Math 334 Fall 2011 Review

Nov. 30, 2011

Disclaimer: Many minor issues or details (of formulas, say) are not included here, but they may still be involved in the final exam. Some topics have been covered in the review of the 2nd midterm and are thus just mentioned briefly.

You are strongly **discouraged** to use this review as a "formula sheet" and try to memorize everything (in fact many formulas are not here). Use it as a guide or a reminder instead.

1. First Order Equations

- Foundation: Exact equations. (Lecture 3)

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0. \tag{1}$$

  - To check:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

  - To solve: Find $u(x,y)$ such that $\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N$.

    1. $u(x,y) = \int M\,\mathrm{d}x + g(y)$ (or $\int N\,\mathrm{d}y + g(x)$)
    2. Determine g through

$$N(x,y) = \frac{\partial u}{\partial y}. \tag{3}$$

    3. General solution

$$u(x,y) = C. \tag{4}$$

    4. If initial value problem $y(x_0) = y_0$, then $C = u(x_0, y_0)$.

- Extensions.
  - Idea: Multiply the equation by a function (integrating factor) such that the resulting equation is exact.
  - Linear equation: (Lecture 4)

$$y' + p(x)\,y = g(x) \tag{5}$$

    Integrating factor: $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$.

  - Separable equation: (Lecture 5)

$$y' = p(y)\,g(x). \tag{6}$$

    Integrating factor: $\mu(y) = p(y)^{-1}$. Remember to check zeros when dividing!

  - Homogeneous equation: (Lecture 5)

$$y' = H(y/x). \tag{7}$$

    Becomes separable when introducing $v = y/x$.

  - General non-exact equation: (Lecture 6)

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{8}$$

    but

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \tag{9}$$

Find integrating factor by solving

$$M\,\frac{\partial\mu}{\partial y} - N\,\frac{\partial\mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu. \tag{10}$$

How to solve: Do the following one by one (in whatever order you see appropriate)

- Is there a solution depending on x only? (That is $\mu(x)$)
- Is there a solution depending on y only?
- Is there a solution depending on $x\,y$ only?
- and so on...

2. Second and Higher Order Equations

2.1. Homogeneous equations.

- Constant coefficients: Formulas (Lectures 9, 14);
- Variable coefficients (2nd order): Reduction of order. (Lecture 12)
  1. Guess y_1.
  2. $y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}$. (Remember to write equation into standard form $y'' + p(x)\,y' + q(x)\,y = 0$ first!)
- Variable coefficients (2nd and higher order): Power series method (Lectures 19, 20)

2.2. Nonhomogeneous equations.

- Constant coefficients:
  - Undetermined coefficients. (Lectures 10, 11, 15)
    - Only works for special right hand sides: $e^{at}\,P_n(t)$, $e^{\alpha t}\cos\beta t\,P_n(t) + e^{\alpha t}\sin(\beta t)\,Q_n(t)$, or sum of such terms.
  - Variation of parameters. (Lecture 12)
    - $y_p = u_1\,y_1 + u_2\,y_2$ with $u_1 = \int \frac{-y_2\,g}{W[y_1,\,y_2]}$, $u_2 = \int \frac{y_1\,g}{W[y_1,\,y_2]}$.
    - Remember:
      - Write the equation in standard form $y'' + p(x)\,y' + q(x)\,y = g(x)$ first.
    - Works for all right hand sides.
  - Power series method.
  - Laplace transform method.
    - Transforming simple functions and derivatives (Lecture 22)
    - Use inverse transform to obtain solution (Lectures 23, 24).
      - Partial fraction + table of Laplace transforms of simple functions;
      - Convolution formula

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f*g = \int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau. \tag{11}$$

        + table of Laplace transforms of simple functions.
- Variable coefficients
  - Variation of parameters. (Lecture 12)
    - May need to be combined with reduction of order.
  - Power series method (Lectures 17 – 20)

3. First Order Constant Coefficient Linear Homogeneous Systems

- Problem:

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x}, \tag{12}$$

possibly with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

- Formulas:
  - To solve (Lecture 33)

$$\dot{\boldsymbol{x}} = A\,\boldsymbol{x} \tag{13}$$

1. Solve

$$\det(A - \lambda I) = 0 \tag{14}$$

to obtain all the eigenvalues;

2. For each eigenvalue, find all corresponding eigenvectors, represented as

$$a\,\boldsymbol{x}_1 + b\,\boldsymbol{x}_2 + \cdots \tag{15}$$

with $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ linearly independent.

3. If overall we have n eigenvectors already then the general solution is

$$C_1\,e^{\lambda_1 t}\,\boldsymbol{x}_0^{(1)} + \cdots + C_n\,e^{\lambda_n t}\,\boldsymbol{x}_0^{(n)}. \tag{16}$$

where $\boldsymbol{x}_0^{(1)}, \ldots, \boldsymbol{x}_0^{(n)}$ are the n eigenvectors, and $\lambda_1, \ldots, \lambda_n$ (may or may not be distinct) are the corresponding eigenvalues.

   – In the case of complex eigenvalues, we have to do the following. Let $\lambda = \alpha + i\,\beta$ be a complex eigenvalue with a set of linearly independent eigenvectors $\boldsymbol{x}_1 + i\,\boldsymbol{y}_1$, $\boldsymbol{x}_2 + i\,\boldsymbol{y}_2, \ldots$. Then we have to replace the $e^{\lambda t}(\boldsymbol{x}_i + i\,\boldsymbol{y}_i)$ and $e^{\bar{\lambda}}(\boldsymbol{x}_i - i\,\boldsymbol{y}_i)$ terms in the general solution formula by

$$e^{\alpha t}\left[(\cos\beta t)\,\boldsymbol{x}_i - (\sin\beta t)\,\boldsymbol{y}_i\right] \text{ and } e^{\alpha t}\left[(\sin\beta t)\,\boldsymbol{x}_i + (\cos\beta t)\,\boldsymbol{y}_i\right] \tag{17}$$

  - What to do if we do not have enough eigenvectors: (Please see Lecture 34)
- Understanding the formulas through matrix exponential (Please see Lecture 35)

4. Separate Topics

- Existence and uniqueness of first order systems (Lecture 7, Lecture 30)
  - $\dot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x})$. $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$. Condition: all $\frac{\partial F_i}{\partial x_j}$ remain bounded in a region containing $(t_0, \boldsymbol{x}_0)$.
  - The above applies to all equations in this class:
    - $\dot{x} = F(t, x)$ is just the case $n = 1$;
    - Any single equation of order n can be re-written into a first order system with n equations.
- General solution for linear equations of order n:

$$y = C_1\,y_1 + \cdots + C_n\,y_n + y_p \tag{18}$$

- General solution for 1st order linear system consisting of n equations:

$$\boldsymbol{x} = C_1\,\boldsymbol{x}^{(1)} + \cdots + \boldsymbol{x}^{(n)} + \boldsymbol{x}_p. \tag{19}$$

- Wronskian for linear homogeneous system: (Lecture 8, Lecture 30)

$$W\left[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}\right] = \det\left(\begin{array}{ccc} \boldsymbol{x}^{(1)} & \cdots & \boldsymbol{x}^{(n)} \end{array}\right). \tag{20}$$

Special case (treating n-th order linear equation as a special case of 1st order linear system)

$$W[y_1, \ldots, y_n] = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}. \tag{21}$$

- ◦ Key property: Solutions form a fundamental set if and only if W is nonzero at t_0.

- Analytic functions. (Lecture 20)
  - ◦ f is analytic at $x_0 \Longleftrightarrow f(x) =$ Its Taylor expansion at x_0 over a small interval containing x_0.
- Radius of convergence. (Lecture 17)
- Ordinary, regular singular, and irregular singular points, Frobenius method (Lecture 20, 21)
- Euler equations (Lecture 20)
- Equations whose right hand sides involve jumps or δ's. (Lectures 26, 27, 28)
  - ◦ Laplace transform:

$$\mathcal{L}\{u(t-a)\, f(t)\} = e^{-as}\, \mathcal{L}\{f(t+a)\}; \qquad \mathcal{L}\{\delta(t-a)\, f(t)\} = f(a)\, e^{-as}. \tag{22}$$

  - ◦ Inverse transform of functions involving e^{-as}:

$$\mathcal{L}^{-1}\{e^{-as}\, F(s)\} = u(t-a)\, f(t-a). \tag{23}$$