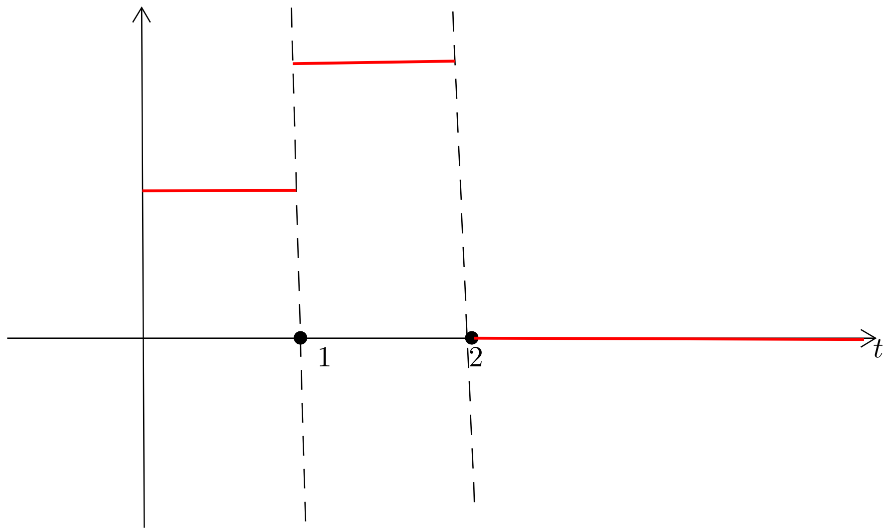
1. This can be replaced by "continuous in its domain", but it seems the proof will become much more technical.

Answers:

- Problem 1

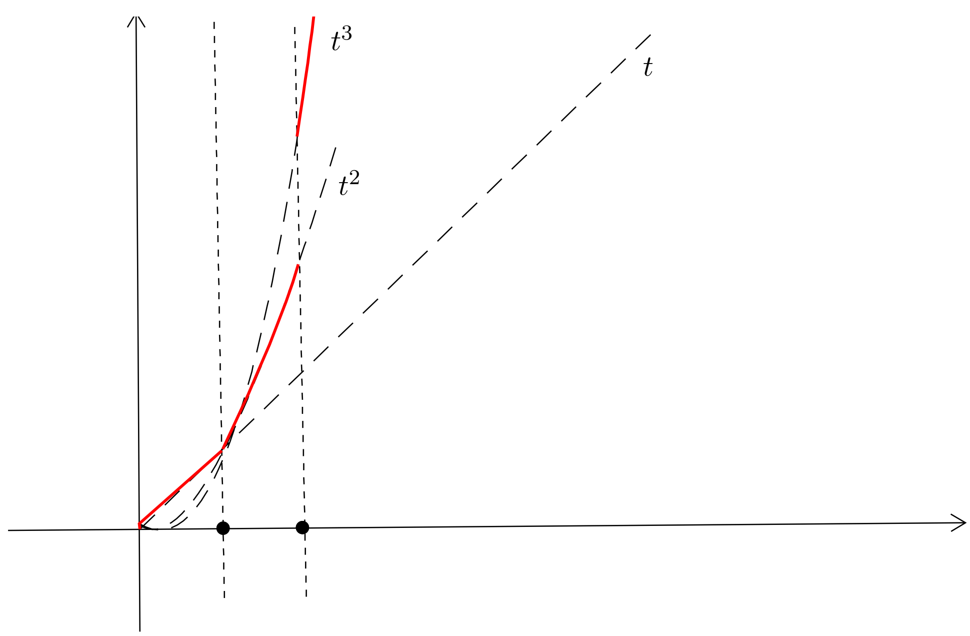
  a)

$$g(t) = 1 + (2-1)\,u(t-1) + (0-2)\,u(t-2) = 1 + u(t-1) - 2\,u(t-2). \tag{5}$$



  b)

$$g(t) = t + (t^2 - t)\,u(t-1) + (t^3 - t^2)\,u(t-2). \tag{6}$$



- Problem 2:

  a) $e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$

b) $\frac{\sqrt{2}}{2}\, e^{-\frac{\pi}{8}s}\, \frac{s-2}{s^2+4} + e^{-2s}\left(\frac{18}{s^3} + \frac{38}{s^2} + \frac{39}{s}\right)$

- Problem 3: $(\cos 1)\, e^{-s}$
- Problem 4:

  a) $2\, e^{t-2} \cos(t-2)\, u(t-2)$.

  b) $u(t-1) + u(t-2) - u(t-3) - u(t-4)$.

- Problem 5: $y = \frac{t}{2} + \frac{1}{2}\sin t - \left[\frac{t}{2} - 3 - \frac{1}{2}\sin(t-6)\right] u(t-6)$.
- Problem 6: $y = \sin t\, [1 + u(t - 2\pi)]$.