Solving Second Order Linear ODEs

Table of contents

1. Introduction: A few facts.

We make a few things clear in this short introduction.

1. We will discuss linear equations only, as nonlinear equations of order two and higher are too hard.[1]

2. Contrary to first order equations, there is no clever transformation that leads to direct integration. For example, there is no way to transform the equation

$$y'' - 3\,y' - 4\,y = f(x) \tag{3}$$

into the directly integrable form

$$Y'' = F(X) \tag{4}$$

through introducing new unknown Y and new variable X.

3. Linear homogeneous equations with constant coefficients are easy. Such equations of order 2 are very very easy. Such equations of order higher than 2 are reasonably easy. In one word, easy.

4. Linear nonhomogeneous equations with constant coefficients are conceptually still easy, but the calculation becomes complicated.

5. Linear equations with variable coefficients are hard. We will discuss them in the "Series method" section.

6. In theory, there is not much difference between 2nd order and higher order equations.

We will start from 2nd order constant coefficient equations.

2. Linear, constant-coefficient, homogeneous: exp is all we need .

That is, general solution to

$$a\,y'' + b\,y' + c\,y = 0 \tag{5}$$

where a, b, c are constants.

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1. A special class of 2nd order equation $y'' = f(y, y')$ – those that x does not appear explicitly – can be transformed to 1st order by setting $v = y'$. Then we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v; \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = f(y, v) \tag{1}$$

which leads to

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{v}{f(y, v)}. \tag{2}$$

Side: Exploring linearity.

The key property that enables us to solve a linear homogeneous equation is the following.

Proposition 1. *If $y_1, y_2, \ldots, y_k$ solves a linear homogeneous equation, so does $c_1\, y_1 + c_2\, y_2 + \cdots + c_k\, y_k$ for any constants $c_1, \ldots, c_k$.*

Thus there is possibility of representing a large amount (actually infinitely many – as can be easily seen) of solutions by only a few. Now we ask

1. Is it possible to represent all solution by the linear combinations of finitely many?
2. If yes, how many is needed?

The answers are: 1. Yes; 2. Same as the order of the equation.

Therefore, to find the general solution for

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \tag{6}$$

we only need to find two solutions y_1, y_2, and then the general solution is

$$c_1\, y_1 + c_2\, y_2. \tag{7}$$

Or in the general case

$$a_n(x)\, y^{(n)} + \cdots + a_0(x)\, y = 0 \tag{8}$$

we only need to find n solutions $y_1, \ldots, y_n$ and then write

$$c_1\, y_1 + \cdots + c_n\, y_n. \tag{9}$$

However, there is a catch.

The n solutions must be linearly independent.

Definition 2. (Linear dependence/independence) *k functions $y_1, \ldots, y_k$ are said to be linearly dependent if there are constants $c_1, \ldots, c_k$ such that*

$$c_1\, y_1(x) + \cdots + c_k\, y_k(x) = 0. \tag{10}$$

Otherwise it is said to be linearly independent.

Checking linear independence is in general no easy task. But our case is special: $y_1, \ldots, y_k$ are not just arbitrary functions, they are solutions to linear ODEs. As a consequence, there is a neat way of checking linear dependence/independence through some magical quantity called the "Wronskian". We will discuss the Wronskian in a few days.

Now back to solving

$$a\, y'' + b\, y' + c\, y = 0 \tag{11}$$

where a, b, c are constants. Recall that all we need to do is to find two solutions that are linearly independent. Note that **it doesn't matter how we obtain them**.

It turns out we will obtain the two solutions through considering solutions of the special form $y = e^{rx}$ for some constant r. Substituting this into the equation leads to

$$e^{rt}\left[a\, r^2 + b\, r + c\right] = 0. \tag{12}$$

Therefore, $y = e^{rt}$ solves the equation if and only if $a\, r^2 + b\, r + c = 0$, or r is either

$$r_1 = \frac{-b + \sqrt{b^2 - 4\, a\, c}}{2\, a}, \qquad \text{or} \quad r_2 = \frac{-b - \sqrt{b^2 - 4\, a\, c}}{2\, a}. \tag{13}$$

Recall from the theory of quadratic equations that there are three cases: Both r_1, r_2 real and $r_1 \neq r_2$; $r_1 = r_2$ real; r_1, r_2 complex. We discuss them one by one.

2.1. $r_1 \neq r_2$ real.

This is the easiest case. We have two solutions $e^{r_1 t}$ and $e^{r_2 t}$. If they are linearly independent, then all solutions to

$$a\, y'' + b\, y' + c\, y = 0 \tag{14}$$

can be written as

$$y = c_1\, e^{r_1 t} + c_2\, e^{r_2 t} \tag{15}$$

and the equation is solved.

But are they linearly independent? Yes they are. We can either use the to-be-introduced Wronskian or prove directly.[2]

Example 3. Solve

$$y'' + 5\, y' + 6\, y = 0. \tag{21}$$

Solution. Substituting $y = e^{r x}$ into the equation, we reach

$$r^2 + 5\, r + 6 = 0 \Longrightarrow r_1 = -3, r_2 = -2. \tag{22}$$

As a consequence, the general solution is given by

$$y = c_1\, e^{-3t} + c_2\, e^{-2t}. \tag{23}$$

2.2. $r_1 = r_2$ real.

In this case $e^{r_1 t} = e^{r_2 t}$, that is we can only find one solution of the form $e^{r t}$. We need another solution which is linearly independent. It turns out that this other solution can be taken to be $t\, e^{r t}$.[3] [4]It is easy to show that $e^{r t}$ and $t\, e^{r t}$ are linearly independent. Thus when $r_1 = r_2 = r$ real, the general solution to

$$a\, y'' + b\, y' + c\, y = 0 \tag{24}$$

is given by

$$y = c_1\, e^{r t} + c_2\, t\, e^{r t}. \tag{25}$$

Example 4. Solve

$$y'' + 8\, y' + 16\, y = 0. \tag{26}$$

Solution. Substituting $y = e^{r x}$ we easily see that

$$r^2 + 8\, r + 16 = 0 \Longrightarrow r_1 = r_2 = -4. \tag{27}$$

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2. Assume that there are constants C_1, C_2 such that

$$0 = C_1\, e^{r_1 t} + C_2\, e^{r_2 t}. \tag{16}$$

Differentiating with respect to t, we have

$$0 = C_1\, r_1\, e^{r_1 t} + C_2\, r_2\, e^{r_2 t}. \tag{17}$$

In other words, the constants C_1, C_2 solve the system

$$\begin{aligned} e^{r_1 t}\, x + e^{r_2 t}\, y &= 0 \qquad (18)\\ r_1\, e^{r_1 t}\, x + r_2\, e^{r_2 t}\, y &= 0. \qquad (19) \end{aligned}$$

Solving this system shows that the only solutions are $x = y = 0$. Therefore $C_1 = C_2 = 0$. We have shown that

$$\text{If } C_1, C_2 \text{ are such that } 0 = C_1\, e^{r_1 t} + C_2\, e^{r_2 t}, \text{then } C_1 = C_2 = 0. \tag{20}$$

This is exactly the linearly independence of $e^{r_1 t}$ and $e^{r_2 t}$.

3. Note that $t\, e^{r t} = \frac{\partial}{\partial r}(e^{r t})$. This is not a coincidence! Hint: In the case of $r_1 \neq r_2$, $e^{r_1 t}$, $e^{r_2 t}$ are solutions, so is $\frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}$.

4. Another way is to use "reduction of order". That is we look for the 2nd solution of the form $u(t)\, e^{r t}$. Then the equation for u will be just first order and can be easily solved, yielding $u(t) = a\, t + b$.

Thus the general solution is given by

$$y = c_1 e^{-4t} + c_2\, t\, e^{-4t}. \tag{28}$$

2.3. r_1, r_2 complex.

As

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{29}$$

with a, b, c real, the only possibility for r_1, r_2 to be complex is that

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta \tag{30}$$

where $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$ are both real.

In this case, what we have are two linearly independent complex solutions

$$e^{(\alpha + i\beta)t}, \qquad e^{(\alpha - i\beta)t}. \tag{31}$$

The task now is the produce two linearly independent **real** solutions out of them.

The idea is the use the properties of complex exponentials together with Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{32}$$

to obtain

$$e^{(\alpha + i\beta)t} = e^{\alpha t}\cos\beta t + i\, e^{\alpha t}\sin\beta t, \qquad e^{(\alpha - i\beta)t} = e^{\alpha t}\cos\beta t - i\, e^{\alpha t}\sin\beta t. \tag{33}$$

Recall that due to the linearity of the equation, if y_1, y_2 are solutions, so is $c_1 y_1 + c_2 y_2$ for any constants c_1, c_2. Inspecting the formulas for $e^{(\alpha \pm i\beta)t}$ we easily figure out that

$$e^{\alpha t}\cos\beta t = \frac{1}{2} e^{(\alpha + i\beta)t} + \frac{1}{2} e^{(\alpha - i\beta)t} \tag{34}$$

and

$$e^{\alpha t}\sin\beta t = \frac{1}{2} e^{(\alpha + i\beta)t} - \frac{1}{2} e^{(\alpha - i\beta)t} \tag{35}$$

are also solutions.

Now we show that they are linearly independent. Assume that C_1, C_2 are constants such that

$$C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t = 0. \tag{36}$$

This gives

$$\frac{C_1 + C_2}{2} e^{(\alpha + i\beta)t} + \frac{C_1 - C_2}{2} e^{(\alpha - i\beta)t} = 0. \tag{37}$$

As $e^{(\alpha \pm i\beta)t}$ are linearly independent, necessarily

$$\frac{C_1 + C_2}{2} = \frac{C_1 - C_2}{2} = 0 \tag{38}$$

from which $C_1 = C_2 = 0$ immediately follows.

Example 5. Solve

$$y'' + 9y = 0. \tag{39}$$

Solution. Substituting $y = e^{rt}$ into the equation, we reach

$$r^2 + 9 = 0 \Longrightarrow r_1 = 3i,\ \ r_2 = -3i. \tag{40}$$

Thus we obtain two complex solutions

$$e^{3it} = \cos 3t + i\sin 3t \quad \text{and} \quad e^{-3it} = \cos 3t - i\sin 3t. \tag{41}$$

From this we obtain two real solutions

$$\cos 3t, \sin 3t. \tag{42}$$

The general solution is then given by

$$y = c_1 \cos 3t + c_2 \sin 3t. \tag{43}$$

Side: Checking linear independence, the Wronskian.

Given two solutions y_1, y_2, we can compute their Wronskian, defined as the determinant:

$$W(y_1, y_2) := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{44}$$

Then

- y_1, y_2 are linearly dependent over an interval I if and only if $W(y_1, y_2) \equiv 0$ over I.[5]

When y_1, y_2 solves a 2nd order linear homogeneous ODE

$$y'' + p(t)\, y' + q(t)\, y = 0, \tag{45}$$

we have furthermore Abel's Theorem (assuming $p(t)$ is continuous over the interval I under consideration):

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp\left[-\int_{t_0}^{t} p(s)\, \mathrm{d}s\right] \tag{46}$$

which enables us to find out the value of the Wronskian at any point without actually solving the equation. Furthermore, Abel's Theorem implies

- Either $W(y_1, y_2) \equiv 0$ for all $x \in I$, or $W(y_1, y_2) \neq 0$ for any $x \in I$.

Finally, the above remains true for higher order equations, where we define the Wronskian as

$$W(y_1, \ldots, y_n) := \det\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \tag{47}$$

where n is the order of the equation.

2.4. More examples.

We first recall the procedure of solving linear 2nd order constant-coefficient equations:

$$a\, y'' + b\, y' + c\, y = 0. \tag{48}$$

1. Write down and solve the auxiliary equation

   $$a\, r^2 + b\, r + c = 0 \Longrightarrow r_1, r_2. \tag{49}$$

   Or equivalently, substitute $y = e^{rt}$ into the equation.

2. Three cases.

   a. $r_1 \neq r_2$, both real. General solution is given by

   $$y = c_1\, e^{r_1 t} + c_2\, e^{r_2 t}. \tag{50}$$

   b. $r_1 = r_2$ real. General solution is given by

   $$y = c_1\, e^{r_1 t} + c_2\, t\, e^{r_1 t}. \tag{51}$$

5. This is not true if y_1, y_2 are not solutions to the same linear ODE. That is, if y_1, y_2 are just two functions, $W(y_1, y_2) \equiv 0 \not\Longrightarrow y_1, y_2$ linearly dependent!

c. r_1, r_2 complex. In this case $r_1 = \alpha + i\,\beta, r_2 = \alpha - i\,\beta$ and the general solution is given by

$$y = c_1\, e^{\alpha t} \cos \beta\, t + c_2\, e^{\alpha t} \sin \beta\, t. \tag{52}$$

Now we look at more examples.

Example 6. Solve

$$y'' - y' - 11\, y = 0. \tag{53}$$

Solution. Try $y = e^{r t}$. We have

$$r^2 - r - 11 = 0 \Longrightarrow r_1 = \frac{1 + \sqrt{1 + 44}}{2} = \frac{1 + 3\sqrt{5}}{2}, \quad r_2 = \frac{1 - 3\sqrt{5}}{2}. \tag{54}$$

As a consequence, the general solution is given by

$$y = c_1\, e^{\frac{1 + 3\sqrt{5}}{2} t} + c_2\, e^{\frac{1 - 3\sqrt{5}}{2} t}. \tag{55}$$

Example 7. Solve

$$y'' + 2\, y' - 8\, y = 0; \quad y(0) = 3, \quad y'(0) = -12 \tag{56}$$

Solution. First we find the general solution. Substituting $y = e^{r t}$ leads to

$$r^2 + 2\, r - 8 = 0 \Longrightarrow r_1 = -4, r_2 = 2. \tag{57}$$

So the general solution is given by

$$y(t) = c_1\, e^{-4t} + c_2\, e^{2t}. \tag{58}$$

To determine the constants we use the initial values:

$$3 = y(0) = c_1 + c_2; \qquad -12 = y'(0) = -4\, c_1 + 2\, c_2. \tag{59}$$

Solving this linear system (with c_1, c_2 as unknowns) gives

$$c_1 = 3, \qquad c_2 = 0. \tag{60}$$

So finally the solution to the initial value problem is given by

$$y(t) = 3\, e^{-4t}. \tag{61}$$

Example 8. Solve

$$4\, w'' + 20\, w' + 25\, w = 0. \tag{62}$$

Solution. Substituting $w = e^{r t}$ leads to

$$4\, r^2 + 20\, r + 25 = 0 \Longrightarrow r_1 = r_2 = -\frac{5}{2}. \tag{63}$$

Thus the two linearly independent solutions are

$$e^{-\frac{5}{2} t} \text{ and } t\, e^{-\frac{5}{2} t}. \tag{64}$$

A a consequence, the general solution is

$$w(t) = c_1\, e^{-\frac{5}{2} t} + c_2\, t\, e^{-\frac{5}{2} t}. \tag{65}$$

Example 9. Solve

$$y'' + 2\, y' + y = 0; \quad y(0) = 1, \quad y'(0) = -3. \tag{66}$$

Solution. First we find the general solution. Substituting $y = e^{r t}$ leads to

$$r^2 + 2\, r + 1 = 0 \Longrightarrow r_1 = r_2 = -1. \tag{67}$$

Thus the general solution is given by

$$y(t)=c_1\,e^{-t}+c_2\,t\,e^{-t}. \tag{68}$$

Using the initial values:

$$1=y(0)=c_1; \qquad -3=y'(0)=-c_1+c_2. \tag{69}$$

This gives

$$c_1=1, \qquad c_2=-2. \tag{70}$$

Therefore the solution to the initial value problem is given by

$$y(t)=1\,e^{-t}-2\,t\,e^{-t}. \tag{71}$$

Example 10. Solve

$$y''+2\,y'+5\,y=0. \tag{72}$$

Solution. Substititing $y=e^{rt}$ leads to

$$r^2+2\,r+5=0 \Longrightarrow r_1=-1+2\,i, \quad r_2=-1-2\,i. \tag{73}$$

The two linearly independent complex solutions are then given by

$$e^{(-1+2i)t}=e^{-t}\,[\cos 2\,t+i\sin 2\,t], \qquad e^{-(-1-2i)t}=e^{-t}\,[\cos 2\,t-i\sin 2\,t]. \tag{74}$$

From this we can obtain the real solutions:

$$e^{-t}\cos 2\,t, \qquad e^{-t}\sin 2\,t. \tag{75}$$

Finally the general solution is given by

$$y(t)=c_1\,e^{-t}\cos 2\,t+c_2\,e^{-t}\sin 2\,t. \tag{76}$$

Example 11. Solve

$$y''+2\,y'+2\,y=0; \qquad y(0)=2, \quad y'(0)=1. \tag{77}$$

Solution. First we find the general solution. Substituting $y=e^{rt}$ gives

$$r^2+2\,r+2=0 \Longrightarrow r_1=-1+i, \quad r_2=-1-i. \tag{78}$$

The complex solutions are then

$$e^{(-1+i)t}, e^{(-1-i)t} \tag{79}$$

which yield the real solutions

$$e^{-t}\cos t, \quad e^{-t}\sin t. \tag{80}$$

So the general solution is given by

$$y(t)=c_1\,e^{-t}\cos t+c_2\,e^{-t}\sin t. \tag{81}$$

Now we use the initial values:

$$2=y(0)=c_1, \qquad 1=y'(0)=-c_1+c_2. \tag{82}$$

Solving this we obtain

$$c_1=2, \qquad c_2=3. \tag{83}$$

The solution to the initial value problem is then given by

$$y(t)=2\,e^{-t}\cos t+3\,e^{-t}\sin t. \tag{84}$$

2.5. Linear, homogeneous, constant coefficient equations of higher order.

The above theory and method naturally extends to linear homogeneous equations of higher orders:

$$a_n\,y^{(n)}+\cdots+a_1\,y'+a_0\,y=0 \tag{85}$$

We have

- If $y_1, \ldots, y_n$ are solutions that are linearly independent[6], then the general solution is given by
$$y = c_1\, y_1 + \cdots + c_n\, y_n. \tag{86}$$
- To find $y_1, \ldots, y_n$, we try $y = e^{r t}$ to reach
$$a_n\, r^n + \cdots + a_1\, r + a_0 = 0 \Longrightarrow r = r_1, \ldots, r_n \tag{87}$$
Then if $r_1, \ldots, r_n$ are all different and real,
$$e^{r_1 t}, \ldots, e^{r_n t} \tag{88}$$
are the n solutions we want.
- If some r_i's coincide, or some are complex, we use similar tricks. If r is a root of multiplicity m, then we put
$$e^{r t}, t\, e^{r t}, \ldots, t^{m-1}\, e^{r t} \tag{89}$$
in our list of linearly independent solutions. If $\alpha \pm i\, \beta$ are complex roots, then we put
$$e^{\alpha t} \cos \beta\, t, \quad e^{\alpha t} \sin \beta\, t \tag{90}$$
into our list of linearly independent solutions.[7]
- There is a new situation here: What if some r_i's are both multiple roots and complex? The trick is still the same. For example, suppose $\alpha + i\, \beta$ is a double root (note that $\alpha - i\, \beta$ must also be a double root). Then the four solutions corresponding to $\alpha \pm i\, \beta$ are:
$$e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta\, t, t\, e^{\alpha t} \cos \beta\, t, t\, e^{\alpha t} \sin \beta\, t. \tag{91}$$
This is a result of
$$e^{(\alpha \pm i \beta) t}, t\, e^{(\alpha \pm i \beta) t} \tag{92}$$
being solutions.

Example 12. Solve
$$y''' + y'' - 6\, y' + 4\, y = 0. \tag{93}$$
Solution. Substituting $y = e^{r t}$ gives
$$r^3 + r^2 - 6\, r + 4 = 0. \tag{94}$$
Inspection suggests $r = 1$ is a solution. Thus $r^3 + r^2 - 6\, r + 4 = (r - 1)\, [\cdots]$. Assuming $\cdots$ to be $a\, r^2 + b\, r + c$ and then solving a, b, c, we obtain
$$r^3 + r^2 - 6\, r + 4 = (r - 1)\left[r^2 + 2\, r - 4\right] = (r - 1)\left(r - \left(-1 + \sqrt{5}\right)\right)\left(r - \left(-1 - \sqrt{5}\right)\right). \tag{95}$$
Thus we have three different roots:
$$r_1 = 1, \qquad r_2 = -1 + \sqrt{5}, \qquad r_3 = -1 - \sqrt{5}. \tag{96}$$
The general solution is then given by
$$y(t) = c_1\, e^t + c_2\, e^{\left(-1 + \sqrt{5}\right) t} + c_3\, e^{\left(-1 - \sqrt{5}\right) t}. \tag{97}$$

Example 13. Solve
$$z''' + 2\, z'' - 4\, z' - 8\, z = 0. \tag{98}$$
Solution. Substituting $z = e^{r t}$ leads to
$$r^3 + 2\, r^2 - 4\, r - 8 = 0. \tag{99}$$

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6. Meaning: If constants $C_1, \ldots, C_n$ makes $C_1\, y_1 + \cdots + C_n\, y_n = 0$, then $C_1 = C_2 = \cdots = C_n = 0$.

7. It can be shown that if $\alpha + i\, \beta$ is a root, so is $\alpha - i\, \beta$. That is, complex roots have to appear in conjugate pairs (otherwise we are in trouble!).

Inspecting for a while we realize that

$$r^3+2\,r^2-4\,r-8=(r+2)\left[r^2-4\right]=(r+2)^2\,(r-2). \tag{100}$$

Thus the roots are

$$r_1=r_2=-2, \qquad r_3=2 \tag{101}$$

and the general solution is given by

$$z(t)=c_1\,e^{-2t}+c_2\,t\,e^{-2t}+c_3\,e^{2t}. \tag{102}$$

3. Nonhomogeneous equations – how to find the particular solution.

Having understood linear 2nd order constant-coefficient homogeneous equations, we turn to equations with nonzero right hand side:

$$a\,y''+b\,y'+c\,y=f(t). \tag{103}$$

Now the crucial observation is the following. Let y_p be a solution to this equation. Then for any other solution $\tilde{y}$, $y:=\tilde{y}-y_p$ solves

$$a\,y''+b\,y'+c\,y=0. \tag{104}$$

In other words, once we find one solution to the nonhomogeneous equation (this solution will be called "particular solution"), what remains is just solving the homogeneous equation.

More specifically, if $y_p(t)$ is a particular solution, y_1, y_2 are two linearly independent solutions to the homogeneous equation, then the general solution for the nonhomogeneous equation is given by

$$y=c_1\,y_1+c_2\,y_2+y_p \tag{105}$$

As we have already know how to solve the homogeneous equation, the question is

How to find the particular solution y_p?

It turns out that finding this one solution is quite difficult. There is a general method called "variation of parameters". However the calculation involved may get a bit messy in practice – in particular, it involves integration of possibly complicated functions. On the other hand, for several class of special $f(t)$'s, one can take "short-cuts" by cleverly guess the form of the solution. In the following we will first discuss these special cases, and then turn to the general method.

3.1. The method of undetermined coefficients.

The basic idea is as follows. Since functions like $e^{a t}$, $\{\sin\beta\,t, \cos\beta\,t\}$, $\{t^n\}$ are "closed" under differentiation, when such functions appear on the right hand side, we guess that there are solutions of similar form.

Example 14. Find a particular solution of

$$2\,z''+z=9\,e^{2t}. \tag{106}$$

Solution. We guess a solution of the form $z=A\,e^{2t}$. Substituting into the equation we have

$$2\,z''+z=8\,A\,e^{2t}+A\,e^{2t}=9\,A\,e^{2t}. \tag{107}$$

It is clear that taking $A=9/5$ leads to a particular solution

$$z(t)=e^{2t}. \tag{108}$$

Example 15. Find a particular solution of

$$2\,x'+x=3\,t^2. \tag{109}$$

Solution. As $(t^n)'=n\;t^{n-1}$, we would like to guess that there is a particular solution of the form $A\;t^n$. Substituting into the left hand side, we find

$$2\,x'+x=2\,A\,n\,t^{n-1}+A\,t^n. \tag{110}$$

This clearly cannot be equal to the right hand side. But this first try tells us two things:

1. To obtain $3\ t^2$, we cannot use a single power t^n. We have to use a polynomial $A_n\ t^n + A_{n-1}\, t^{n-1} + \cdots$.
2. The highest power on the left hand side is t^n, coming from x. It cannot be cancelled by any term coming from x'. As a consequence, $n = 2$.

Therefore we would like to try

$$z = A\, t^2 + B\, t + C. \tag{111}$$

Substituting into the equation, we have

$$2\, x' + x = 2\, [2\, A\, t + B] + A\, t^2 + B\, t + C = A\, t^2 + (4\, A + B)\, t + \big(2\, B + C\big). \tag{112}$$

Equating with the right hand side:

$$A\, t^2 + (4\, A + B)\, t + \big(2\, B + C\big) = 3\, t^2 \tag{113}$$

gives the following system:

$$\begin{aligned} A &= 3 && (114)\\ 4\, A + B &= 0 && (115)\\ 2\, B + C &= 0 && (116) \end{aligned}$$

whose solution is

$$A = 3, \quad B = -12, \quad C = 24. \tag{117}$$

As a consequence, the particular solution can be taken as

$$x_p(t) = 3\, t^2 - 12\, t + 24. \tag{118}$$

Example 16. Solve

$$y'' - y' + 9\, y = 3 \sin 3\, t. \tag{119}$$

Solution. Recall that differentiating sin one obtains cos, and differentiating cos one obtains sin. Thus the set $\{A \sin 3\, t + B \cos 3\, t\}$ is closed under differentiations.

Inspired by this, we guess

$$y = A \sin 3\, t + B \cos 3\, t. \tag{120}$$

Substituting into the left hand side, we have

$$\begin{aligned} y'' - y' + 9\, y &= [-9\, A \sin 3\, t - 9\, B \cos 3\, t] - [3\, A \cos 3\, t - 3\, B \sin 3\, t] + 9\, [A \sin 3\, t + B \cos 3\, t] \\ &= 3\, B \sin 3\, t - 3\, A \cos 3\, t. && (121) \end{aligned}$$

Equating with the right hand side:

$$3\, B \sin 3\, t - 3\, A \cos 3\, t = 3 \sin 3\, t \tag{122}$$

we obtain

$$B = 1, \qquad A = 0. \tag{123}$$

Thus the particular solution is given by

$$y_p(t) = \cos 3\, t. \tag{124}$$

Example 17. Find a particular solution for

$$y''(x) + y(x) = 2^x. \tag{125}$$

Solution. Writing $2^x = e^{(\ln 2) x}$, we naturally guess

$$y(x) = A\, e^{(\ln 2) x}. \tag{126}$$

Substituting into the equation, we reach

$$y'' + y = A\,(\ln 2)^2\, e^{(\ln 2)x} + A\, e^{(\ln 2)x} = A\left[(\ln 2)^2 + 1\right] 2^x. \tag{127}$$

Equating with the right hand side, we have

$$A = \left[(\ln 2)^2 + 1\right]^{-1} \tag{128}$$

which leads to

$$y_p(x) = \left[(\ln 2)^2 + 1\right]^{-1} e^{(\ln 2)x} = \left[(\ln 2)^2 + 1\right]^{-1} 2^x. \tag{129}$$

Thus we can find the general solution to equations with right hand side of the above forms.

Example. Find the general solution to

$$y'' - y = t. \tag{130}$$

Solution. Recall that all we need to do is to find one particular solution and the general solution to the corresponding homogeneous equation

$$y'' - y = 0. \tag{131}$$

- Finding one particular solution. Try $y = A\,t + B$. Substituting into the left hand side, we have

  $$y'' - y = 0 - (A\,t + B). \tag{132}$$

  Equating with the right hand side, we find out $A = -1$, $B = 0$. Thus $y_p = -t$ is a particular solution.

- Finding general solution to the homogeneous equation. Substituting $y = e^{rt}$ gives

  $$r^2 - 1 = 0 \Longrightarrow r_1 = 1, r_2 = -1. \tag{133}$$

  Thus the general solution is given by

  $$\tilde{y} = c_1\, e^t + c_2\, e^{-t}. \tag{134}$$

Putting everything together, we see that the general solution to the non-homogeneous equation is

$$y(t) = c_1\, e^t + c_2\, e^{-t} - t. \tag{135}$$

Example 18. Find the general solution to

$$2\,y'' + 3\,y' - 4\,y = 2\,t + \sin^2 t + 3. \tag{136}$$

Solution. The idea now is to use superposition principle, that is, if y_i solves

$$a\,y'' + b\,y' + c\,y = f_i, \tag{137}$$

then $c_1\, y_1 + \cdots + c_k\, y_k$ solves

$$a\,y'' + b\,y' + c\,y = c_1\, f_1 + c_2\, f_2 + \cdots + c_k\, f_k. \tag{138}$$

Guided by this, what we need to do (to solve the problem) are the following: First we simplify the right hand side using the trignometric formula

$$\cos 2\,t = 1 - 2\sin^2 t \Longrightarrow \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2\,t. \tag{139}$$

Thus the right hand side becomes

$$2\,t - \frac{1}{2}\cos 2\,t + \frac{7}{2}. \tag{140}$$

We need to

- Find a particular solution y_{p1} for

  $$2\,y'' + 3\,y' - 4\,y = 2\,t; \tag{141}$$

- Find a particular solution y_{p2} for
$$2\,y''+3\,y'-4\,y=-\frac{1}{2}\cos 2\,t; \tag{142}$$
- Find a particular solution y_{p3} for
$$2\,y''+3\,y'-4\,y=\frac{7}{2}; \tag{143}$$
- Find the general solution $\tilde{y}$ for
$$2\,y''+3\,y'-4\,y=0. \tag{144}$$

Then the general solution to the nonhomogeneous equation is given by
$$y=\tilde{y}+y_{p1}+y_{p2}+y_{p3}. \tag{145}$$
Now we carry this plan out.

- Find a particular solution y_{p1} for
$$2\,y''+3\,y'-4\,y=2\,t; \tag{146}$$
Based on the right hand side, we guess $y=A\,t+B$. Substituting into the equation we reach
$$-4\,A\,t+3\,A-4\,B=2\,t. \tag{147}$$
Thus
$$A=-\frac{1}{2},\quad B=-\frac{3}{8} \tag{148}$$
and
$$y_{p1}=-\frac{1}{2}\,t-\frac{3}{8}. \tag{149}$$
- Find a particular solution y_{p2} for
$$2\,y''+3\,y'-4\,y=-\frac{1}{2}\cos 2\,t; \tag{150}$$
We use the ansatz
$$y=A\cos 2\,t+B\sin 2\,t. \tag{151}$$
Then
$$\begin{aligned} 2\,y''+3\,y'-4\,y &= 2\,[-4\,A\cos 2\,t-4\,B\sin 2\,t]+3\,[-2\,A\sin 2\,t+2\,B\cos 2\,t] \\ &\quad -4\,[A\cos 2\,t+B\sin 2\,t] \\ &= (-12\,A+6\,B)\cos 2\,t+(-12\,B-6\,A)\sin 2\,t. \end{aligned} \tag{152}$$
Equating with the right hand side
$$-12\,A+6\,B=-\frac{1}{2},\quad -12\,B-6\,A=0. \tag{153}$$
This gives
$$A=\frac{1}{30},\quad B=-\frac{1}{60}. \tag{154}$$
Therefore we can take
$$y_{p2}=\frac{1}{30}\cos 2\,t-\frac{1}{60}\sin 2\,t. \tag{155}$$
- Find a particular solution y_{p3} for
$$2\,y''+3\,y'-4\,y=\frac{7}{2}; \tag{156}$$
As the right hand side is a constant, we try
$$y=A. \tag{157}$$
This quickly leads to
$$y_{p3}=-\frac{7}{8}. \tag{158}$$
- Find the general solution $\tilde{y}$ for
$$2\,y''+3\,y'-4\,y=0. \tag{159}$$

Substituting $y = e^{rt}$ leads to

$$2\,r^2 + 3\,r - 4 = 0 \Longrightarrow r_1 = \frac{-3 + \sqrt{41}}{4}, \quad r_2 = \frac{-3 - \sqrt{41}}{4}. \tag{160}$$

We have two distinct real roots, thus the general solution is given by

$$\tilde{y} = c_1\, e^{\frac{-3+\sqrt{41}}{4}t} + c_2\, e^{\frac{-3-\sqrt{41}}{4}}. \tag{161}$$

Putting everything together, the solution to the original problem is

$$y = \tilde{y} + y_{p1} + y_{p2} + y_{p3} = c_1\, e^{\frac{-3+\sqrt{41}}{4}t} + c_2\, e^{\frac{-3-\sqrt{41}}{4}} - \frac{1}{2}\,t + \frac{1}{30}\cos 2\,t - \frac{1}{60}\sin 2\,t - \frac{5}{4}. \tag{162}$$

From the above experience we see that, basically, when the right hand side is of the form $e^{a\,t}$, we guess the solution to be of the form $A\,e^{a\,t}$ (Note that constant right hand side belongs to this case); When the RHS is $\sin\beta\,t$ or $\cos\beta\,t$, we guess the solution to be $A\cos\beta\,t + B\sin\beta\,t$; When the RHS is of the form t^m, we guess the solution to be a polynomial of degree m, that is $A_m\,t^m + A_{m-1}\,t^{m-1} + \cdots + A_1\,t + A_0$.

However life is not so simple, as is shown by the following example.

Example 19. (4.4.17) Solve

$$y'' - 2\,y' + y = 8\,e^t. \tag{163}$$

Solution. As usual we try

$$y = A\,e^t. \tag{164}$$

Substituting into the equation, we have

$$y'' - 2\,y' + y = A\left[\left(e^t\right)'' - 2\,e^t + e^t\right] = 0 \tag{165}$$

which clearly cannot be made $8\,e^t$! Inspecting the above, we see that the problem is that e^t is a solution to the corresponding homogeneous equation, thus everything cancelled out and we have nothing left to adjust.

The idea now is to put factor of the form t^k in front of e^t. Let's see what is the appropriate k. Substituting $y = A\,t^k\,e^t$ we have

$$\begin{aligned} y'' - 2\,y' + y &= A\left[t^k\,e^t + 2\,k\,t^{k-1}\,e^t + k\,(k-1)\,t^{k-2}\,e^t\right] - 2\,A\left[t^k\,e^t + k\,t^{k-1}\,e^t\right] + A\,t^k\,e^t \\ &= A\,t^k\left[e^t - 2\,e^t + e^t\right] + A\,t^{k-1}\left[2\,k\,e^t - 2\,k\,e^t\right] + A\,k\,(k-1)\,t^{k-2}\,e^t \\ &= A\,k\,(k-1)\,t^{k-2}\,e^t. \end{aligned} \tag{166}$$

As the right hand side is $8\,e^t$, we see that the appropriate k is 2.

Substituting $y = A\,t^2\,e^t$ into the equation we reach

$$2\,A\,e^t = 8\,e^t \Longrightarrow A = 4. \tag{167}$$

So the desired particular solution is

$$y = 4\,t^2\,e^t. \tag{168}$$

Looking at the above example again, we see that e^t corresponds to $r = 1$, which is a double root of the auxiliary equation

$$r^2 - 2\,r + 1 = 0 \tag{169}$$

of the homogeneous equation

$$y'' - 2\,y' + y = 0. \tag{170}$$

That is, the multiplicity of the root $r = 1$ is 2. We notice that this is exactly the value of k we need to take in the ansatz $t^k\,e^t$. **This is no coincidence!**

Now we introduce the full version of the method.

Method of undetermined coefficients

To find a particular solution to the differential equation

$$a\,y'' + b\,y' + c\,y = P_m(t)\,e^{rt}, \tag{171}$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_m\,t^m + \cdots + A_1\,t + A_0)\,e^{rt}; \tag{172}$$

where

- r is not a root of the associated auxiliary equation $\Longrightarrow s = 0$;
- r is a simple root $\Longrightarrow$ $s = 1$;
- r is a double root $\Longrightarrow$ $s = 2$.

To find a particular solution to the differential equation

$$a\,y'' + b\,y' + c\,y = P_m(t)\,e^{\alpha t}\cos\beta\,t + Q_n(t)\,e^{\alpha t}\sin\beta\,t \tag{173}$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$\begin{aligned} y_p(t) &= t^s\big(A_k\,t^k + \cdots + A_1\,t + A_0\big)\,e^{\alpha t}\cos\beta\,t \\ &\quad + t^s\left(B_k\,t^k + \cdots + B_1\,t + B_0\right)e^{\alpha t}\sin\beta\,t. \end{aligned} \tag{174}$$

where k is the larger of m and n (that is $k = \max\{m, n\}$), and s is determined by

- $\alpha + \beta\,i$ is not a root of the associated auxiliary equation $\Longrightarrow$ $s = 0$;
- $\alpha + \beta\,i$ is a root $\Longrightarrow$ $s = 1$.[8]

Side: Making sense of this mess.

Somehow I feel the above is too mysterious, so I will try to make sense out of it. The following is what I would call "pseudo-math" in the sense of "pseudo-code" – looks like math argument but actually is not. Anyone who's satisfied with the above rules can skip the following.

In the method of undetermined coefficients, we guess the form of the particular solution involving a number of unknown constant coefficients, plug it into the equation, and try to determine the value of these constants by equating the "same terms" on the left and the right. Each term yields an equation that has to be satisfied.

To be able to carry this out successfully, we need

> The number of equations (for those unkonwn coefficients) is the same as the number of the unknown coefficients.

In particular, if the number of the unknown coefficients is less than the number of equations, we won't be able to find our particular solution.

Now let's count. Say let's have

$$y'' + 2\,y = e^t. \tag{175}$$

The right hand side is a single term e^t. So we expect one equation. Therefore we guess $y_p = A\,e^t$ and it works.

Of course things are not so simple. Consider

$$y'' + 2\,y = \sin t. \tag{176}$$

8. Think: What happens to the "double root" case?

The right hand side is a single term $\sin t$ and we guess $y_p = A \sin t$ and it won't work. Why? Because the real number of terms on the right hand side is not one, but two. Indeed, when counting the number of terms, we have to also

count the "hidden ones" that can be obtained through differentiation.

In the above example, $\sin t$ is the one that appears explicitly, but if we differentiate $\sin t$, we obtain $\cos t$, we have to count this one too. As differentiating $\cos t$ we obtain $\sin t$ again, the total number of terms on the right hand side is 2 and our y_p should be assumed as $y_p = A \cos t + B \sin t$.

Same philosophy applies to polynomial right hand sides:

$$y'' + y = t^{10}. \tag{177}$$

The right hand side is a polynomial, so we guess y_p is also a polynomial. But of what order? t^{10} is just one term, but differentiating it yields 10 "hidden terms" $t^9, t^8, \ldots, t, 1$. So there are 11 equations to be satisfied. We need 11 unknowns. So $y_p = a_{10}\, t^{10} + \cdots + a_1\, t + a_0$, exactly 11 unknowns.

Now consider the general case

$$a\, y'' + b\, y' + c\, y = P_m(t)\, e^{\alpha t} \cos \beta t + Q_n(t)\, e^{\alpha t} \sin \beta t. \tag{178}$$

We know

$$y_p = \left(A_k\, t^k + \cdots + A_1\, t + A_0\right) e^{\alpha t} \cos \beta t + \left(B_k\, t^k + \cdots + B_1\, t + B_0\right) e^{\alpha t} \sin \beta t. \tag{179}$$

But what is k? We see that the right hand side has $m + 1 + n + 1$ explicit terms. But (assuming $n > m$) differentiating the second term yields $n - m$ more "hidden" ones. So we expect a total of $2\,(n + 1)$ equations. So we should take k to be $n = \max\{m, n\}$.

Finally the real test. Consider

$$a\, y'' + b\, y' + c\, y = P_m(t)\, e^{r t} \tag{180}$$

with r a single root. We see that the right hand side contains $m + 1$ terms. So we need $m + 1$ unknowns. Guess

$$y_p = (A_m\, t^m + \cdots + A_1\, t + A_0)\, e^{r t} \tag{181}$$

But it won't work. Why? Because $A_m, \ldots, A_0$ are not all "effective" unknowns. The term $A_0\, e^{r t}$ does not survive the homogeneous equation and disappears. So effectively we have m unknowns $A_m, \ldots, A_1$ trying to satisfy $m + 1$ relations, which is bound to fail.

The fix? We add one term,

$$y_p = \left(A_{m+1}\, t^{m+1} + A_m\, t^m + \cdots + A_1\, t + A_0\right) e^{r t}. \tag{182}$$

As A_0 is useless anyway, we drop it, to write

$$y_p = t\, (A_{m+1}\, t^m + \cdots + A_1)\, e^{r t}. \tag{183}$$

Same idea applies to double roots.

This way of counting is very easy to extend to higher order equations. Consider

$$y^{(4)} + 2\, y'' + y = 3 \sin t - 5 \cos t. \tag{184}$$

Turns out the general solution to the homogeneous equation is

$$y = C_1 \sin t + C_2 \cos t + C_3\, t \sin t + C_4\, t \cos t. \tag{185}$$

Now following the form of the RHS we guess

$$y_p = \left(A_k\, t^k + \cdots + A_1\, t + A_0\right) \sin t + \left(B_k\, t^k + \cdots + B_1\, t + B_0\right) \cos t. \tag{186}$$

The RHS contains 2 terms. We need to balance it with two **effective** unknowns. Clearly A_1, A_0, B_1, B_0 are not effectively. Therefore the correct guess is

$$y_p = A\, t^2 \sin t + B\, t^2 \cos t. \tag{187}$$

Armed with the general procedure, we now study a few more examples.

Example 20. Find a particular solution to

$$y''(x) + y(x) = 4\,x\cos x. \tag{188}$$

Solution. First we need to figure out the correct form of the solution. Comparing with the general procedure above, we write the right hand side as

$$4\,x\,e^{0x}\cos x. \tag{189}$$

We see that the polynomial involved is x, which is of degree one. This suggests that a factor $(A_1\,x + A_0)$ is in the solution. Thus the solution is of the form

$$x^s\,(A_1\,x + A_0)\cos x + t^s\,(B_1\,x + B_0)\sin x. \tag{190}$$

To fix the power s, we need to check whether $0 + i$ is a root to the auxiliary equation

$$r^2 + 1 = 0. \tag{191}$$

Clearly the answer is yes. Therefore we need to take $s = 1$.

Substituting

$$y = x\,[(A_1\,x + A_0)\cos x + (B_1\,x + B_0)\sin x] \tag{192}$$

into the equation, we have

$$\begin{aligned} 4\,x\cos x &= y'' + y \\ &= \left[\left(A_1\,x^2 + A_0\,x\right)\cos x + \left(B_1\,x^2 + B_0\,x\right)\sin x\right]'' \\ &\quad + \left(A_1\,x^2 + A_0\,x\right)\cos x + \left(B_1\,x^2 + B_0\,x\right)\sin x \\ &= \left[2\,A_1\cos x - 2\,(2\,A_1\,x + A_0)\sin x - \left(A_1\,x^2 + A_0\,x\right)\cos x\right] \\ &\quad + \left[2\,B_1\sin x + 2\,(2\,B_1\,x + B_0)\cos x - \left(B_1\,x^2 + B_0\,x\right)\sin x\right] \\ &\quad + \left(A_1\,x^2 + A_0\,x\right)\cos x + \left(B_1\,x^2 + B_0\,x\right)\sin x \\ &= 4\,B_1\,x\cos x + (-4\,A_1)\,x\sin x + (2\,A_1 + 2\,B_0)\cos x + (2\,B_1 - 2\,A_0)\sin x. \end{aligned} \tag{193}$$

Thus

$$4\,B_1 = 4 \tag{194}$$

$$-4\,A_1 = 0 \tag{195}$$

$$2\,A_1 + 2\,B_0 = 0 \tag{196}$$

$$2\,B_1 - 2\,A_0 = 0. \tag{197}$$

Solving this gives

$$B_1 = 1,\;\; A_1 = 0,\;\; B_0 = 0,\;\; A_0 = 1. \tag{198}$$

So the particular solution is given by

$$y_p = x\,[\cos x + x\sin x]. \tag{199}$$

Example 21. Find a particular solution to

$$y'' - 2\,y' + 3\,y = \cosh t. \tag{200}$$

Solution. First recall that

$$\cosh t = \frac{1}{2}\left[e^t + e^{-t}\right]. \tag{201}$$

Thus to find the particular solution, we need to find a particular solution for

$$y'' - 2\,y' + 3\,y = \frac{1}{2}\,e^t \text{ and } y'' - 2\,y' + 3\,y = \frac{1}{2}\,e^{-t} \tag{202}$$

respectively.

- Particular solution for
$$y'' - 2\,y' + 3\,y = \frac{1}{2}\,e^t. \tag{203}$$
Guided by the method of undetermined coefficients, we look for solutions of the form
$$y = A\,t^s\,e^t. \tag{204}$$
To determine s, we need to check the relation between $r = 1$ and the auxiliary equation
$$r^2 - 2\,r + 3 = 0. \tag{205}$$
We see that $r = 1$ is not a solution. Thus $s = 0$. Substituting $y = A\,e^t$ into the equation we reach
$$\frac{1}{2}\,e^t = y'' - 2\,y' + 3\,y = 2\,A\,e^t \Longrightarrow A = \frac{1}{4} \Longrightarrow y_{p1} = \frac{1}{4}\,e^t. \tag{206}$$
- Particular solution for
$$y'' - 2\,y' + 3\,y = \frac{1}{2}\,e^{-t}. \tag{207}$$
We look for solutions of the form
$$y = A\,t^s\,e^{-t}. \tag{208}$$
This time $r = -1$ is also not a solution to the auxiliary equation
$$r^2 - 2\,r + 3 = 0 \tag{209}$$
and consequently $s = 0$. Substituting $y = A\,e^{-t}$ into the equation leads to
$$\frac{1}{2}\,e^{-t} = y'' - 2\,y' + 3\,y = 6\,A\,e^{-t} \Longrightarrow A = \frac{1}{12} \Longrightarrow y_{p2} = \frac{1}{12}\,e^{-t}. \tag{210}$$

Putting everything together, the particular solution is given by
$$y_p = \frac{1}{4}\,e^t + \frac{1}{12}\,e^{-t}. \tag{211}$$

The same idea works for higher order equations.

Example 22. Find a particular solution to
$$y''' + y'' - 2\,y = t\,e^t. \tag{212}$$

Solution. We look for solutions of the form
$$y = t^s\,[A_1\,t + A_0]\,e^t. \tag{213}$$
To determine s we need to check the relation between $r = 1$ and the auxiliary equation
$$r^3 + r^2 - 2 = 0 \Longleftrightarrow (r-1)\left(r^2 + 2\,r + 2\right) = 0. \tag{214}$$
We see that $r = 1$ is a simple root. Therefore $s = 1$. Substituting $y = t\,(A_1\,t + A_0)\,e^t$ into the equation we have
$$\begin{aligned} t\,e^t &= y''' + y'' - 2\,y \\ &= \left[\left(A_1\,t^2 + A_0\,t\right)e^t\right]''' + \left[\left(A_1\,t^2 + A_0\,t\right)e^t\right]'' - 2\left(A_1\,t^2 + A_0\,t\right)e^t \\ &= \left[A_1\,t^2 + (6\,A_1 + A_0)\,t + 6\,A_1 + 3\,A_0\right]e^t + \left[A_1\,t^2 + (4\,A_1 + A_0)\,t + 2\,A_1 + 2\,A_0\right]e^t \\ &\quad - 2\left(A_1\,t^2 + A_0\,t\right)e^t \\ &= 10\,A_1\,t\,e^t + (8\,A_1 + 5\,A_0)\,e^t. \end{aligned} \tag{215}$$

Thus
$$A_1 = \frac{1}{10}, \qquad A_0 = -\frac{4}{25}. \tag{216}$$

The special solution is then given by

$$y_p(t) = t\left[\frac{t}{10} - \frac{4}{25}\right]e^t = \left(\frac{t^2}{10} - \frac{4\,t}{25}\right)e^t. \tag{217}$$

3.2. Variation of parameters.

What if the right hand side is not a combination of powers, exponential, sin's and cos's? In this section we present a more general method: variation of parameters. This method yields a particular solution for any linear equation **as long as all solutions to the corresponding homogeneous equation are already known.**

Consider the nonhomogeneous problem

$$a\,y'' + b\,y' + c\,y = g(t). \tag{218}$$

As we can easily solve the homogeneous equation

$$a\,y'' + b\,y' + c\,y = 0, \tag{219}$$

we would like to check the possibility of obtaining the particular solution from modifying the general solutions of the homogeneous problem.

More specifically, let y_1, y_2 be two linearly independent solutions to the homogeneous problem. We know that the general solution of the homogeneous problem is given by

$$y = c_1\,y_1 + c_2\,y_2. \tag{220}$$

Now the idea is to replace the constants $c_{1,2}$ by two functions and try to find particular solution of the form

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t). \tag{221}$$

Substituting this into the nonhomogeneous equation, we obtain

$$\begin{aligned} g(t) &= a\,y'' + b\,y' + c\,y \\ &= a\,[v_1\,y_1 + v_2\,y_2]'' + b\,[v_1\,y_1 + v_2\,y_2]' + c\,[v_1\,y_1 + v_2\,y_2] \\ &= a\left[v_1''\,y_1 + 2\,v_1'\,y_1' + v_1\,y_1'' + v_2''\,y_2 + 2\,v_2'\,y_2' + v_2\,y_2''\right] \\ &\quad + b\left[v_1'\,y_1 + v_1\,y_1' + v_2'\,y_2 + v_2\,y_2'\right] + c\,[v_1\,y_1 + v_2\,y_2] \\ &= v_1\left[a\,y_1'' + b\,y_1' + c\,y_1\right] + v_2\left[a\,y_2'' + b\,y_2' + c\,y_2\right] \\ &\quad + a\left[v_1''\,y_1 + 2\,v_1'\,y_1' + v_2''\,y_2 + 2\,v_2'\,y_2'\right] + b\left[v_1'\,y_1 + v_2'\,y_2\right] \\ &= a\left[v_1''\,y_1 + 2\,v_1'\,y_1' + v_2''\,y_2 + 2\,v_2'\,y_2'\right] + b\left[v_1'\,y_1 + v_2'\,y_2\right]. \end{aligned} \tag{222}$$

Thus as long as we can find v_1, v_2 such that

$$a\left[v_1''\,y_1 + 2\,v_1'\,y_1' + v_2''\,y_2 + 2\,v_2'\,y_2'\right] + b\left[v_1'\,y_1 + v_2'\,y_2\right] = g(t), \tag{223}$$

we are done.

This equation is quite complicated. We need to simplify it. Note that we have two unknowns v_1, v_2 and just one equation, it is likely that we can require v_1, v_2 to satisfy another equation without losing existence of solutions.

One good choice is requiring

$$v_1'\,y_1 + v_2'\,y_2 = 0. \tag{224}$$

Note that this not only makes the term containing b vanish, but also greatly simplify the first term through

$$0 = \left(v_1'\,y_1 + v_2'\,y_2\right)' = v_1''\,y_1 + v_1'\,y_1' + v_2''\,y_2 + v_2'\,y_2'. \tag{225}$$

Thus our equations for v_1, v_2 are

$$a\left[v_1'\,y_1' + v_2'\,y_2'\right] = g(t) \tag{226}$$

$$v_1'\,y_1 + v_2'\,y_2 = 0. \tag{227}$$

Rewrite this system to

$$\begin{aligned} y_1' v_1' + y_2' v_2' &= g(t)/a \qquad (228) \\ y_1 v_1' + y_2 v_2' &= 0. \qquad (229) \end{aligned}$$

This system admits a unique solution when y_1, y_2 are linearly independent.[9] The solution is given by

$$v_1' = \frac{-g(t)\, y_2(t)}{a\,[y_1\, y_2' - y_1'\, y_2]}, \qquad v_2' = \frac{g(t)\, y_1(t)}{a\,[y_1\, y_2' - y_1'\, y_2]}. \qquad (234)$$

Then v_1, v_2 can be obtained from integrating the above.

Example 23. Find the general solution to the differential equation

$$y'' + 4\,y = \tan 2\,t. \qquad (235)$$

Solution. We need to do two things: finding general solution to the homogeneous equation, and finding one particular solution to the nonhomogeneous equation.

- General solution to the homogeneous problem. The auxiliary equation is

  $$r^2 + 4 = 0 \Longrightarrow r_1 = 2\,i, r_2 = -2\,i \Longrightarrow z_1 = e^{2it} = \cos 2\,t + i \sin 2\,t, z_2 = \cos 2\,t - i \sin 2\,t. \qquad (236)$$

  Thus the general solution to the homogeneous equation is given by

  $$\tilde{y} = c_1 \cos 2\,t + c_2 \sin 2\,t. \qquad (237)$$

- Particular solution to the nonhomogeneous problem. As the right hand side is $\tan 2\,t$, it is not possible to use the method of undetermined coefficients. We use variation of parameters instead. We have

  $$y_1 = \cos 2\,t, \qquad y_2 = \sin 2\,t, \qquad g(t) = \tan 2\,t = \frac{\sin 2\,t}{\cos 2\,t}, \qquad a = 1. \qquad (238)$$

  We compute

  $$y_1\, y_2' - y_1'\, y_2 = 2. \qquad (239)$$

  Thus the equations for v_1, v_2 are

  $$v_1' = \frac{-g(t)\, y_2(t)}{a\,[y_1\, y_2' - y_1'\, y_2]} = -\frac{1}{2}\frac{\sin^2 2\,t}{\cos 2\,t}, \qquad v_2' = \frac{g(t)\, y_1(t)}{a\,[y_1\, y_2' - y_1'\, y_2]} = \frac{1}{2}\sin 2\,t. \qquad (240)$$

  Integrating, we have

  $$\begin{aligned} v_1(t) &= -\frac{1}{2}\int \frac{\sin^2 2\,t}{\cos 2\,t}\,\mathrm{d}t \\ &= -\frac{1}{2}\int \frac{1-\cos^2 2\,t}{\cos 2\,t}\,\mathrm{d}t \\ &= -\frac{1}{2}\int \frac{1}{\cos 2\,t}\,\mathrm{d}t + \frac{1}{2}\int \cos 2\,t\,\mathrm{d}t. \qquad (241) \end{aligned}$$

9. Recall from linear algebra that the system

$$\begin{aligned} a\,x + b\,y &= e \qquad (230) \\ c\,x + d\,y &= f \qquad (231) \end{aligned}$$

has a unique solution for all e, f if and only if

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d - b\,c \neq 0. \qquad (232)$$

When this is the case, the unique solution is given by Cramer's rule

$$x = \frac{\det\begin{pmatrix} e & b \\ f & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}, \qquad y = \frac{\det\begin{pmatrix} a & e \\ c & f \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}. \qquad (233)$$

We evaluate

$$\frac{1}{2}\int \cos 2\,t\,\mathrm{d}t = \frac{1}{4}\sin 2\,t. \tag{242}$$

$$\begin{aligned} -\frac{1}{2}\int \frac{1}{\cos 2\,t}\,\mathrm{d}t &= -\frac{1}{2}\int \frac{\cos 2\,t}{\cos^2 2\,t}\,\mathrm{d}t \\ &= -\frac{1}{4}\int \frac{\mathrm{d}\sin 2\,t}{1-\sin^2 2\,t} \\ &= -\frac{1}{8}\left[\int \frac{\mathrm{d}\sin 2\,t}{1-\sin 2\,t} + \int \frac{\mathrm{d}\sin 2\,t}{1+\sin 2\,t}\right] \\ &= -\frac{1}{8}\left[\ln|1+\sin 2\,t| - \ln|1-\sin 2\,t|\right]. \end{aligned} \tag{243}$$

Thus

$$v_1(t) = \int \frac{\sin^2 2\,t}{\cos 2\,t}\,\mathrm{d}t = -\frac{1}{8}\left[\ln|1+\sin 2\,t| - \ln|1-\sin 2\,t|\right] + \frac{1}{4}\sin 2\,t. \tag{244}$$

On the other hand,

$$v_2(t) = \frac{1}{2}\int \sin 2\,t\,\mathrm{d}t = -\frac{1}{4}\cos 2\,t. \tag{245}$$

Putting things together, we have

$$\begin{aligned} y_p(t) &= v_1\,y_1 + v_2\,y_2 \\ &= \left[-\frac{1}{8}\left[\ln|1+\sin 2\,t| - \ln|1-\sin 2\,t|\right] + \frac{1}{4}\sin 2\,t\right]\cos 2\,t \\ &\quad -\frac{1}{4}\cos 2\,t\sin 2\,t \\ &= -\frac{1}{8}\left[\ln|1+\sin 2\,t| - \ln|1-\sin 2\,t|\right]\cos 2\,t. \end{aligned} \tag{246}$$

Thus the desired general solution is given by

$$y(t) = c_1\cos 2\,t + c_2\sin 2\,t - \frac{1}{8}\left[\ln|1+\sin 2\,t| - \ln|1-\sin 2\,t|\right]\cos 2\,t. \tag{247}$$

Example 24. Solve

$$2\,y'' - 2\,y' - 4\,y = 2\,e^{3t}. \tag{248}$$

Solution. Clearly we should first simplify the equation to

$$y'' - y' - 2\,y = e^{3t}. \tag{249}$$

- General solution to

  $$y'' - y' - 2\,y = 0. \tag{250}$$

  The auxiliary equation is

  $$r^2 - r - 2 = 0 \Longrightarrow r_1 = 2, \quad r_2 = -1. \tag{251}$$

  Thus the general solution is given by

  $$\tilde{y} = c_1\,e^{2t} + c_2\,e^{-t}. \tag{252}$$

- Finding one particular solution to

  $$y'' - y' - 2\,y = e^{3t}. \tag{253}$$

  We use variation of parameters. We have

  $$y_1 = e^{2t}, \quad y_2 = e^{-t}, \quad g = e^{3t}, \quad a = 1. \tag{254}$$

  We compute

  $$y_1\,y_2' - y_1'\,y_2 = -e^t - 2\,e^t = -3\,e^t. \tag{255}$$

Thus we have

$$v_1' = \frac{-g(t)\,y_2(t)}{a\,[y_1\,y_2' - y_1'\,y_2]} = \frac{1}{3}\,e^t, \qquad v_2' = \frac{g(t)\,y_1(t)}{a\,[y_1\,y_2' - y_1'\,y_2]} = -\frac{1}{3}\,e^{4t}. \tag{256}$$

Integrating, we have

$$v_1(t) = \frac{1}{3}\,e^t, \qquad v_2(t) = -\frac{1}{12}\,e^{4t}. \tag{257}$$

The particular solution is then given by

$$y_p(t) = v_1\,y_1 + v_2\,y_2 = \frac{1}{4}\,e^{3t}. \tag{258}$$

Finally, the general solution to the nonhomogeneous problem is

$$x(t) = c_1\,e^{2t} + c_2\,e^{-t} + \frac{1}{4}\,e^{3t}. \tag{259}$$

Remark 25. Clearly we can also find the particular solution to

$$x'' - x' - 2\,x = e^{3t}. \tag{260}$$

using the method of undetermined coefficients: Guess $x = A\,e^{3t}$ and sutstitute into the equation:

$$[9\,A - 3\,A - 2\,A]\,e^{3t} = e^{3t} \Longrightarrow A = \frac{1}{4} \Longrightarrow x_p = \frac{1}{4}\,e^{3t}. \tag{261}$$

Remark 26. As we can see from the above calculation, when the method of undetermined coefficients can be applied, usually it's a bit simpler than the method of variation of parameters[10]. On the other hand, variation of parameters can be applied to any right hand side. More importantly, it's theoretically much cleaner and can be generalized to the case of variable-coefficient equations.

	Undetermined Coefficients	Variation of Parameters
Applicable to	Some g	All g
Need to solve $a\,y'' + b\,y' + c\,y = 0$?	Sometimes	All times
Complexity of calculation	Low	High
Extension to variable coefficient equations?	No	Yes

Table 1. Solving $a\,y'' + b\,y' + c\,y = g(t)$, UC vs. VP.

Remark 27. A few more remarks.

1. Although variation of parameters can be extended to variable coefficient equations, in practice it is rarely used for the following reasons:
   - There is no good method getting formulas for solutions y_1, y_2 of the homogeneous equation $a(t)\,y'' + b(t)\,y' + c(t)\,y = 0$. Such formulas may not even exist;
   - The "series" method can get y_1, y_2 in the form of infinite sums. But they are hard to use in variation of parameters. Furthermore the same "series" method can get y_p directly without any more difficulty than getting y_1, y_2.
2. Back to constant coefficient equations. We will introduce a third method, the method of Laplace transform, which is in some sense in between the current two methods: Simpler calculation than variation of parameters, but more complicated than undetermined coefficients, etc.

4. Variable-coefficient equations.

The theory is still true: For an nth order linear equation

$$a_n(t)\,y^{(n)} + \cdots + a_0(t)\,y = g(t), \tag{262}$$

10. The higher the order, the simpler (compared to variation of parameters).

the following are true on intervals over which a_i/a_n, $i=0,\ldots,n-1$, are simultaneously continuous:

- if $g(t)=0$ (homogeneous), the general solution is
$$c_1\,y_1+\cdots+c_n\,y_n \tag{263}$$
where $y_1,\ldots,y_n$ are linearly independent solutions;
- Otherwise, the general solution is
$$c_1\,y_1+\cdots+c_n\,y_n+y_p \tag{264}$$
where y_p is a particular solution and $y_1,\ldots,y_n$ are as above.
- Linear dependence/independence can still be checked using Wronskian.
- The method of variation of parameters can be generalized to the variable-coefficient case almost without any modification. For example, consider the case of second order equation
$$a(t)\,y''+b(t)\,y'+c(t)\,y=g(t). \tag{265}$$
Let y_1, y_2 be two linearly independent solutions to the homogeneous equation, then $v_1\,y_1+v_2\,y_2$ is a particular solution, where
$$v_1'=\frac{-\,g\,y_2}{a\,W[y_1,y_2]}, \qquad v_2'=\frac{g\,y_1}{a\,W[y_1,y_2]}. \tag{266}$$

However, finding y_1, ..., y_n becomes much much harder. We have to restrict ourselves in special cases or rely on luck (combined with educated guesses whose accuracy depends on experience). Furthermore, for any important variable coefficient equations, the solutions cannot be expressed via classical functions (polynomial, trignometric, exponential, logarithm, etc.).

Example 28. (Cauchy-Euler equaitoin) Solve
$$t^2\,y''+7\,t\,y'-7\,y=0. \tag{267}$$

Solution. Guided by the theory, we only need to find two linearly independent solutions. The key now is to realize the following property of t^r: $(t^r)^{(k)}\,t^k=C\,t^r$. Substitute $y=t^r$ into the equation, we have
$$0=t^2\,y''+7\,t\,y'-7\,y=[r\,(r-1)+7\,r-7]\,t^r=0\Longleftrightarrow r^2+6\,r-7=0\Longrightarrow r_1=-7, r_2=1. \tag{268}$$
Thus the general solution is
$$y=c_1\,t^{-7}+c_2\,t. \tag{269}$$

One easily sees that this strategy would work for equations of the form
$$a\,t^2\,y''+b\,t\,y'+c\,y=0 \tag{270}$$
where a,b,c are constants. Such equations, together with the nonhomogeneous version
$$a\,t^2\,y''+b\,t\,y'+c\,y=g(t) \tag{271}$$
are called Cauchy-Euler, or Equidimensional, equations.

Consider the homogeneous Cauchy-Euler equation
$$a\,t^2\,y''+b\,t\,y'+c\,y=0 \tag{272}$$
Substituting $y=t^r$ leads to
$$a\,r\,(r-1)+b\,r+c=0. \tag{273}$$
This is called the associated *characteristic equation*.

As the characteristic equation is quadratic, we have three cases:

- Two distinct real roots r_1, r_2. In this case the two linearly independent solutions are
$$t^{r_1}, \quad t^{r_2}. \tag{274}$$

- A double root $r = r_1 = r_2$. As a, b, c are real, r is real. In this case t^r is one solution and we need another one. It turns out we can take the second one to be

$$y_2 = t^r \ln t. \tag{275}$$

- A pair of complex roots $r_{1,2} = \alpha \pm i\,\beta$. Then we have

$$t^{r_1} = e^{(\alpha + i\beta)\ln t} = e^{\alpha \ln t}\,[\cos(\beta \ln t) + i \sin(\beta \ln t)], \quad t^{r_2} = e^{\alpha \ln t}\,[\cos(\beta \ln t) - \sin(\beta \ln t)]. \tag{276}$$

  Using the linearity of the solution, we see that two linearly independent solutions are given by

$$y_1 = e^{\alpha \ln t} \cos(\beta \ln t) = t^\alpha \cos(\beta \ln t), \quad y_2 = t^\alpha \sin(\beta \ln t). \tag{277}$$

Remark 29. The above looks similar to our theory for linear constant-coeffcient equations. This similarity becomes more striking if we introduce a new variable $x = \ln t$: The three cases become

- Two distinct roots $\Longrightarrow$ $e^{r_1 x}, e^{r_2 x}$;
- One double root $\Longrightarrow$ $e^{r x}, x\, e^{r x}$;
- Complex roots $\Longrightarrow e^{\alpha x} \cos \beta x,\ e^{\alpha x} \sin \beta x$.

This is no coincidence! In fact, setting $x = \ln t$ gives

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = t^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}, \quad y' = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[t^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right]\frac{\mathrm{d}x}{\mathrm{d}t} = t^{-2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - t^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{278}$$

Substituting into the equation

$$0 = a\,t^2\,y'' + b\,t\,y' + c\,y = a\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (b - a)\,\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y. \tag{279}$$

Thus we have transformed the Euler-Cauchy equation into a constant-coefficient equaiton. Furthermore, the auxiliary equation for this equation is

$$a\,r^2 + (b - a)\,r + c = a\,r\,(r - 1) + b\,r + c \tag{280}$$

which is exactly the characteristic equation of the Cauchy-Euler equation!

Example 30. Solve the following initial value problem for the Cauchy-Euler equation

$$t^2\,y'' - 4\,t\,y' + 4\,y = 0; \qquad y(1) = -2, \quad y'(1) = -11. \tag{281}$$

Solution. Substituting $y = t^r$ gives

$$r\,(r - 1) - 4\,r + 4 = 0 \Longleftrightarrow r^2 - 5\,r + 4 = 0 \Longrightarrow r_1 = 4, \quad r_2 = 1. \tag{282}$$

Thus the general solution is given by

$$y(t) = c_1\,t^4 + c_2\,t. \tag{283}$$

Using the initial values, we have

$$-2 = y(1) = c_1 + c_2; \quad -11 = y'(1) = 4\,c_1 + c_2. \tag{284}$$

Solving this we reach

$$c_1 = -3, \quad c_2 = 1. \tag{285}$$

Thus the solution to the initial value problem is given by

$$y(t) = -3\,t^4 + t. \tag{286}$$

Example 31. solve

$$t^2\,y'' - 3\,t\,y' + 4\,y = 0. \tag{287}$$

Solution. Substituting $y = t^r$ we have

$$r\,(r - 1) - 3\,r + 4 = 0 \Longleftrightarrow r^2 - 4\,r + 4 = 0 \Longrightarrow r_1 = r_2 = 2. \tag{288}$$

This is double root so the general solution is given by

$$y = c_1\, t^2 + c_2\, t^2 \ln t. \tag{289}$$

Example 32. Solve

$$y'' - \frac{1}{t}\, y' + \frac{5}{t^2}\, y = 0. \tag{290}$$

Solution. Multiply both sides by t^2:

$$t^2\, y'' - t\, y' + 5\, y = 0. \tag{291}$$

Substituting $y = t^r$ gives

$$r\,(r-1) - r + 5 = 0 \Longleftrightarrow r^2 - 2\,r + 5 = 0 \Longrightarrow r_1 = 1 + 2\,i, r_2 = 1 - 2\,i. \tag{292}$$

The general solution is then given by

$$y = c_1\, t \cos(2 \ln t) + c_2\, t \sin(2 \ln t). \tag{293}$$

Cauchy-Euler equation is just one special class of equations. For most equations, however, we will not be able to obtain two linearly independent solutions from guessing. On the other hand, we may be able to find one single solution via guesswork. Fortunately, we can use this one solution as leverage and pull the following trick to obtain a 2nd solution which is linearly independent of the guessed one, and thus obtain the general solution.

Consider the equation

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{294}$$

Let y_1 be a solution. The idea is to find $v(t)$ such that $y_2 = v\, y_1$ is also a solution. Note that this clearly requires $y_1 \neq 0$, since otherwhse $y_2 = y_1 = 0$ no matter what v is. On the other hand, as long as v is not a constant, y_1 and y_2 are linearly independent.[11]

Substituting $y = v\, y_1$ into the equation we obtain

$$\begin{aligned} 0 &= y'' + p(t)\, y' + q(t)\, y \\ &= (v\, y_1)'' + p(t)\,(v\, y_1)' + q(t)\, v\, y_1 \\ &= v\, y_1'' + 2\, v'\, y_1' + v''\, y_1 + p\, v\, y_1' + p\, v'\, y_1 + q\, v\, y_1 \\ &= v\left[y_1'' + p\, y_1' + q\, y_1\right] + v''\, y_1 + 2\, v'\, y_1' + p\, v'\, y_1 \\ &= v''\, y_1 + \left(2\, y_1' + p\, y_1\right) v'. \end{aligned} \tag{295}$$

Thus all we need to do is to solve

$$y_1\, v'' + \left(2\, y_1' + p\, y_1\right) v' = 0. \tag{296}$$

This is a linear first order equation (after setting $w = v'$). Thus we should multiply both sides by

$$e^{\int (2 y_1' + p y_1)/y_1} = e^{2 \int \frac{y_1'}{y_1}}\, e^{\int p} = y_1^2\, e^{\int p} \tag{297}$$

to obtain

$$\left(y_1^2\, e^{\int p}\, v'\right)' = 0 \Longrightarrow y_1^2\, e^{\int p}\, v' = C \Longrightarrow v' = \frac{C\, e^{-\int p}}{y_1^2} \Longrightarrow v = C \int \frac{e^{-\int p(t) \mathrm{d}t}}{y_1(t)^2}\, \mathrm{d}t. \tag{298}$$

As all we need is v satisfying

$$y_1\, v'' + \left(2\, y_1' + p\, y_1\right) v' = 0. \tag{299}$$

we have the freedom to take any constant C. For example, we can take $C = 1$ to obtain the following theorem.

Theorem 33. (Reduction of order) *Consider the equation*

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{300}$$

11. Another linear algebra exercise!

Let y_1 be a solution, not identically zero. Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)\mathrm{d}t}}{y_1(t)^2} \mathrm{d}t \tag{301}$$

is a second, linearly independent solution.

Example 34. Solve

$$t\, y'' - (t+1)\ y' + y = t^2, \qquad t > 0. \tag{302}$$

Solution. Based on the above theory, we proceed as follows.

1. Find the general solution to the homogeneous equation

$$t\, y'' - (t+1)\ y' + y = 0, \qquad t > 0. \tag{303}$$

   This step consists of two substeps:

   a. Find one nonzero solution through guessing.

   b. Find a second one via reduction of order.

2. Find a particular solution to the nonhomogeneous equation using variation of parameters.

Now we carry this plan out.

1. Find the general solution to the homogeneous equation

$$t\, y'' - (t+1)\ y' + y = 0, \qquad t > 0. \tag{304}$$

   a. Find one nonzero solution through guessing. We notice that $t - (t+1) + 1 = 0$. Thus if we can find y such that $y'' = y' = y$, this y is a solution. Fortunately such y exists: $y = e^t$.

   b. Find a second one via reduction of order. We have found $y_1 = e^t$. Rewrite the equation into

$$y'' - \frac{t+1}{t}\, y' + \frac{1}{t}\, y = 0. \tag{305}$$

   Thus

$$p(t) = -\left(1 + \frac{1}{t}\right), \qquad q(t) = \frac{1}{t}. \tag{306}$$

   Calculate

$$\begin{aligned} y_2(t) &= y_1(t) \int \frac{e^{-\int p(t)\mathrm{d}t}}{y_1(t)^2} \mathrm{d}t \\ &= e^t \int \frac{e^{-\int\ -(1+1/t)}}{e^{2t}} \\ &= e^t \int \frac{e^t\, e^{\ln t}}{e^{2t}} \quad \text{(Note that as } t > 0, |t| = t) \\ &= e^t \int t\, e^{-t}\, \mathrm{d}t \\ &= e^t \left[-\int t\, \mathrm{d}e^{-t} \right] \\ &= e^t \left[-t\, e^{-t} + \int e^{-t}\, \mathrm{d}t \right] \\ &= -(t+1). \end{aligned} \tag{307}$$

   As the above calculation is quite complicated, it is a good idea to substitute $y = -(t+1)$ into the equation and see that it is indeed a solution.

   As the equation is linear, $-c\,(t+1)$ can also serve as y_2 for arbitrary constant $c \neq 0$. We take $c = -1$ to make things simple.

   Thus the general solution to the homogeneous equation is given by

$$y = c_1\, e^t + c_2\, (t+1). \tag{308}$$

2. Now we apply variation of parameters, that is, we try to find a particular solution to the nonhomogeneous equation

$$t\,y'' - (t+1)\ y' + y = t^2, \qquad t > 0. \tag{309}$$

of the form

$$y = v_1\,y_1 + v_2\,y_2. \tag{310}$$

Here $y_1 = e^t,\ y_2 = (t+1)$.

Compute the Wronskian

$$W[y_1, y_2](t) = y_1\,y_2' - y_1'\,y_2 = e^t - e^t\,(t+1) = -\,t\,e^t. \tag{311}$$

Then we have (Here $a_2 = t, g = t^2$)

$$v_1' = \frac{-\,g\,y_2}{a_2\,W[y_1, y_2]} = e^{-t}\,(t+1), \qquad v_2' = \frac{g\,y_1}{a_2\,W[y_1, y_2]} = -\,1. \tag{312}$$

Integrating, we obtain

$$\begin{aligned} v_1 &= \int\ (t+1)\,e^{-t}\,\mathrm{d}t \\ &= -\int\ (t+1)\,\mathrm{d}e^{-t} \\ &= -\,(t+1)\,e^{-t} + \int e^{-t}\,\mathrm{d}(t+1) \\ &= -\,(t+1)\,e^{-t} + \int e^{-t}\,\mathrm{d}t \\ &= -\,(t+2)\,e^{-t}. \end{aligned} \tag{313}$$

$$v_2 = \int\ -1 = -\,t. \tag{314}$$

Thus the particular solution is given by

$$y_p = v_1\,y_1 + v_2\,y_2 = -\,(t+2) + (-\,t)\,(t+1) = -\,t^2 - 2\,(t+1). \tag{315}$$

As $(t+1)$ is a solution to the homogeneous equation, we can simply take $y_p = -\,t^2$.

Summarizing, we have the general solution to be

$$y = c_1\,e^t + c_2\,(t+1) - t^2. \tag{316}$$

Note that even if we have used $y_p = -\,t^2 - 2\,(t+1)$, the term $-\,2\,(t+1)$ would be absorbed into $c_2\,(t+1)$ here.

Remark 35. (Thanks to Mr. Boblin Travis for showing me this) One can also start from $y_1 = (t+1)$. In that case we need to solve a formidable looking integral:

$$\begin{aligned} v &= \int\ \frac{t\,e^t}{(t+1)^2}\,\mathrm{d}t \\ &= -\int t\,e^t\,\mathrm{d}\frac{1}{t+1} \\ &= -\frac{t\,e^t}{t+1} + \int\ \frac{1}{t+1}\left(e^t\,\mathrm{d}t + t\,e^t\,\mathrm{d}t\right) \\ &= -\frac{t\,e^t}{t+1} + \int e^t \\ &= \frac{e^t}{t+1}. \end{aligned} \tag{317}$$

Thus we obtain $y_2 = v\,y_1 = e^t$.

Remark 36. We see some interesting symmetry here. If we start from $y_1 = e^t$, we obtain $y_2 = -(t+1)$. If we start from $y_1 = t+1$, we reach $y_2 = e^t$. This is not a coincidence. Indeed, suppose we start from a solution y_1, then use reduction of order to get $y_2 = v\, y_1$. Now if we instead start from y_2 and use reduction of order, we will always get $-y_1$. To see this, we investigate the formula

$$y_2 = y_1\, v = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{318}$$

Divide both sides by y_1 and then differentiate, we reach

$$\left(\frac{y_2}{y_1}\right)' = \frac{e^{-\int p}}{y_1^2} \Longleftrightarrow y_2'\, y_1 - y_1'\, y_2 = e^{-\int p}. \tag{319}$$

On the other hand, if we start from y_2 and use reduction of order, we would have

$$\tilde{y}_1 = y_2 \int \frac{e^{-\int p}}{y_2^2} \tag{320}$$

which is equivalent to

$$\tilde{y}_1'\, y_2 - y_2'\, \tilde{y}_1 = e^{-\int p}. \tag{321}$$

Thus y_1 and $\tilde{y}_1$ are related by the equation

$$\tilde{y}_1'\, y_2 - y_2'\, \tilde{y}_1 = -\left(y_1'\, y_2 - y_2'\, y_1\right). \tag{322}$$

Assuming the solution is unique, we see that it has to be $\tilde{y}_1 = -y_1$.

Remark 37. Let y_1 be a given solution. Let y be any other solution. Let $v = y/y_1$. Then v satisfies

$$v'' + \left(2\,\frac{y_1'}{y_1} + p\right) v' = 0. \tag{323}$$

The general solution to this equation is

$$v = C_2 \int \frac{e^{-\int p}}{y_1^2} + C_1. \tag{324}$$

Then we have

$$y = C_2\, y_1 \int \frac{e^{-\int p}}{y_1^2} + C_1\, y_1 = C_1\, y_1 + C_2\, y_2 \tag{325}$$

where

$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2} \tag{326}$$

is the solution we obtained through the reduction of order formula.