Solving First Order ODEs

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1. Introduction.

In this section we will investigate 1st order ODEs. By the end of the day, using our two main tools: integration factors and separation of variables, we will be able to solve

1. all 1st order linear equations;
2. a class of nonlinear equations called "exact equations";
3. several other classes of nonlinear equations through transforming them into the previous two.

We will also discuss various ways to extract useful information without the help of explicit formulas of the solutions.

The most general 1st order ODEs we will study are of the form

$$y' = f(x, y). \tag{1}$$

As f can be nonlinear in y (say $f(x, y) = x\,y^2$), we are not restricted to linear ODEs. However, it should be emphasized that (1) does not cover all nonlinear 1st order ODEs. One can see this, for example, by noticing that the dependence on y' is linear. To address this fact, (1) is often called "semilinear".

Remark 1. The reason for restricting to semilinear equations is that such equations have satisfactory theory guaranteeing the existence and uniqueness of solutions. More on this later.

Remark 2. One definition of semilinear equations goes as follows: A differential equation is semilinear if the order of any nonlinear term is lower than that of any linear term. For example, the equation

$$y'' + 3\,x\,y' = y^5 \tag{2}$$

is semilinear, as the nonlinear term is 0th order, while the two linear terms are 2nd and 1st respectively. As "order is power" in differential equations, linearity rules[1] in semilinear equations. As a consequence, semilinear equations usually have satisfactory theory.

Side: Two ways of writing a semilinear 1st order ODE.

There are two ways to write a semilinear 1st order ODE, they are slightly different. The first is of course either (1) or its equivalent form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y). \tag{3}$$

The second way of writing it is obtained through "multiplying both sides by $\mathrm{d}x$"[2]:

$$\mathrm{d}y = f(x, y)\,\mathrm{d}x \tag{4}$$

The difference is two-fold.

1. Philosophically, writing $\frac{\mathrm{d}y}{\mathrm{d}x} = f$ emphasizes that y is a function while x is a variable, while $\mathrm{d}y = f(x, y)\,\mathrm{d}x$ is more "symmetric" in the role of x and y. In fact the second form often appears as

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{5}$$

in which x and y are totally equal.

2. Practically, the second form can be seen as a "short-hand" of

$$M(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}t} + N(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{6}$$

and may have more solutions. To see this, note that any curve in the x-y plane may be a solution of (6) while onlly those curves that are graphs of a function $y = Y(x)$ are eligible for a solution of $y' = \frac{M(x, y)}{N(x, y)}$. For example, $x \equiv 0$ is a solution for $\frac{1}{x}\,\mathrm{d}x + \mathrm{d}y = 0$ but not a solution of $y' = -\frac{1}{x}$.

Despite the above, we will **not** emphasize the difference of the two forms, and treat $f(x, y)\,\mathrm{d}x + \mathrm{d}y = 0$ as the same as $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$.

2. Integrating factors and separation of variables.

There are two general methods for 1st order ODEs: Integrating factors and separation of variables. Both tries to transform the equation into a form that can be directly integrated. We illustrate them through solving the simplest differential equation possible.

Example 3. Solve

$$y' + 3\,y = 0. \tag{7}$$

Method 1. We realize that

$$\left(e^{3x}\,y\right)' = 3\,e^{3x}\,y + e^{3x}\,y' = e^{3x}\,(y' + 3\,y). \tag{8}$$

This inspires us to multiply the equation by e^{3x}:

$$\left(e^{3x}\,y\right)' = e^{3x}\,(y' + 3\,y) = 0 \Longrightarrow e^{3x}\,y = C \tag{9}$$

where C is an arbitrary constant. Now the solution is

$$y(x) = C\,e^{-3x}. \tag{10}$$

Remark. The key to our success is finding an appropriate function such that, when multiplied to the equation, turns the left hand side into an exact derivative. Such a function is called an "integrating factor". We will see later that, for first order linear ODEs, there is a systematic way of finding it.

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1. Unfortunately, only in theory (existence, uniqueness, etc.), not in practice (finding explicit formulas for solutions).
2. A purely formal operation right now, will be explained more later.

Method 2. We move $3\,y$ to the other side, then divide both sides by y (Remember that we need to check whether $y=0$ is a solution!):

$$y' = -3\,y \Longrightarrow \frac{y'}{y} = -3. \tag{11}$$

Integrating, we reach

$$\ln|y| = -3\,x + C \text{ (Don't forget the arbitrary constant!)} \tag{12}$$

This gives

$$|y| = e^C e^{-3x}. \tag{13}$$

To remove the absolute value we need to write things case by case:

$$y = \begin{cases} e^C e^{-3x} & y > 0 \\ -e^C e^{-3x} & y < 0 \end{cases}. \tag{14}$$

But as C is arbitrary, e^C and $-e^C$ together can cover every real number except 0. So the above is equivalent to the compact form

$$y = C\,e^{-3x}, \qquad C \neq 0. \tag{15}$$

Now we need to check $y=0$. We substitute $y=0$ into the equation and see that it is indeed a solution. So the solutions to the equation should be written as

$$y = C\,e^{-3x}, \quad C \neq 0 \qquad \text{and} \qquad y = 0. \tag{16}$$

But luckily $y=0$ is exactly the function obtained from $C\,e^{-3x}$ by setting $C=0$. So finally we can write the solution in one single formula:

$$y = C\,e^{-3x}. \tag{17}$$

Remark. The key to the success of our second approach is that we can make one side of the equation involving the unknown y and its derivatives only, while everything on the other side is known. Not all equations can be manipulated to reach this state. Those can are called "separable". We will discuss separable equations soon.

Remark 4. We see that one arbitrary constant is involved in the solution. This is universal in first order linear ODEs. In general, the number of constants involved is the same as the order of the equation.

Side: One must be careful when multiplying an ODE by a function.

In the above example, we turn the equation into more convenient forms by multiplying both sides with certain functions (in the first approach e^{3x}, in the second $1/y$). However, one should be careful that during such manipulation, solutions may be lost or gained. For example, consider the two equations $y' = y$ and $\frac{y'}{y} = 1$. The first can be obtained from the second by multiplying y, while the second from the first by multiplying $1/y$. We notice that $y=0$ is a solution to the first but not a solution to the second.

Therefore whenever we multiply an ODE by a function $\mu(x, y)$, we have to

1. Check all $y_1(x)$, $y_2(x)$, ... satisfying $\mu(x, y(x)) = 0$. These are clearly solutions to the new equation but may not solve the original one.
2. Check all $y_1(x)$, $y_2(x)$, ... satisfying $\mu(x, y(x)) = \infty$. These are not solutions to the new equation but may be solutions to the original equation.

Side: The road ahead.

- The method of integrating factors naturally applies to linear equations:

$$a_1(x)\,y' + a_0(x)\,y = b(x). \tag{18}$$

- The method of separation of variables naturally applies to equations of the form (called "separable equations")

$$y' = g(x)\,p(y). \tag{19}$$

Note that such equations can be nonlinear.

- Separable equations turns out to be a special case of so-called exact equations.
- Some non-exact equations can be transformed either into exact equations or into linear equations.

3. Linear 1st order equations.

We start by a very simple example.

Example 5. Solve

$$y' + 3\,y = 5. \tag{20}$$

Solution. The only difference with the equation we have just solved is that the right hand side is 5 instead of 0. We try the same strategy.

$$\left(e^{3x}\,y\right)' = e^{3x}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + 3\,y\right] = 5\,e^{3x}. \tag{21}$$

Integrating, we have

$$e^{3x}\,y = \int 5\,e^{3x} = \frac{5}{3}\,e^{3x} + C \tag{22}$$

which leads to

$$y = \frac{5}{3} + C\,e^{-3x}. \tag{23}$$

Finally we check that, multiplying e^{ax} will not add or lose any solution.

In the above example, the key trick is to multiply the equation by an appropriate function. It turns out that the same idea still works for general linear 1st order ODEs – not necessarily with constant coefficients. Let's look at an example.

Example 6. Solve

$$y' + 3\,x\,y = 5\,x. \tag{24}$$

Solution. We try to find a function $G(x)$ such that

$$G(x)\,(y' + 3\,x\,y) = (G(x)\,y)'. \tag{25}$$

As the right hand side can be expanded to

$$(G(x)\,y)' = G(x)\,y' + G(x)'\,y. \tag{26}$$

Thus we need this particular G to satisfy

$$3\,x\,G = G'. \tag{27}$$

Dividing both sides by G we reach

$$3\,x = (\ln|G|)' \Longrightarrow |G| = e^{C}\,e^{\frac{3}{2}x^2}. \tag{28}$$

Since we only need one such G, we can simply take $G(x) = e^{\frac{3}{2}x^2}$. Multiply our equation by this G, we have

$$\left(e^{\frac{3}{2}x^2}\,y\right)' = 5\,x\,e^{\frac{3}{2}x^2}. \tag{29}$$

Integrating, we obtain

$$\begin{aligned} e^{\frac{3}{2}x^2}\,y &= \int 5\,x\,e^{\frac{3}{2}x^2} \\ &= \frac{5}{3}\int \mathrm{d}e^{\frac{3}{2}x^2} \\ &= \frac{5}{3}\,e^{\frac{3}{2}x^2} + C. \end{aligned} \tag{30}$$

Therefore

$$y(x) = C\,e^{-\frac{3}{2}x^2} + \frac{5}{3}. \tag{31}$$

Substituting back into the equation, we have

$$y' + 3\,x\,y = -3\,x\,C\,e^{-\frac{3}{2}x^2} + 3\,x\,C\,e^{-\frac{3}{2}x^2} + 3\,x\,\frac{5}{3} = 5\,x. \tag{32}$$

Now we are confident that we have found the solution.

Note. It is a good habit to always substitute your solution back into the equation to check.

Now we are ready to discuss the general case. Consider the general linear 1st order ODE

$$a_1(x)\,y' + a_0(x)\,y = b(x). \tag{33}$$

Note that if $a_1(x) = 0$ then the equation is not a DE anymore. Thus we can assume $a_1(x) \neq 0$ and divide both sides by $a_1(x)$ and obtain

$$y' + P(x)\,y = Q(x) \tag{34}$$

where

$$P(x) = \frac{a_0(x)}{a_1(x)}, \qquad Q(x) = \frac{b(x)}{a_1(x)}. \tag{35}$$

Now apply the same idea: We try to multiply the equation by $e^{f(x)}$ so that the LHS (left hand side) can be written as a single derivative. We compute

$$\left[e^{f(x)}\,y\right]' = e^{f(x)}\,[y' + f'(x)\,y]. \tag{36}$$

It is clear that this trick will work if

$$f'(x) = P(x). \tag{37}$$

This is equivalent to

$$f(x) = \int P(x) + C. \tag{38}$$

Thus we see that for any $P(x)$, the appropriate $f(x)$ exists and can be taken to be $\int P(x)$ (we are free to choose the value of the constant C. Usually $C = 0$ is the best choice.).

Taking $f = \int P$, we have

$$\left[e^{\int P}y\right]' = e^{\int P}\,[y' + P(x)\,y] = e^{\int P}Q(x). \tag{39}$$

Integrating, we have

$$e^{\int P}y = \int e^{\int P}Q + C \implies y = e^{-\int P}\int e^{\int P}Q + C\,e^{-\int P}. \tag{40}$$

Thus we have obtained a formula for the general solution of general linear 1st order ODEs.

Remark 7. To some it may be easier to remember the idea – multiplying both sides by $e^{f(x)}$ for appropriate f, than to remember the formula for the solution.

Caution: Don't forget to multiply the right hand side by $e^{\int P}$ too[3]!

Example 8. Solve

$$\left(x^2 + 1\right)y' + x\,y - x = 0, \qquad y(3) = 2. \tag{41}$$

Solution. We rewrite it as

$$y' + \frac{x}{x^2 + 1}\,y = \frac{x}{x^2 + 1}. \tag{42}$$

3. How can anyone forget that? My experience shows that such things happen after solving too many homogeneous equations – those with right hand side 0 which will remain 0 anyway!

Thus $P(x) = \frac{x}{1+x^2}$, $Q(x) = \frac{x}{1+x^2}$. We compute

$$\int P(x) = \int \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}x^2}{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right) + C = \ln\left[\left(1+x^2\right)^{1/2}\right] + C. \tag{43}$$

This gives the correct multiplier to be (clearly taking a nonzero C will not benefit us)

$$e^{\int P} = \left(1+x^2\right)^{1/2}. \tag{44}$$

We have

$$\left[\left(1+x^2\right)^{1/2} y\right]' = x\left(1+x^2\right)^{-1/2}. \tag{45}$$

(At this stage, it is a good habit to check that we have found the correct integrating factor by expanding $\left[\left(1+x^2\right)^{1/2} y\right]'$).

Integrating, we have

$$\left(1+x^2\right)^{1/2} y = \int \frac{x\,\mathrm{d}x}{(1+x^2)^{1/2}} = \int \frac{\mathrm{d}x^2}{(1+x^2)^{1/2}} = \left(1+x^2\right)^{1/2} + C. \tag{46}$$

Dividing both sides by $\left(1+x^2\right)^{1/2}$, we obtain the general solution

$$y(x) = 1 + C\left(1+x^2\right)^{-1/2} \tag{47}$$

where C is an arbitrary constant.

Now we fix the value of C using $y(3) = 2$. Replacing x by 3 in the above formula we have

$$2 = y(3) = 1 + C\left(1+3^2\right)^{-1/2} \Longrightarrow 1 = C\,10^{-1/2} \Longrightarrow C = 10^{1/2}. \tag{48}$$

Therefore the solution is

$$y(x) = 1 + 10^{1/2}\left(1+x^2\right)^{-1/2}. \tag{49}$$

We substitute this formula into the original equation as well as the condition $y(3) = 2$, and check that we have indeed found the solution.

Remark 9. A problem like the above is called an "initial value problem", a name reflecting the mechanical origina of differential equations. For example, consider an equation governs the motion of the particle x. while location of x at an "initial" time $t = 2$ is $x = 0$. Then the "initial condition" is $x(2) = 0$.

Solving an initial value problem consists of two steps.

1. Find the general solution of the equation;
2. Substitute this general formula into the initial condition and fix the arbitrary constant.

Summary: Solving 1st order linear equations.

Follow these steps.

1. Write the equation into
$$y' + P(x) = Q(x). \tag{50}$$
2. Compute
$$\int P(x). \tag{51}$$
3. Write down the integrating factor
$$\mu(x) = e^{\int P(x)}. \tag{52}$$
4. (Optional – skip it if you are not afraid of wasting time using the wrong factor) Check
$$(\mu(x)\,y)' = \mu(x)\,[y' + P(x)\,y] \tag{53}$$

indeed holds.

5. Integrate

$$(\mu(x)\,y)' = \mu(x)\,Q(x) \Longrightarrow \mu(x)\,y = \int \mu(x)\,Q(x) + C. \tag{54}$$

6. Simplify the formulas if possible.

7. (Optional) Substitute you solution back into the original equation and check that it is indeed a solution.

8. (If you are dealing with an initial value problem, with initial condition $y(x_0) = y_0$). In your solution formula replace all y by y_0 and all x by x_0 to determine the value of C.

4. Separable equations.

We have completely solved all linear 1st order ODEs. But what if the equation is not linear? As it is not possible to explicitly solve all 1st order ODEs, all we can do is to find special cases that can be solved explicitly.

The first case is the so-called separable equations, which has the general form:

$$y' = g(x)\,p(y). \tag{55}$$

It is easy to sees that, if we divide both sides by $p(y)$, then the LHS only involves y and RHS only x, consequently both sides can be integrated.

Before we carry this idea out, we need to take care of those y such that $p(y) = 0$.[4] Any constant y such that $p(y) = 0$, if a solution, is lost during dividing by $p(y)$. Fortunately in this case we do not need to check them one by one in every specific problem. Just notice the following.

- Any function $y(x)$ satisfying $p(y) = 0$ is a constant. Let's denote them as $y_1, y_2, \ldots$.
- For any such constant solution y_i, we have $(y_i)' = 0 = g(x)\,p(y_i)$.
- So any solution of $p(y) = 0$ is a solution to the original equation.

Example 10. Solve

$$y' = y\,(2 + \sin x). \tag{56}$$

Solution. It is clear that we should divide both sides by y. First we check that $y = 0$ is a constant solution.

Next we consider the case $y \neq 0$. In this case we can divide both sides by y and then reach

$$\frac{y'}{y} = (2 + \sin x)\,. \tag{57}$$

Integrating, we have

$$\ln|y| = 2\,x - \cos x + C \Longrightarrow \ |y| = e^C\,e^{2x - \cos x}. \tag{58}$$

Here C is an arbitrary constant.

Now we need to see if it's possible to remove the absolute value sign. There are two cases $y > 0$ and $y < 0$. In the first case we have

$$y = e^C\,e^{2x - \cos x} \qquad \text{for arbitrary real number } C \tag{59}$$

which is equivalent to

$$y = C\,e^{2x - \cos x} \qquad \text{for arbitrary } \textbf{positive} \text{ real number } C. \tag{60}$$

4. The case $p(y) = \infty$ is not a problem here. We notice that, if a constant y_i is such that $p(y_i) = \infty$, then

1. $y = y_i$ is not a solution to the original equation as $(y_i)' = 0 \neq g(x)\,p(y_i)$;
2. $y = y_i$ is not a solution to the new equation as $p(y_i)\,(y_i)' = 0 \neq g(x)$.

Similar discussion for the $y<0$ case leads to

$$y = C\, e^{2x-\cos x} \qquad \text{for arbitrary } \textbf{negative} \text{ real number } C. \tag{61}$$

Thus the formula

$$|y| = e^C\, e^{2x-\cos x} \qquad \text{for arbitrary real number } C \tag{62}$$

is equivalent to

$$y = C\, e^{2x-\cos x} \qquad \text{for arbitrary real number } C \neq 0. \tag{63}$$

As a consequence the solutions to the original equation are given by

$$y = 0 \text{ together with } y = C\, e^{2x-\cos x} \qquad \text{for arbitrary real number } C \neq 0 \tag{64}$$

But $y=0$ can be included into the formula $y = C\, e^{2x-\cos x}$ by setting $C=0$. Thus for this problem we can cover all solutions by one single formula

$$y = C\, e^{2x-\cos x} \qquad \text{for arbitrary constant } C. \tag{65}$$

Remark 11. Whenever integrating terms of the form $\frac{Y'}{Y}$, we should write the primitive as $\ln|Y|$ and proceed as above. In particular, we should keep track of the range of values C can take.

In the following examples, though, we will not write the steps as detailed as in the above one.

Example 12. Solve

$$y' = \frac{\sec^2 y}{1+x^2}. \tag{66}$$

Solution. We have

$$\frac{y'}{\sec^2 y} = \frac{1}{1+x^2}. \tag{67}$$

Note that as $\sec y = (\cos y)^{-1}$, $\sec^2 y \neq 0$ so there is no constant solution.

To integrate the LHS, write

$$\frac{y'}{\sec^2 y} = (\cos^2 y)\, y'. \tag{68}$$

The usual trick for integrating nth powers of sin and cos is using the formulas for $\sin n\, x$ and $\cos n\, x$. In our case $n=2$. We recall

$$\cos 2\, y = \cos^2 y - \sin^2 y = 2\cos^2 y - 1 \Longrightarrow \cos^2 y = \frac{1+\cos 2\, y}{2}. \tag{69}$$

Integrating

$$\frac{1+\cos 2\, y}{2}\, y' \tag{70}$$

gives

$$\frac{y}{2} + \frac{\sin 2\, y}{4}. \tag{71}$$

The RHS may be a bit elusive to most people. There is no other way but to remember that[5]

$$(\arctan x)' = \frac{1}{1+x^2}. \tag{72}$$

Thus integrating the RHS gives

$$\arctan x. \tag{73}$$

Putting everything together, we have the (implicit) formula for the solution

$$\frac{y}{2} + \frac{1}{4}\sin 2\, y = \arctan x + C. \tag{74}$$

5. It is a good exercise to compute the derivative of $\arctan x$. The idea is to let $y = \arctan x$ and thus $x = \tan y = \frac{\sin y}{\cos y}$. Differentiating gives the result.

It is not possible to solve equations like

$$\frac{y}{2}+\frac{1}{4}\sin 2\,y=a \tag{75}$$

by formulas. We have to be content with the implicit formula.

Summary: Solving separable equations.

In general, when we have an equation

$$y'=g(x)\,p(y), \tag{76}$$

we do the following:

1. Divide both sides by $p(y)$;
2. Find $F(y)$ such that $F'=1/p$ (F is the primitive of $1/p$); Find $G(x)$ such that $G'=g$.
3. The solution is then given by $F(y)=G(x)+C$ (simplify when possible[6]) together with $y=y_i$ where y_i's are solutions to the algebraic equation $p(y)=0$.

Remark. From the above we clearly see the hierarchical structure of mathematics. In carrying out the above steps, we need to solve two calculus problems: Finding the primitive (anti-derivative) of $1/p$ and g. So in some sense, inside every differential equations problem there are always one or more calculus problems.

5. Exact equations.

As we will see soon, the above case of separable equations is in fact a special case of the more general case of "exact equations". However, it is hard to see how generalizations can be made through the above approach. It will become clear though when we take a slightly different approach.

Side: The operator "d".

The operator "d" is the so-called "total differential". It acts on functions according to the following rules:

$$\mathrm{d}f(x)=f'(x)\,\mathrm{d}x; \qquad \mathrm{d}f(x,y)=\frac{\partial f}{\partial x}\,\mathrm{d}x+\frac{\partial f}{\partial y}\,\mathrm{d}y; \quad \ldots \tag{77}$$

The "..." means similar rules hold for functions with more variables.

Now the key advantage of the operator "d" is the following: It doesn't matter whether x is a function or a basic variable. More specifically,

$$\mathrm{d}f(x)=f'(x)\,\mathrm{d}x \tag{78}$$

is always true, be x a variable or a function, say $x=x(t)$.

Another important property is this,

$$\mathrm{d}f(x)=0 \Longleftrightarrow f(x)=C \tag{79}$$

or equivalently,

$$\mathrm{d}f(x)=\mathrm{d}g(x) \Longleftrightarrow f(x)=g(x)+C. \tag{80}$$

Turning back to our separable equation

$$y'=g(x)\,p(y). \tag{81}$$

Previously we re-write it as

$$F(y)'=G(x)' \tag{82}$$

6. Sometimes it is not possible to rewrite this into an explicit formula for the solution: $y=\cdots$. In those cases we have to be satisfied with an implicit formula.

and then integrate. Clearly, the only equations that can be written this way are separable ones. In other words, there is no way to generalize starting from $F(y)' = G(x)' = 0$.

Now instead of using $\cdot'$, we use "d". The key property of separable equations is then they can be written as

$$\mathrm{d}F(y) = \mathrm{d}G(x). \tag{83}$$

Or equivalently

$$\mathrm{d}[F(y) - G(x)] = 0 \tag{84}$$

which then gives the solutions in the form

$$F(y) - G(x) = C. \tag{85}$$

But now we notice that any equation of the form

$$\mathrm{d}u(x, y) = 0 \tag{86}$$

is immediately solvable, with general solution given by

$$u(x, y(x)) = C. \tag{87}$$

Expanding $\mathrm{d}u$, we see that any equation of the form

$$\frac{\partial u(x, y)}{\partial x}\,\mathrm{d}x + \frac{\partial u(x, y)}{\partial y}\,\mathrm{d}y = 0 \tag{88}$$

is solvable.

In other words, given a first order equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{89}$$

if we can find a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N, \tag{90}$$

then the solution is given by[7]

$$u(x, y(x)) = C. \tag{91}$$

This inspires the notion of "exact equations".

Definition 13. *The equation is called "exact" if there is u such that*

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y} \tag{92}$$

Example 14. Solve

$$(2\,x\,y + 3)\,\mathrm{d}x + \left(x^2 - 1\right)\mathrm{d}y = 0. \tag{93}$$

Solution. We need to find $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2\,x\,y + 3, \qquad \frac{\partial u}{\partial y} = x^2 - 1. \tag{94}$$

Integrating the first one, we have

$$u(x, y) = x^2\,y + 3\,x + g(y) \tag{95}$$

where $g(y)$ is a function to be determined. To determin g we use the other equation

$$x^2 - 1 = \frac{\partial u}{\partial y} = x^2 + g'(y) \Longrightarrow g'(y) = -1 \Longrightarrow g(y) = -y + C. \tag{96}$$

Putting everything together, we have

$$u(x, y) = x^2\,y + 3\,x - y + C. \tag{97}$$

7. You need to simplify when possible.

The solution is then given by

$$x^2\,y+3\,x-y=C\Longrightarrow y=\frac{3\,x-C}{1-x^2}. \tag{98}$$

So far so good. However, the next example shows that not all equations are exact.

Example 15. Solve

$$\frac{1}{y}\,\mathrm{d}x-\left(3\,y-\frac{x}{y^2}\right)\mathrm{d}y=0. \tag{99}$$

Solution. We try to find $u(x,y)$ such that

$$\frac{\partial u}{\partial x}=\frac{1}{y},\quad \frac{\partial u}{\partial y}=-\left(3\,y-\frac{x}{y^2}\right). \tag{100}$$

Integrating the first equation, we have

$$u(x,y)=\frac{x}{y}+g(y). \tag{101}$$

To determine $g(y)$ we compute

$$-\left(3\,y-\frac{x}{y^2}\right)=\frac{\partial u}{\partial y}=-\frac{x}{y^2}+g'(y)\implies g'(y)=\frac{2\,x}{y^3}-3\,y. \tag{102}$$

But this is not possible as $g'(y)$, being a function of y only, cannot have any dependence on x.

From the above examples, we see that a simple criterion of telling when an equation is exact is in need, which would save us from carrying out many integrations just to find out at the end that that the method does not work.

Recall the definition of "exactness":

$$M=\frac{\partial u}{\partial x},\qquad N=\frac{\partial u}{\partial y}. \tag{103}$$

It follows that

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial N}{\partial x}. \tag{104}$$

Thus an equation cannot be exact unless $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

This condition is easy to check. For the above two examples, we check

$$(2\,x\,y+3)\,\mathrm{d}x+\left(x^2-1\right)\mathrm{d}y=0\implies M=2\,x\,y+3,\ N=x^2-1\Longrightarrow\frac{\partial M}{\partial y}=2\,x=\frac{\partial N}{\partial x} \tag{105}$$

and

$$\frac{1}{y}\,\mathrm{d}x-\left(3\,y-\frac{x}{y^2}\right)\mathrm{d}y=0\Longrightarrow M=\frac{1}{y},\ N=-\left(3\,y-\frac{x}{y^2}\right)\Longrightarrow\frac{\partial M}{\partial y}=-\frac{1}{y^2}\neq\frac{1}{y^2}=\frac{\partial N}{\partial x}. \tag{106}$$

which explains the phenomena we observe above.

However, so far the condition is only necessary. A necessary condition can only give us negative information (such as a certain equation is not exact) but not positive information (a certain equation is exact). We have to check whether it is sufficient, that is, if an equation satisfies the condition, is it exact? The following theorem answers it affirmatively.

Theorem 16. *Consider the equation*

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0. \tag{107}$$

If

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}, \tag{108}$$

then the equation is exact. That is, there is $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y}. \tag{109}$$

Proof. The proof is constructive. First note that any function of the form

$$\psi(x, y) = \int M \,\mathrm{d}x + g(y) \tag{110}$$

satisfies

$$M = \frac{\partial F}{\partial x}. \tag{111}$$

Differentiating with respect to y, we have

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} \int M \,\mathrm{d}x + g'(y) = \int \frac{\partial M}{\partial y} \,\mathrm{d}x + g'(y) = \int \frac{\partial N}{\partial x} \,\mathrm{d}x + g'(y). \tag{112}$$

Thus all we need to do is to show the existence of $g(y)$ such that

$$g'(y) = N - \int \frac{\partial N}{\partial x} \,\mathrm{d}x. \tag{113}$$

As

$$\frac{\partial}{\partial x}\left[N - \int \frac{\partial N}{\partial x} \,\mathrm{d}x\right] = \frac{\partial N}{\partial x} - \frac{\partial N}{\partial x} = 0 \tag{114}$$

we know that

$$N - \int \frac{\partial N}{\partial x} \,\mathrm{d}x \tag{115}$$

is a function of y only (remember why we failed when trying to solve the non-exact equation!). Thus the existence of g is proved. □

Remark 17. Readers well-versed in calculus may have already noticed that several conditions, without which the proof cannot hold, are missing in the statement of the theorem. We briefly discuss these conditions here.

1. We need to require the continuity of $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$.
2. If we are discussing the problem in a particular region (instead of the whole x-y plane), then this region has to be simply connected, that is it cannot have "holes".

When the first condition is not satisfied, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is no longer guaranteed to hold and everything fall apart; When the second condition is not satisfied, the construction of u will fail.

Example 18. (Separable equations are exact) Consider the separable equation

$$g(x) \,\mathrm{d}x + f(y) \,\mathrm{d}y = 0. \tag{116}$$

We have

$$\frac{\partial g(x)}{\partial y} = 0 = \frac{\partial f(y)}{\partial x}. \tag{117}$$

Thus the equation is exact.

6. Non-exact equations.

So far we have managed to be able to solve two classes of equations

- Linear equations:

$$y' + P(x)\, y = Q(x); \tag{118}$$

- Exact equations:

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{119}$$

with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

But wait a minute: Do we really have two different classes here? We know that an exact equation may not be linear, but how about the other direction? Are linear equations exact? Let's check.

First we write a linear equation into the form that's easy to check exactness:

$$y' + P(x)\, y = Q(x) \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\, y = Q(x) \Longrightarrow (P(x)\, y - Q(x))\, \mathrm{d}x + \mathrm{d}y = 0. \tag{120}$$

So $M = P(x)\, y - Q(x)$, $N = 1$. Compute

$$\frac{\partial M}{\partial y} = P(x) \neq 0 = \frac{\partial N}{\partial x}. \tag{121}$$

So indeed a general linear equation is not exact.

But we somehow still managed to solve it, by multiplying both sides by the "integrating factor" $e^{\int P}$, so that the equation becomes

$$e^{\int P} y' + e^{\int P} P(x)\, y = e^{\int P} Q(x). \tag{122}$$

Rewrite to

$$\left[e^{\int P} P(x)\, y - e^{\int P} Q(x)\right] \mathrm{d}x + e^{\int P}\, \mathrm{d}y = 0. \tag{123}$$

This time

$$M = e^{\int P} P(x)\, y - e^{\int P} Q(x), \qquad N = e^{\int P} \tag{124}$$

and...

$$\frac{\partial M}{\partial y} = e^{\int P} P = \frac{\partial N}{\partial x}! \tag{125}$$

So we have learned:

> A non-exact equation can be made exact through multiplying an appropriate "integrating factor".

Now back to our general form

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0. \tag{126}$$

Let's say the equation failed the "exactness test": It turns out that

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \tag{127}$$

The idea now is to notice the following: If $y(x)$ is a solution to the above equation, it also solves

$$\mu(x, y)\, M(x, y)\, \mathrm{d}x + \mu(x, y)\, N(x, y)\, \mathrm{d}y = 0. \tag{128}$$

for any function $\mu(x, y)$.[8] But this latter equation may be exact. In other words, there may exist an appropriate μ such that

$$\frac{\partial(\mu\, M)}{\partial y} = \frac{\partial(\mu\, N)}{\partial x}. \tag{129}$$

Taking the partial derivatives inside the products, we reach the following equation

$$N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} = \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right). \tag{130}$$

All we need to do is to solve this equation. And indeed we have

Theorem 19. *Under reasonable assumptions on M, N (say differentiable), the integrating factor μ always exists.*

8. As long as $\mu(x, y(x))$ is not ∞ everywhere.

In other words, for any first order equation

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{131}$$

there is always a function $\mu(x,y)$ such that

$$(\mu\,M)\,\mathrm{d}x+(\mu\,N)\,\mathrm{d}y=0 \tag{132}$$

is exact, and thus solvable.

Unfortunately, in practice, the PDE

$$N\,\frac{\partial\mu}{\partial x}-M\,\frac{\partial\mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right). \tag{133}$$

despite being the simplest PDE – first order, linear, homogeneous, is in general harder to solve than our nonlinear ODE.[9] Therefore, although in theory integrating factors always exist (and consequently we can solve all 1st order ODEs), in practice most of them are impossible to find – finding them is as hard as solving the original ODE.

Nevertheless, there are special cases in which the PDE

$$N\,\frac{\partial\mu}{\partial x}-M\,\frac{\partial\mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right). \tag{134}$$

is solvable without any knowledge of the solution to the original ODE

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0.$$

Example 20. Solve

$$\left(3\,x^2+y\right)\mathrm{d}x+\left(x^2\,y-x\right)\mathrm{d}y=0. \tag{135}$$

Solution. We first check whether it is exact.

$$M(x,y)=3\,x^2+y\Longrightarrow\frac{\partial M}{\partial y}=1;\qquad N(x,y)=x^2\,y-x\Longrightarrow\frac{\partial N}{\partial x}=2\,x\,y-1 \tag{136}$$

Thus the equation is not exact.

Next we try to find an integration factor. The equation for μ is

$$\left(x^2\,y-x\right)\frac{\partial\mu}{\partial x}-\left(3\,x^2+y\right)\frac{\partial\mu}{\partial y}=\mu\,(2-2\,x\,y). \tag{137}$$

Now we check whether $\mu=\mu(x)$ is possible. In this case we have

$$\mu'=\mu\,\frac{2-2\,x\,y}{x^2\,y-x}=-\,\mu\,\frac{2}{x}\Longleftrightarrow\frac{\mu'}{\mu}=-\,\frac{2}{x}\Longleftrightarrow\ln|\mu|=-\,2\ln x. \tag{138}$$

It worked! As only one μ is needed, we choose

$$\mu(x,y)=-\,\frac{1}{x^2}. \tag{139}$$

Multiplying $\mu(x,y)$ to the equation we reach

$$\left(-3-\frac{y}{x^2}\right)\mathrm{d}x+\left(-y+\frac{1}{x}\right)\mathrm{d}y=0. \tag{140}$$

As

$$\frac{\partial}{\partial y}\left(-3-\frac{y}{x^2}\right)=-\,\frac{1}{x^2}=\frac{\partial}{\partial x}\left(-y+\frac{1}{x}\right), \tag{141}$$

the equation is exact now.

9. In fact, when we look at the standard "method of characteristics" which solves such PDEs, we will see that we need to solve $M\,\mathrm{d}x+N\,\mathrm{d}y=0$ first!

Now we set out to solve this equation. We need $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x}=-3-\frac{y}{x^2}, \qquad \frac{\partial\psi}{\partial y}=-y+\frac{1}{x}. \tag{142}$$

Integrating the first, we have

$$\psi(x,y)=-3x+\frac{y}{x}+g(y). \tag{143}$$

Differentiating:

$$\frac{1}{x}+g'(y)=\frac{\partial\psi}{\partial y}=-y+\frac{1}{x}\Longleftrightarrow g'(y)=-y\Longleftrightarrow g(y)=-\frac{y^2}{2}. \tag{144}$$

Thus $\psi(x,y)=-3x+\frac{y}{x}-\frac{y^2}{2}$ and the solution is given by

$$-3x+\frac{y}{x}-\frac{y^2}{2}=C. \tag{145}$$

Finally we need to check whether multiplying μ adds or loses any solutions. In our cases, $\mu=-\frac{1}{x^2}$ is never 0, therefore no solution is gained; On the other hand, it is ∞ for $x=0$, so $x=0$ may be a solution that is lost. Indeed $x=0$ satisfies the original equation.[10]

Example 21. Solve

$$\left(3x+\frac{6}{y}\right)\mathrm{d}x+\left(\frac{x^2}{y}+3\frac{y}{x}\right)\mathrm{d}y=0. \tag{146}$$

Solution. We have

$$M=3x+\frac{6}{y}, \qquad N=\frac{x^2}{y}+3\frac{y}{x}. \tag{147}$$

Compute

$$\frac{\partial M}{\partial y}=-\frac{6}{y^2}, \qquad \frac{\partial N}{\partial x}=\frac{2x}{y}-\frac{6y}{x^2}. \tag{148}$$

So this equation is not exact.

Next we write it as

$$\left(\frac{x^2}{y}+3\frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+3x+\frac{6}{y}=0 \tag{149}$$

and conclude that it is not linear.

Thus the only thing to do is to find an integrating factor. The equation for μ is

$$N\frac{\partial\mu}{\partial x}-M\frac{\partial\mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\Longleftrightarrow\left(\frac{x^2}{y}+3\frac{y}{x}\right)\frac{\partial\mu}{\partial x}-\left(3x+\frac{6}{y}\right)\frac{\partial\mu}{\partial y}=\left(-\frac{6}{y^2}-\frac{2x}{y}+\frac{3y}{x^2}\right)\mu. \tag{150}$$

Let's starting guessing.

- $\mu=\mu(x)$. This leads to

$$\left(\frac{x^2}{y}+3\frac{y}{x}\right)\mu'=\left(-\frac{6}{y^2}-\frac{2x}{y}+\frac{6y}{x^2}\right)\mu\Longrightarrow\frac{\mu'}{\mu}=\frac{-6x^2-2x^3y+3y^3}{xy\,(x^3+3y)} \tag{151}$$

  Clearly the RHS is not a function of x alone. So such μ (a function of x alone) does not exist.

- $\mu=\mu(y)$. This leads to

$$-\left(3x+\frac{6}{y}\right)\mu'=\left(-\frac{6}{y^2}-\frac{2x}{y}+\frac{6y}{x^2}\right)\mu\Longrightarrow\frac{\mu'}{\mu}=-\frac{6}{3xy+6}\,\frac{-6x^2-2x^3y+3y^3}{x^2y^2} \tag{152}$$

  Clearly the RHS is not a function of y alone. So we fail again.

- $\mu=\mu(xy)$.[11] This time we have

$$\left(\frac{x^2}{y}+3\frac{y}{x}\right)y\,\mu'-\left(3x+\frac{6}{y}\right)x\,\mu'=\left(-\frac{6}{y^2}-\frac{2x}{y}+\frac{6y}{x^2}\right)\mu \tag{153}$$

10. As we do not emphasize this point, either including $x=0$ into solutions or not is OK.

11. In hindsight, we can say that this guess is inspired by the repeated appearance of xy in previous calculations.

Simplify we have

$$\left(x^2+\frac{3\,y^2}{x}-3\,x^2-\frac{6\,x}{y}\right)\mu'=\left(-\frac{6}{y^2}-\frac{2\,x}{y}+\frac{6\,y}{x^2}\right)\mu \tag{154}$$

This leads to

$$\frac{\mu'}{\mu}=\frac{1}{x\,y}. \tag{155}$$

Now remember that μ is a function of $x\,y$. Let $z=x\,y$. We have

$$\frac{\mu'(z)}{\mu(z)}=\frac{1}{z}\Longleftrightarrow \ln|\mu|=\ln|z|+C. \tag{156}$$

As we only need one μ, just take

$$\mu=z=x\,y. \tag{157}$$

So we have found our integrating factor.

Now we multiply both sides of the equation by $\mu=x\,y$, to obtain

$$\left(3\,x^2\,y+6\,x\right)\mathrm{d}x+\left(x^3+3\,y^2\right)\mathrm{d}y=0. \tag{158}$$

We check

$$\frac{\partial\left(3\,x^2\,y+6\,x\right)}{\partial y}=3\,x^2=\frac{\partial\left(x^3+3\,y^2\right)}{\partial x}. \tag{159}$$

So we now are sure that we have found a correct integrating factor.

Now we find $u(x,y)$ such that

$$\frac{\partial u}{\partial x}=3\,x^2\,y+6\,x;\qquad \frac{\partial u}{\partial y}=x^3+3\,y^2. \tag{160}$$

Integrating the first equation, we have

$$u(x,y)=x^3\,y+3\,x^2+g(y). \tag{161}$$

Substituting into the second,

$$x^3+3\,y^2=\frac{\partial u}{\partial y}=x^3+g'(y)\Longrightarrow g'(y)=3\,y^2\Longrightarrow g(y)=y^3. \tag{162}$$

So the general solution to the new equation (Remember we multiplied the original equation by $x\,y$!) is

$$x^3\,y+3\,x^2+y^3=C. \tag{163}$$

Finally, we need to check whether multiplying both sides by $x\ y$ brings in any new solutions. The only possibility is the y such that $x\,y=0$, that is $y=0$. But it is easy to see that $y=0$ is not in the above formula. So we know that the solution to the original equation is also

$$x^3\,y+3\,x^2+y^3=C. \tag{164}$$

Remark 22. We have the following criterions. Here we have used the short hands $f_y=\frac{\partial f}{\partial y}$, $f_x=\frac{\partial f}{\partial x}$ to make things look better.

- If $\frac{M_y-N_x}{N+M}$ is a function of $x-y$ alone, then so is μ;
- If $\frac{M_y-N_x}{y\,N-x\,M}$ is a function of $x\,y$ alone, then so is μ;
- If $\frac{(M_y-N_x)\,y^2}{y\,N+x\,M}$ is a function of x/y alone, then so is μ;
- If $\frac{M_y-N_x}{2\,(x\,N-y\,M)}$ is a function of x^2+y^2 alone, then so is μ;
- If $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=N\,g(x)-M\,h(y)$ for some functions g,h, then $\mu=X(x)\,Y(y)$ with $X=e^{\int g(x)\,\mathrm{d}x}$, $Y=e^{\int h(y)\,\mathrm{d}y}$.

The idea of finding such criterions is as follows. First assume μ has the special form, then put it into the equation. Then the consistency of the equation becomes a condition on M and N. For example, assuming μ has the special form $\mu(g(x,y))$, we compute

$$N\frac{\partial\mu}{\partial x}-M\frac{\partial\mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \tag{165}$$

to get

$$N\mu'\frac{\partial g}{\partial x}-M\mu'\frac{\partial g}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\Longrightarrow\frac{N g_x-M g_y}{M_y-N_x}=\frac{\mu}{\mu'}. \tag{166}$$

As the right hand side is of the form of a composite function $F(g(x,\,y))$, the left hand side has to be of the same form too.

Remark 23. In practice, instead of remembering all the above rules, one may use the following approach:

1. Guess a form of the integrating factor, for example $\mu=\mu(x)$.
2. Compute $\frac{\partial(\mu M)}{\partial y}$ and $\frac{\partial(\mu N)}{\partial x}$.
3. Is it possible to find $\mu(x)$ such that the two equal?[12]
4. If yes, find such $\mu(x)$ and use it to solve the problem; If no, go back to 1 and make another guess ($\mu=\mu(y)$, $\mu=\mu(x\,y)$; $\mu=\mu(x^2\,y)$; ...);
5. Give up when run out of guesses.

Recall that we have used "integrating factors" for 1st order linear ODEs. Is it the same thing? It turns out it is a special case of what we are discussing here.

Example 24. Solve the linear equation

$$y'+P(x)\,y=Q(x). \tag{167}$$

Solution. First write it as

$$\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)\,y-Q(x)=0\Longrightarrow(P(x)\,y-Q(x))\,\mathrm{d}x+\mathrm{d}y=0. \tag{168}$$

We set $M(x,\,y)=P(x)\,y-Q(x)$, $N(x,\,y)=1$. It is easy to see that $M_y\neq N_x$ thus the equation is not exact.

However we can find an integrating factor $\mu(x,y)$. We compute

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=P(x). \tag{169}$$

Thus $\frac{M_y-N_x}{N}=P(x)$ is a function of x alone, which suggests the existence of an integrating factor depending on x alone. The equation for this $\mu(x)$ is then

$$N\mu'=\mu\,(M_y-N_x)\Longrightarrow\mu'=P(x) \tag{170}$$

which leads to $\mu(x)=e^{\int P(x)}$, exactly the integrating factor in the theory of 1st order equations.

Remark 25. Now we see that, what we have been doing in all these previous lectures is just one thing:

Make the equation exact and then solve it.

This is the only strategy we have to solve differential equations.

Remark 26. We can solve the equation as soon as we know two integrating factors μ_1,μ_2 such that their ratio is not a constant. Then $\frac{\mu_1}{\mu_2}=C$ gives the solutions.

12. Put all μ's on one side and all others on the other side. If "the other side" does not only depend on x ($y, x\,y, x^2\,y$ – varies with your guesses), then μ with such particular form does not exist.

This claim can be proved through calculation, but can also be understood geometrically. A curve given by $u(x, y) = C$ solves $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ means the vector $\binom{M}{N}$ is perpendicular to the tangential direction of the curve. But the tangential direction is just $\binom{-u_y}{u_x}$. Thus $u = C$ is a solution $\Longleftrightarrow N\,u_x - M\,u_y = 0$.

7. Solvable equations in disguise: Transformation method.

There are equations which are at first sight not included in any of the classes discussed above, yet can be transformed into one of them.

7.1. Homogeneous equation.

Consider the 1st order equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{171}$$

where $f(x, y)$ is a homogeneous function, that is satisfies

$$f(\lambda\,x, \lambda\,y) = f(x, y) \tag{172}$$

for all real numbers $\lambda \neq 0$. If we take the particular $\lambda = \frac{1}{x}$, we conclude

$$f(x, y) = f\left(1, \frac{y}{x}\right) := G\left(\frac{y}{x}\right). \tag{173}$$

Thus naturally we would like to define a new unknown function $v = y/x$ so that the RHS becomes $G(v)$ which seems hopeful for separation. We check the LHS

$$v = y/x \Longrightarrow y = x\,v \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\,\frac{\mathrm{d}v}{\mathrm{d}x}. \tag{174}$$

Thus the equation becomes

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} = G(v) - v \tag{175}$$

which is separable and therefore can be solved explicitly.

Example 27. Solve

$$\left(y^2 - x\,y\right)\mathrm{d}x + x^2\,\mathrm{d}y = 0. \tag{176}$$

Solution. Write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2 - x\,y}{x^2} \tag{177}$$

One can check that it is homogeneous. Let $v = y/x$. The equation becomes

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} = -v^2 \Longrightarrow \frac{\mathrm{d}v}{-v^2} = \frac{\mathrm{d}x}{x} \Longrightarrow \frac{1}{v} = \ln|x| + C \Longrightarrow v = (\ln|x| + C)^{-1}. \tag{178}$$

As we have divided both sides by v^2, we should check the possibility of $v \equiv 0$. It is easy to see that $v \equiv 0$ is a solution. Thus the solution to the original problem is

$$y = \frac{x}{\ln|x| + C}, \qquad y = 0. \tag{179}$$

C is an arbitrary constant.

Example 28. Solve

$$y' = \frac{x^2 + x\,y + y^2}{x^2}. \tag{180}$$

Solution. Clearly it is homogeneous. We write $y = x\,v$ and obtain

$$x\,v' = 1 + v^2 \Longrightarrow \frac{v'}{1 + v^2} = \frac{1}{x} \Longrightarrow \arctan(v) = \ln|x| + C \tag{181}$$

which leads to the solution

$$\arctan(y/x) = \ln|C\,x|. \tag{182}$$

7.2. $\frac{\mathrm{d}y}{\mathrm{d}x} = G(a\,x + b\,y)$.

For this equation naturally we try

$$z = a\,x + b\,y. \tag{183}$$

the equation becomes

$$\frac{\mathrm{d}z}{\mathrm{d}x} = b\,G(z) + a \tag{184}$$

which is again separable.

Example 29. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x + y} - 1. \tag{185}$$

Solution. Let

$$z = x + y. \tag{186}$$

The equation is transformed to

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{z}. \tag{187}$$

Divide both sides by $\sqrt{z}$ (Check: $z \equiv 0$ is a solution) and multiply both sides by $\mathrm{d}x$, we have

$$\frac{\mathrm{d}z}{\sqrt{z}} = \mathrm{d}x \Longrightarrow -2\sqrt{z} = x + C \Longrightarrow z = \frac{(x + C)^2}{4}. \tag{188}$$

Finally we can write the solutions to the original problem:

$$y = \frac{(x + C)^2}{4} - x, \qquad y = -x. \tag{189}$$

7.3. Bernoulli equation.

Bernoulli equations have the following form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x)\,y^n. \tag{190}$$

where n is a real number.

It looks very similar to the linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x) \tag{191}$$

except a factor y^n on the RHS. Inspired by this similarity, we divide both sides by y^n to make the RHS exactly $Q(x)$:

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y^{1-n} = Q(x). \tag{192}$$

To make it look like the linear case, we set

$$v = y^{1-n}. \tag{193}$$

We are fortunate as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{1-n}\right) = (1-n)\,y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{194}$$

Thus the equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)\,v = Q(x) \tag{195}$$

or equivalently

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)\,P(x)\,v = (1-n)\,Q(x). \tag{196}$$

Remark 30. Since we divide by y^n, we need to check those y such that y^n is zero. There are two cases

- $n \leqslant 0$. y^n cannot be 0.

- $n > 0$. $y^n = 0$ for $y = 0$.

Thus when $n > 0$, we need to add the solution $y = 0$ back.

Remark 31. Since we divide by $1 - n$, we need to take a look at the case $n = 1$. But in that case the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (Q(x) - P(x))\, y \tag{197}$$

which is separable.

Also note that the case $n = 0$ is exactly the linear case.

Example 32. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\,y}{x} - x^2\, y^2. \tag{198}$$

Solution. First note that $y = 0$ is a solution. When $y \neq 0$, divide both sides by y^{-2} and let $v = y^{-1}$. We have

$$y^{-2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x\,y} - x^2 \Longleftrightarrow \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2}{x}\,v = x^2. \tag{199}$$

Recalling the method of solving linear equations, we compute

$$\int \frac{2}{x} = \ln x^2 \Longrightarrow e^{\int \frac{2}{x}} = x^2 \tag{200}$$

Thus multiply both sides by x^2, we reach

$$x^4 = x^2\,\frac{\mathrm{d}v}{\mathrm{d}x} + 2\,x\,v = \frac{\mathrm{d}}{\mathrm{d}x}\big(x^2\,v\big). \tag{201}$$

Integrating, we obtain

$$x^2\,v = \frac{1}{5}\,x^5 + C \Longrightarrow v = \frac{1}{5}\,x^3 + C\,x^{-2}. \tag{202}$$

Summarizing, the solutions are

$$y = \left(\frac{1}{5}\,x^3 + C\,x^{-2}\right)^{-1}, \qquad y = 0. \tag{203}$$

7.4. Equations with linear coefficients (NOT REQUIRED FOR 334 FALL 2010).

Consider the following type of equation

$$(a_1\,x + b_1\,y + c_1)\,\mathrm{d}x + (a_2\,x + b_2\,y + c_2)\,\mathrm{d}y = 0. \tag{204}$$

The idea is to transform this equation to a form that we can solve: linear, exact, etc. We notice that

- When $b_2 = 0$, the equation is linear.
- When $b_1 = a_2$, the equation is exact.[13]
- When $a_1\,b_2 = a_2\,b_1$, the equation is of the form

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = G(a\,x + b\,y). \tag{205}$$

  Those who want to test their linear algebra knowledge should try to prove this.
- Writing the equation as

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{a_1\,x + b_1\,y + c_1}{a_2\,x + b_2\,y + c_2}, \tag{206}$$

  we see that when $c_1 = c_2 = 0$, the equation is homogeneous.

It turns out that in general we can transform the equation into the last case, as long as $a_1\,b_2 \neq a_2\,b_1$. In the following we consider the case $a_1\,b_2 \neq a_2\,b_1$.

13. The case $b_1 = b_2 = 0$ can also be made exact via multiplying appropriate $\mu(x, y)$.

Thus we will try to get rid of the two constants. To do this, notice that replacing x, y by $x + C, y + C'$ leave the LHS unchanged, and produce extra constants on both the nominator and denominator on the RHS. Inspired by this, we set

$$x = u + h, \qquad y = v + k \tag{207}$$

where v is the new unknown, u is the new variable, and h, k are constants. The equation the becomes

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{a_1 u + b_1 v + [a_1 h + b_1 k + c_1]}{a_2 u + b_2 v + [a_2 h + b_2 k + c_2]}. \tag{208}$$

We see that as long as we can find h, k satisfying

$$a_1 h + b_1 k + c_1 = 0 \tag{209}$$

$$a_2 h + b_2 k + c_2 = 0 \tag{210}$$

the equation is transformed to the homogeneous case. As $a_1 b_2 \neq a_2 b_1$, such h, k always exist and are unique.[14]

Example 33. (2.6.31) Solve

$$(2x - y)\,\mathrm{d}x + (4x + y - 3)\,\mathrm{d}y = 0. \tag{211}$$

Solution. Write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x - y}{4x + y - 3}. \tag{212}$$

Set

$$x = u + h, \quad y = v + k. \tag{213}$$

We obtain

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{2u - v + [2h - k]}{4u + v + [4h + k - 3]}. \tag{214}$$

Thus we need to find h, k such that

$$2h - k = 0 \tag{215}$$

$$4h + k - 3 = 0. \tag{216}$$

The solution is

$$h = 1/2, \quad k = 1. \tag{217}$$

Next we solve

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{2u - v}{4u + v}. \tag{218}$$

As this is a homogeneous equation, set

$$z = \frac{v}{u}. \tag{219}$$

We have

$$u\frac{\mathrm{d}z}{\mathrm{d}u} + z = -\frac{2 - z}{4 + z} \Longrightarrow u\frac{\mathrm{d}z}{\mathrm{d}u} = -\frac{z^2 + 3z + 2}{4 + z} \Longrightarrow -\frac{4 + z}{z^2 + 3z + 2}\,\mathrm{d}z = \frac{\mathrm{d}u}{u}. \tag{220}$$

As usual for separable equations, we check possible zeros of $\frac{z^2 + 3z + 2}{4 + z}$. We find two constant solutions $z = -1, z = -2$.

To integrate the LHS, we write

$$-\frac{4 + z}{z^2 + 3z + 2} = \frac{a}{z + 1} + \frac{b}{z + 2} = \frac{(a + b)z + (2a + b)}{z^2 + 3z + 2}. \tag{221}$$

Setting

$$a + b = -1, \qquad 2a + b = -4 \tag{222}$$

14. Another linear algebra test!

we have

$$a = -3, \qquad b = 2. \tag{223}$$

Thus

$$\begin{aligned} -\frac{4+z}{z^2+3z+2}\,\mathrm{d}z &= -\frac{3}{z+1}\,\mathrm{d}z + \frac{2}{z+2}\,\mathrm{d}z \\ &= \mathrm{d}[-3\ln|z+1| + 2\ln|z+2|]. \end{aligned} \tag{224}$$

Consequently

$$-3\ln|z+1| + 2\ln|z+2| = \ln|u| + C \tag{225}$$

which gives

$$(z+2)^2 = e^C\,|z+1|^3\,|u|. \tag{226}$$

As C is arbitrary, e^C is an arbitrary positive real number. So the above, together with the solution $z = -2$, is equivalent to

$$(z+2)^2 = C\,(z+1)^3\,u \tag{227}$$

where C is an arbitrary real number. Recalling $z = \frac{v}{u}$, the above becomes

$$(v+2u)^2 = C\,(v+u)^3. \tag{228}$$

Now as $v = y - k = y - 1$, $u = x - h = x - 1/2$, we have

$$(y+2x-2)^2 = C\,(y+x-3/2)^3 \tag{229}$$

which is equivalent to

$$(y+2x-2)^2 = C\,(2y+2x-3)^3 \tag{230}$$

due to the fact that C is an arbitrary real number.

Note that the above formula already includes the case $z = -2$. The case $z = -1$ corresponds to $v = -u$ or equivalently $y = \frac{3}{2} - x$. Substituting back into the original equation

$$(2x - y)\,\mathrm{d}x + (4x + y - 3)\,\mathrm{d}y = 0. \tag{231}$$

we see that it is indeed a solution.

Summarizing, the solutions to our problem are

$$y = \frac{3}{2} - x, \qquad (y+2x-2)^2 = C\,(2y+2x-3)^3. \tag{232}$$

Remark 34. Note that the above approach also works for the more general case

$$y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right). \tag{233}$$

8. Direction field, numerics, and well-posedness theory.

Although many equations can be solved with techniques introduced above, there are many others that cannot.[15] Therefore we need to find some way to investigate these "un-solvable" equations. The standard approach is to study the "direction field" of the equation, either by hand-eye or by a computer. We will now briefly introduce this approach, along the way indicating its relation with general well-posedness theory of 1st order semilinear ODEs.

8.1. Direction field.

15. Recall that only in special cases can we find appropriate integrating factors explicitly; Furthermore, even if we have found an integrating factor, the functions we need to integrate may not have a primitive with explicit formula consisting of elementary functions. This may have been mentioned in your calculus class – most of the functions cannot be integrated explicitly.

The idea of "direction fields" comes from the geometrical interpretation of ODEs. Consider our semi-linear equation

$$y' = f(x, y). \tag{234}$$

We notice the following.

1. At any point (x, y) in the plane, one can define a vector $(1, f(x, y))$. Thus for every point in the plane, we have one vector "attached" to it. In other words we have defined a vector field. We call this vector field "direction field" or "slope field".
2. For any curve $(x, y(x))$, its tangent vector at point $(x, y(x))$ is given by $(1, y'(x))$.

Combining these two observations, we reach

- A curve in the x-y plane represents a solution $\Longleftrightarrow$ The solution curve is tangent to the vector $(1, f(x, y))$ at every point on this curve $(x, y(x))$.

Now for any given equation

$$y' = f(x, y) \tag{235}$$

we can do the following:

1. Plot the vector field $(1, f(x, y))$. As we cannot plot the vectors for every point, what we usually do is imagine a fine grid and only plot those vectors on the grid points.
2. Try to "fit" one or more curves to the vector field such that they are everywhere tangent to the vectors.

Example 35. Study

$$y' = a\,y + b \tag{236}$$

and discuss the role of r in the behavior of the solution.

Remark 36. Clearly there is much ambiguity in "fitting" such curves. Why should the curve pass this point but not that one? The following are some guidelines:

1. Passing every point there should be a curve.
2. The curves should look as smooth as possible;
3. The curves should not cross (this is obvious) or touch (this is not obvious) each other.

All the above claims rely on well-posedness theory. 1. is existence, 2. is regularity, 3 is uniqueness.

In many cases $f(x, y)$ is of the form of a ratio $f(x, y) = \frac{M(x, y)}{N(x, y)}$ with M, N nice functions. In these cases it is more convenient to plot the fields $(N(x, y), M(x, y))$ instead of $(1, f(x, y))$.

Example 37. Consider

$$y' = \frac{x}{y} \tag{237}$$

and discuss the asymptotic behavior (that is, what happens at infinity) of solutions.

Remark 38. When plotting solutions, the very first thing to do is to find those solutions that are straight lines. These lines immediately divide the plane into smaller regions containing other solutions, as a consequence of the theoretical result that no other solutions can tough/cross these lines.

Remark 39. Nowadays the task of plotting direction fields is always done by computers. When analyzing equations by hand, usually only the following is done:

1. Finding "straight-line" solutions. These solutions divide the plane into several regions;

2. In each region, try to determine the rough directions of the vector fields;
3. Form some rough idea of what's going on.

If 3. seems difficult, get a computer. On the course webpage there is also a link to an online "vector field plotter" written in Java.

8.2. Numerics.

It is obvious that the above method of direction field won't help much when the direction field looks complicated, or when we need more than a very rough qualitative understanding of how the solution behaves. In these cases we do numerical computation.

To do this we need to write the equation

$$y' = f(x, y) \tag{238}$$

into approximate forms that can be solved by computers. Each such "form" is called a "numerical scheme". The simplest one is the forward Euler scheme:

$$y_{n+1} = y_n + h\, f(x_n, y_n) \tag{239}$$

where y_n is an approximate value of $y(x_n)$ and $x_{n+1} = x_n + h$. That this is an approximation of the original equation can be seen from re-writing the scheme into

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n) \tag{240}$$

and recall the Taylor expansion

$$y(x_{n+1}) = y(x_n) + y'(x_n)\, h + \text{higher order terms.} \tag{241}$$

Numerical computation is very powerful. However it has to be guided by theory. For example, let's solve

$$y' = 2\, y^{1/2}, \qquad y(0) = 0 \tag{242}$$

using forward Euler. Clearly $y_n = 0$ for all n, which corresponds to the solution $y(x) \equiv 0$. But we have a second solution $y(x) = x^2$. This solution can never be found by forward Euler scheme. We see that numerical computation fails when the solution is not unique, furthermore, the computation does not even tell us that some thing may be wrong.

8.3. Some theory.

The fundamental theory for differential equations is the theory for well-posedness, regarding:

1. Existence: Given a point (x_0, y_0), is there a solution passing it at all?
2. Uniqueness: Given a point (x_0, y_0), is there at most one solution passing it?
3. Continuous dependence: Consider two solutions passing (x_1, y_1) and $(x_2 . y_2)$ respectively, will the solutions remain close to each other if the two initial points are close?

For first order semilinear equation

$$y' = f(x, y), \tag{243}$$

the answer is quite clear.

Theorem 40. *Consider the equation $y' = f(x, y)$ and a point (x_0, y_0). Then if f is continuous and $\frac{\partial f}{\partial y}$ is uniformly bounded in a region R containing (x_0, y_0), then*

1. *There is a solution curve passing (x_0, y_0);*
2. *The solution is unique;*
3. *The solution depends continuously on the location of the point (x_0, y_0).*

Remark 41. The requirement on the uniform boundedness of $\frac{\partial f}{\partial y}$ is sharp. Consider the problem

$$y' = \begin{cases} y^\alpha & y \geqslant 0 \\ -|y|^\alpha & y < 0 \end{cases}, \qquad y(0) = 0. \tag{244}$$

There are two distinct solutions. Thus uniqueness does not hold.

On the other hand, existence is true under much relaxed conditions. For example, as long as f is continuous, existence holds.

When f is not even continuous, the situation is complicated. For example, the initial value problem

$$y' = \frac{2}{x}(y-1), \qquad y(0) = 0 \tag{245}$$

has no solution, while a slight change of the initial value

$$y' = \frac{2}{x}(y-1), \qquad y(0) = 1 \tag{246}$$

leads to infinite number of solutions $y(x) = 1 + C x^2$.

Remark 42. For general nonlinear ODEs $F(x, y, y') = 0$, the situation is much more complicated, even when F is a very nice function. For example, consider the initial value problem

$$(y')^2 = 1, \qquad y(0) = 0. \tag{247}$$

Here $F(x, y, p) = p^2 - 1$ is very nice, but clearly the solution is not unique. Furthermore, if we consider the boundary value problem

$$(y')^2 = 1, \qquad y(0) = y(1) = 0, \tag{248}$$

then existence fails.[16]

9. Summary.

The most concise summary of this section is the following:

Is the equation exact? Yes $\Longrightarrow$ Integrate; No $\Longrightarrow$ Make it exact, then integrate.

Now we turn to a more detailed summary. Suppose we are given a semilinear ODE

$$y' = f(x, y). \tag{249}$$

We check the following case by case:

1. Is it linear? If yes, write it in form $y' + P(x)\, y = Q(x)$, multiply by $e^{\int P}$ and integrate.
2. Is it separable? If yes, write it in form $y' = g(x)\, p(y)$, divide both sides by $p(y)$, integrate, then add back the constant solutions y_i by solving $p(y) = 0$.
3. Is it exact? If yes, write it in form $M\, \mathrm{d}x + N\, \mathrm{d}y = 0$, integrate $\int M(x, y)\, \mathrm{d}x$ and use N to determine the arbitrary function $g(y)$.
4. Can an integrating factor be found? If yes, find it and transform the equation into an exact one.
5. Is it homogeneous? If yes, use $v = y/x$ as a new unkonwn. The new equation will be separable.
6. Is it of the form $y' = G(a\, x + b\, y)$? If yes, use $z = a\, x + b\, y$ as a new unknown. The new equation will be linear.
7. Is it Bernoulli? If yes, use y^{1-n} as a new unknown. The new equation will be linear.
8. (Not required for Math 334 Fall 2010) Is it of the form $y' = F\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$? If yes, introduce $u = x + h, v = y + k$ and find appropriate h, k so that the equation becomes homogeneous.

16. A good test of calculus knowledge.

There are of course many equations that are not included in the above recipe, yet still can be transformed into exact equations through clever tricks. Like evaluating integral, finding explicit formulas for ODEs is largely an art and relies much on experience.