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1. What is a differential equation?.

1.1. Definitions and examples.

A differential equation (DE) is an equation involving one or more unknown functions and their derivatives.

- An "ODE" (ordinary differential equation):

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x}+y(x)=0 \tag{1}$$

- An "PDE" (partial differential equation − involves partial derivatives!):

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0 \tag{2}$$

- An ODE with some generic "known" function:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x}+y(x)=f(x) \tag{3}$$

Note. From now on we will often use the shorthand y' for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Also, following tradition, when the variable is denoted t instead of x, $\dot{y}$ may be used instead of y'.

1.2. Classifications.

Differential equations is a vast field, and currently there is no hope of reaching a full understanding. Instead, a "greedy" type approach is used to grasp classes of equations here and there. Naturally, certain classifications appear.

Ordinary vs Partial.

One major classification is "ordinary differential equations"(ODE) vs "partial differential equations"(PDE), based on the number of variables involved, or equivalently whether "partial derivatives" appear in the equation or not.[1] For example,

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x}+y(x)=0 \tag{4}$$

is an ODE while

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0 \tag{5}$$

1. Not exactly "equivalent", but equivalent for all our purposes in this course.

is a PDE.

Note. From now on we will use "differential equations" to refer to "ordinary differential equations".

Scalar vs System.

This is a classification based on the number of unknown functions. When there is only one, the equation is a scalar equation; When there are more than one, the equation is a system. As any "nice" equation satisfies the following,

The number of unkonwn functions $=$ The number of equations[2]

we can tell by just looking at the number of equations.

Order.

Another important way of classifying differential equations is to look at the highest order of derivative on the unknown function in the equation, and count the number of derivatives. This number is called the "order" of the equation. For example, let's look at the ODE

$$x^2\, y'' - 3\, x\, y' + 3\, y = 0. \tag{6}$$

The equation involves the following derivatives of the unknown function y: y'', y', y. The highest order of derivative involved is then 2nd order. Consequently the order is 2.

We can also do the same to PDEs. Let's take the Monge-Ampere equation

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = f(x, y) \tag{7}$$

where u is the unknown function.[3] For this equation the following derivatives of the unknown function u are involved: $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\, \partial y}$. All of them are second order (for the last one, one derivative w.r.t. x and one derivative w.r.t. y – adds up to 2). Thus the order of the equation is 2.

Linear vs Nonlinear.

This is another important classification. Roughly speaking, a differential equation is linear if wherever an unknown quantity (the unknown function or its derivatives) appears, it appears alone – not in a product with another unknown term, not inside a exponential, etc. As an example we look at

$$x^2\, y'' - 3\, x\, y' + 3\, y = 0. \tag{8}$$

The unknown quantities are highlighted. Now we check each of them:

- y'' is in the product $x^2\, y''$. However x^2 is not "unknown", thus does not make the equation nonlinear;
- y' is in the product $-3\, x\, y'$, which is also OK;
- y is in the product $3\, y$, also OK.

As a matter of fact, we have

Theorem. *A linear ODE of order n is always of the form*

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_0(x)\, y = b(x). \tag{9}$$

Thus the above equation is clearly linear with $a_2 = x^2, a_1 = -3\, x, a_0 = 3, b = 0$.

On the other hand, we easily see that the Monge-Ampere equation

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = f(x, y) \tag{10}$$

2. The justification (aka proof) of this claim is way beyond what we can cover.

3. Those who are familiar with linear algebra and multi-variable calculus may recognize that the left hand side is just the determinant of the Hessian matrix: $\det(\nabla^2 u)$. This particular determinant is an important quantity in many fields, such as cosmology.

is nonlinear. As two "unknown terms" $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ is multiplied together.

Remark 1. Why do we single out the unknown quantities and discuss whether they appear in a linear or nonlinear manner? Why don't we say the equation is nonlinear when we see terms such as x^2 or $x\, y$? Aren't such products also "nonlinear"? The reason is the following.

> When the unknown quantities appear in a linear manner, the equation can be solved relatively easily: Figure out a few solutions[4] and we automatically obtain all solutions.

This property is not affected by the nonlinear existence of the variable, say terms like x^2. However it is lost when some unkonwn quantities are combined in a nonlinear manner.

For example, to solve the equation

$$y'' + 3\, y' + 5\, y = 7 \tag{11}$$

we only need to figure out one single solution as well as 2 solutions y_1, y_2 of the homogeneous counterpart

$$y'' + 3\, y' + 5\, y = 0, \tag{12}$$

and all solutions are included in the expression $c_1\, y_1 + c_2\, y_2 + y_p$ (c_1, c_2 are constants). If we change $3, 5, 7$ to functions of x, such as

$$y'' + e^{3x}\, y' + 5\, x^2\, y = 7 \sin x \tag{13}$$

Nothing is changed – it still holds that one solution of this equation and two of $y'' + e^{3x}\, y' + 5\, x^2\, y = 0$ are enough to generate all solutions.[5] The (very much) nonlinear terms $e^{3x}\, y'$ or $\sin x$ does not change anything. On the other hand, as soon as we introduce a little bit nonlinearity into the unknown quantities, say

$$y\, y'' + 3\, y' + 5\, y = 7 \tag{14}$$

The good property is lost.

Therefore, to make the adjective "nonlinear" informative, we restrict ourselves to nonlinearity regarding the unknown quantities only.

Constant coefficient vs Variable coefficient.

A linear[6] differential equation is a constant coefficient one if besides unknowns, only constants appear (the right hand side enjoys an exception). For example,

$$3\, y'' + 2\, y' - y = 3\, x \tag{15}$$

is constant coefficient while

$$x^2\, y'' - 3\, x\, y' + 3\, y = 0. \tag{16}$$

is not.

Homogeneous vs Nonhomogeneous.

A linear[7] differential equation is homogeneous if every term involves the unknown function. Otherwise it's nonhomogeneous. For example, the ODE

$$x^2\, y'' - 3\, x\, y' + 3\, y = 0. \tag{17}$$

is homogeneous while the Monge-Ampere equation

$$\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\, \partial y} \right)^2 = f(x, y) \tag{18}$$

is nonhomogeneous.

4. The exact number of solutions needed depends on the order and whether the equation is homogeneous or not.

5. Of course, finding these three guys becomes much harder now.

6. For nonlinear equations the difference between constant or variables coefficients becomes blurred. After all the unknown quantities can be viewed as "coefficients" and they are definitely "variable".

7. As the major advantage of homogeneous equations is that if y_1, y_2 are solutions, so is $a\, y_1 + b\, y_2$ where a, b are constants. Clearly for nonlinear equations, homogeneity becomes meaningless.

Rules of thumb regarding difficulty.

Easier	Harder
ODE	PDE
Scalar	System
Low order	High order
Linear	Nonlinear
Constant coefficient	Variable coefficient
Homogeneous	Nonhomogeneous

Table 1. Relative difficulty of differential equations

From this table we easily see that the simplest differential equation one can has is

ODE - scalar - 1st order - linear - constant coefficient - homogeneous

or in better wording: 1st order linear scalar homogeneous ODE with constant coefficients. As linear ODEs can be characterized as follows,

Theorem. *A linear ODE of order n is always of the form*

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_0(x)\, y = b(x). \tag{19}$$

we easily see that a general form of the "simplest" differential equations is

$$a\, y' + b\, y = 0. \tag{20}$$

We will indeed start from this problem after the introduction.

On the other hand, the hardest problem one can have is clearly: high order nonlinear nonhomogeneous PDE system with variable coefficients. Most equations arising from real world applications belongs here. For example the Navier-Stokes equation describing fluids, or the Einstein equation governing our universe.

2. Roadmap.

- We will start from first order equations, both linear and nonlinear. Two methods will be introduced, although they will be "unified" at the end of the day.
- We then move on to 2nd and higher order equations. It turns out that in this case we are no longer able to systematically solve nonlinear equations. So we focus on linear ones. Homogeneity becomes important here (It's not important for 1st order equations).
- First we study linear homogeneous equations.
    - Constant coefficients: All solutions can be obtained easily;
    - Variable coefficients: Good theory, but can only solve some of them through clever tricks;
- When the equation is non-homogeneous, we introduce the following systematic approaches.
    - Constant coefficients: Laplace transform;
    - Variable coefficients: Series method;

  There is also a method called "variation of parameters".
- All the previous (except for nonlinear 1st order equations) can be seen as special cases of 1st order linear ODE systems;[8]
- Finally, there are many equations we cannot solve analytically (that is writing down a formula for the solution). Nevertheless, much information can be gathered using qualitative theory.

3. Some jargons.

- General solution.

8. Illustrating "system > scalar" in difficulty!

When solving a differential equation, one or more indefinite integrations are usually performed. As a consequence the resulting solution will contain one or more constants, whose values are under no restriction (sometimes referred to as “arbitrary constants” to emphasize this point). Such solutions are called “general solutions”. As we will see, for linear equations the number of constants is the same as the order of the equation. A general solution summarizes infinitely many solutions. If we assign a value to each constant, we get a specific solution; If we assign a different set of values, we get a different solution, etc.

Remark 2. One thing that is quite confusing is: General solutions $\neq$ All solutions. In other words, there may be solutions that are not included in the general solution formula. For example, the general solution for $(y')^2 - x\, y' + y = 0$ is $y(x) = C\, x - C^2$, where C is our “arbitrary constant”, but $y(x) = x^2/4$ is also a solution.

Remark 3. When asked to find the “general solution”, you do not need to worry whether you have found all solutions or not.

- Initial value problem.

  As discussed above, there are usually “arbitrary constants” in the general solutions we obtained. These constants become fixed-value when we specify “initial values”, that is the value of the unknown function as well as an appropriate number of its derivatives at an “initial point”. For example, the general solution to

$$y'' + 3\, y + 7 = 5\, x \tag{21}$$

  will contain two constants, both values will be fixed by specifying initial values

$$y(0) = 1, \qquad y'(0) = 3. \tag{22}$$

  Of course other values will have the same effect.

- Boundary value problem.

  For 2nd order equations, there is another class of problem which specify values at two end points instead of one “initial point”. For example

$$y'' + 3\, y + 7 = 5\, x, \qquad y(0) = 1, \quad y(2) = 7. \tag{23}$$

  Note the equation is solved between 0 and 2. Such “boundary values” will also fix the values of the two constants. Boundary value problems are very important in both theory and practice as it is closely related to partial differential equations.