Sept. 16, 2011

Math 334 Fall 2011 Homework 1 Solutions

Basic

Problem 1. Go to http://www.math.rutgers.edu/~sontag/JODE/JOdeApplet.html, plot the slope fields of the following equations, and then imagine what the integral curves should look like.

a) $y' = 3\,x - 5\,y$;

b) $\dot{x} = (x - 2\,t)\,(x + t)$;

c) $\frac{\mathrm{d}y}{\mathrm{d}x} = \ln|x - y|$ (type "ln(abs(x-y))")

Problem 2. Check solutions.

a) $y = C_1\,e^{-2x} + C_2\,e^{x} + \sin 3\,x$ solves

$$y'' + y' - 2\,y = -11 \sin 3\,x + 3 \cos 3\,x. \tag{1}$$

b) $y = x^3$ solves

$$x^2\,y'' - x\,y' - 3\,y = 0. \tag{2}$$

Problem 3. Solve the following differential equations.

a) $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x \sin x$;

b) $\dot{y} = t \sin t$;

c) $3\,y^2\,y' = x^2$.

Solution.

a) Evaluate

$$\begin{aligned} \int e^x \sin x \,\mathrm{d}x &= \int \sin x \,\mathrm{d}e^x \\ &= e^x \sin x - \int e^x \,\mathrm{d}\sin x \\ &= e^x \sin x - \int \cos x \, e^x \,\mathrm{d}x \\ &= e^x \sin x - \int \cos x \,\mathrm{d}e^x \\ &= e^x \sin x - e^x \cos x + \int e^x \,\mathrm{d}\cos x \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \,\mathrm{d}x. \end{aligned} \tag{3}$$

We have obtained:

$$\int e^x \sin x \,\mathrm{d}x = e^x \sin x - e^x \cos x - \int e^x \sin x \,\mathrm{d}x \tag{4}$$

which means

$$\int e^x \sin x \,\mathrm{d}x = \frac{1}{2}\,(e^x \sin x - e^x \cos x) \tag{5}$$

and the solution is given by

$$y = \frac{1}{2}\,(e^x \sin x - e^x \cos x) + C. \tag{6}$$

b) Evaluate

$$\begin{aligned} \int t \sin t \,\mathrm{d}t &= \int t \,\mathrm{d}(-\cos t) \\ &= -t \cos t + \int \cos t \,\mathrm{d}t \\ &= -t \cos t + \sin t. \end{aligned} \tag{7}$$

So the solution is given by

$$y = -t \cos t + \sin t + C. \tag{8}$$

c) Notice that the $3\,y^2\,y' = (y^3)'$. So the equation becomes

$$(y^3)' = x^2 \Longrightarrow y^3 = \frac{1}{3}\,x^3 + C. \tag{9}$$

The formula is implicit.

INTERMEDIATE

Problem 4. Find the values of α such that $e^{\alpha x}$ solves

$$y'' + 2\,y' + 4\,y = 0. \tag{10}$$

Solution. Substitute $y = e^{\alpha x}$. We have

$$y' = \alpha\,e^{\alpha x}, \qquad y'' = \alpha^2\,e^{\alpha x}. \tag{11}$$

So the equation becomes

$$(\alpha^2 + 2\,\alpha + 4)\,e^{\alpha x} = 0 \tag{12}$$

which leads to

$$\alpha^2 + 2\,\alpha + 4 = 0. \tag{13}$$

The solutions are

$$\frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2} = -1 \pm \sqrt{3}\,i. \tag{14}$$

Problem 5. Find the values of r such that x^r solves

$$x^2\,y'' + 6\,x\,y' + 4\,y = 0. \tag{15}$$

Solution. Substitute $y = x^r$ we get

$$x^2\,r\,(r-1)\,x^{r-2} + 6\,x\,r\,x^{r-1} + 4\,x^r = 0 \tag{16}$$

which simplifies to

$$[r\,(r-1) + 6\,r + 4]\,x^r = 0 \tag{17}$$

so

$$r^2 + 5\,r + 4 = 0 \Longrightarrow r_1 = -4, r_2 = -1. \tag{18}$$

ADVANCED

CHALLENGE

ANSWERS

Problem 3:

a) $y = \frac{1}{2}\,(e^x \sin x - e^x \cos x) + C.$

b) $y = -t\cos t + \sin t + C.$

c) $y^3 = \frac{1}{3}\,x^3 + C.$

Problem 5: $-1 \pm \sqrt{3}\,i.$

Problem 6. $-4, -1.$