(Nov. 15, 12:00pm – 12:50pm, DP6069)

# Math 334 2010 Midterm 2

Name _________________________

ID# _________________________

Signature _________________________

- Only pen/pencil/eraser are allowed. Scratch papers will be provided.
- Please **write clearly**, with intermediate steps to **show sufficient work** even if you can solve the problem in "one go". Otherwise you may not receive full credit.
- Please box, underline, or highlight the most important parts of your answers.

| Problem | Points | Score |
|---|---|---|
| 1 | 35 | |
| 2 | 15 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 5 (extra) | |
| Total | 100+5 | |

**Problem 1. (5.2 14; 35 pts)** Consider

$$2\,y'' + (x+1)\,y' + 3\,y = 0 \tag{1}$$

at $x_0 = 2$.

a) (10 pts) Seek power series solutions at $x_0$, find the recurrence relation.

b) (20 pts) Find the first four nonzero terms in each of two solutions $y_1$ and $y_2$ (Hint: General solution $y = a_0\,y_1 + a_1\,y_2$).

c) (5 pts) By evaluating the Wronskian $W(y_1,\, y_2)(x_0)$, show that $y_1$ and $y_2$ are linearly independent (thus forming a fundamental set of solutions).

**Solution.**

a)

- (3 pts) Know the method: Either let $t = x - x_0 = x - 2$, re-write the equation, and then substitute $y = \sum a_n\,t^n$, or substitute $y = \sum a_n\,(x-2)^n$, to find the recurrence relation.

- (1 pts) Let $t = x - 2$. Then $x + 1 = t + 3$. So the equation becomes

$$2\,y'' + (t+3)\,y' + 3\,y = 0. \tag{2}$$

Now let $y = \sum_0^\infty a_n\,t^n$.

- (1 pts) Substitute into the equation:

$$2\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,t^{n-2} + t\sum_{n=1}^{\infty} n\,a_n\,t^{n-1} + 3\sum_{n=1}^{\infty} n\,a_n\,t^{n-1} + 3\sum_{n=0}^{\infty} a_n\,t^n = 0. \tag{3}$$

- (3 pts) Shift indices correctly.

$$2\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,t^{n-2} = \sum_{n=0}^{\infty} [2\,(n+2)\,(n+1)\,a_{n+2}]\,t^n; \tag{4}$$

$$t\sum_{n=1}^{\infty} n\,a_n\,t^{n-1} = \sum_{n=1}^{\infty} n\,a_n\,t^n; \tag{5}$$

$$3\sum_{n=1}^{\infty} n\,a_n\,t^{n-1} = \sum_{n=0}^{\infty} 3\,(n+1)\,a_{n+1}\,t^n. \tag{6}$$

- (1 pt) We have

$$\sum_{n=0}^{\infty} [2\,(n+2)\,(n+1)\,a_{n+2}]\,t^n + \sum_{n=1}^{\infty} n\,a_n\,t^n + \sum_{n=0}^{\infty} 3\,(n+1)\,a_{n+1}\,t^n + 3\sum_{n=0}^{\infty} a_n\,t^n \tag{7}$$

which can be simplified to

$$4\,a_2 + 3\,a_1 + 3\,a_0 + \sum_{n=1}^{\infty} [2\,(n+2)\,(n+1)\,a_{n+2} + 3\,(n+1)\,a_{n+1} + (n+3)\,a_n]\,t^n = 0. \tag{8}$$

- (1 pt) The recurrence relations are:

$$4\,a_2+3\,a_1+3\,a_0=0 \tag{9}$$

and[1]

$$2\,(n+2)\,(n+1)\,a_{n+2}+3\,(n+1)\,a_{n+1}+(n+3)\,a_n=0 \text{ for } n\geqslant 1. \tag{10}$$

b)

- (4 pts) Know the method.
- (8 pts) Computing $y_1$:
  - (1 pt) Set $a_0=1, a_1=0$. Using the recurrence relation we have
  - (2 pts) $a_2=-\frac{3}{2}\,(a_1+a_0)=-\frac{3}{4}$;
  - (2 pts) $a_3=-\frac{6\,a_2+4\,a_1}{12}=\frac{3}{8}$;
  - (2 pts) $a_4=-\frac{9\,a_3+5\,a_2}{24}=\frac{1}{64}$;
  - (1 pt) So

$$y_1=1-\frac{3}{4}\,(x-2)^2+\frac{3}{8}\,(x-2)^3+\frac{1}{64}\,(x-2)^4+\cdots \tag{11}$$

- (8 pts) Computing $y_2$:
  - (1 pt) Set $a_0=0, a_1=1$.
  - (2 pts) $a_2=-\frac{3}{4}$;
  - (2 pts) $a_3=\frac{1}{24}$;
  - (2 pts) $a_4=\frac{9}{64}$;
  - (1 pt) So

$$y_2=(x-2)-\frac{3}{4}\,(x-2)^2+\frac{1}{24}\,(x-2)^3+\frac{9}{64}\,(x-2)^4+\cdots \tag{12}$$

c)

- (1 pt) Know what Wronskian is:

$$W(y_1,y_2)=y_1\,y_2'-y_1'\,y_2. \tag{13}$$

- (1 pt) $y_1(2)=1, y_1'(2)=0$
- (1 pt) $y_2(2)=0, y_2'(2)=1$
- (1 pt) Compute

$$W(2)=1. \tag{14}$$

- (1 pt) Since $W(2)\neq 0$, the two solutions are linearly independent.

1. Coincidentally, the general relation for $n\geqslant 1$ is also true when $n=0$.

**Problem 2 (5.4 7; 15 pts)** Find the general solution of

$$x^2 y'' + 6 x y' - y = 0. \tag{15}$$

**Solution.**

- (4 pts) Recognize this is Euler equations and know how to solve it (that is the solution should be of the form $C_1 x^{r_1} + C_2 x^{r_2}$ with $r_1, r_2$ roots of the characteristic equation).

- (4 pts) Characteristic equation:

$$r (r - 1) + 6 r - 1 = 0 \tag{16}$$

  which can be simplified to

$$r^2 + 5 r - 1 = 0. \tag{17}$$

- (4 pts) Find roots correctly:

$$r_{1,2} = \frac{-5 \pm \sqrt{29}}{2}. \tag{18}$$

- (3 pts) Write down the solution[2]

$$y = C_1 x^{\left(-5+\sqrt{29}\right)/2} + C_2 x^{\left(-5-\sqrt{29}\right)/2}. \tag{19}$$

2. $|x|$ or $x$ both OK in this course.

**Problem 3 (4.2 23; 20 pts)** Find the general solution

$$y''' - 5\,y'' + 3\,y' + y = 0. \tag{20}$$

**Solution.**

- (4 pts) Know the general procedure.
- (4 pts) Characteristic equation:

$$r^3 - 5\,r^2 + 3\,r + 1 = 0. \tag{21}$$

- (3 pts) Find the first root:

$$r_1 = 1. \tag{22}$$

- (3 pts) Factorize:

$$r^3 - 5\,r^2 + 3\,r + 1 = (r-1)\left(r^2 - 4\,r - 1\right). \tag{23}$$

- (3 pts) Find the other two roots:

$$r_{2,3} = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}. \tag{24}$$

- (3 pts) Write down the solution:

$$y = C_1\,e^t + C_2\,e^{\left(2+\sqrt{5}\right)t} + C_3\,e^{\left(2-\sqrt{5}\right)t}. \tag{25}$$

**Problem 4. (5.5 2; 15 pts)** Consider the equation

$$x^2\, y'' + x\, y' + \left(x^2 - \frac{1}{9}\right) y = 0. \tag{26}$$

a) (7 pts) Show that the given differential equation has a regular singular point at $x = 0$.

b) (8 pts) Determine the indicial equation, and the roots of the indicial equation.

**Solution.**

a)

- (2 pts) Know the definition of a regular singular point.
- (1 pt) Write the equation into standard form
$$y'' + \frac{1}{x}\, y' + \frac{\left(x^2 - 1/9\right)}{x^2}\, y = 0. \tag{27}$$
- (2 pts) Check
$$x\, p = x\left(\frac{1}{x}\right) = 1 \tag{28}$$
is a polynomial and is therefore analytic.
- (2 pts) Check
$$x^2\, q = x^2 - 1/9 \tag{29}$$
is a polynomial and therefore is analytic.

**Remark 1.** I see many of your using the argument $\lim_{x\to 0} x\, p$ and $\lim_{x\to 0} x^2\, q$ are finite so the point 0 is regular singular. This is wrong as can be seen from the example

$$y'' + \ln|x|\, y' + y = 0. \tag{30}$$

where $\lim_{x\to 0} x \ln|x| = 0$ is finite but $x \ln|x|$ is NOT analytic at 0.

Searching in the textbook I found out the source of this mistake: the end of Section 5.4 on page 274. So your mistakes are understandable. However, I will still subtract 1 point (instead of the whole 4 point on checking analyticity) for the following reasons:

1. The "finite limit" criterion is not correct in the general case.
2. The textbook actually acknowledged this: Right above (31) on page 274, it is mentioned that "for equations with more general coefficients", different criterion needs to be used.

In the final, in case you really like this criterion, please remember to write something like "**since *p* and *q* are ratios of analytic functions**" in your solution.

b)

- (2 pts) Know what an indicial equation is.
- (2 pts) Compute
$$p_0 = \text{constant term in the expansion of } x\, p = 1. \tag{31}$$
- (2 pts) Compute
$$q_0 = \text{constant term in the expansion of } x^2\, q = -1/9. \tag{32}$$
- (1 pt) Indicial equation
$$r\,(r-1) + p_0\, r + q_0 = r^2 - 1/9 = 0 \tag{33}$$
- (1 pt) Roots:
$$r_{1,2} = \pm 1/3. \tag{34}$$

**Problem 5 (4.3 7; 15 pts)** Find the general solution of

$$y^{(6)} + y''' = t. \tag{35}$$

**Solution.**

- (3 pts) Know the procedure:
  1. (1 pt) Find the general solution of the homogeneous equation;
  2. (1 pt) Use undetermined coefficients to find one particular solution.
  3. (1 pt) Know what method of undertermined coefficient is.
- (5 pts) Find the general solution of the homogeneous equation

$$y^{(6)} + y''' = 0. \tag{36}$$

  - (1 pt) Characteristic equation:

$$r^6 + r^3 = 0. \tag{37}$$

  - (1 pt) Factorize:

$$r^3 \left(r^3 + 1\right) = 0 \tag{38}$$

    and obtain

$$r_{1,2,3} = 0. \tag{39}$$

  - (2 pts) Find the solutions of $r^3 + 1 = 0$:
    First write $-1 = R\,e^{i\theta}$, $R = |-1| = 1$, $\theta = \pi + 2\,k\,\pi$. Then write

$$r = (-1)^{1/3} = R^{1/3}\,e^{i\theta/3} = 1\,e^{i\frac{(2k+1)\pi}{3}}. \tag{40}$$

    Setting $k = 0, 1, 2$ (or $0, -1, 1$; Any 3 consecutive integers will do) to obtain

$$r_{4,5,6} = -1, \frac{1 \pm \sqrt{3}\,i}{2}. \tag{41}$$

  - (1 pt) General solution for homogeneous equation:[3]

$$y = C_1 + C_2\,t + C_3\,t^2 + C_4\,e^{-t} + C_5\,e^{t/2} \cos\left(\frac{\sqrt{3}}{2}\,t\right) + C_6\,e^{t/2} \sin\left(\frac{\sqrt{3}}{2}\,t\right). \tag{42}$$

- (6 pts) (Apply undetermined coefficients).
  - (2 pts) As the RHS is $t = t\,e^{0t}$, the form of the particular solution should be

$$t^s\,(A\,t + B)\,e^{0t}. \tag{43}$$

  - (2 pts) As 0 is a triple root of the characteristic equation of the homogeneous equation, $s = 3$.
  - (2 pts) Substitute $t^3\,(A\,t + B)$ into the equation to obtain $A = 1/24, B = 0$.
- (1 pt) The general solution is given by

$$C_1 + C_2\,t + C_3\,t^2 + C_4\,e^{-t} + C_5\,e^{t/2} \cos\left(\frac{\sqrt{3}}{2}\,t\right) + C_6\,e^{t/2} \sin\left(\frac{\sqrt{3}}{2}\,t\right) + \frac{t^4}{24}. \tag{44}$$

3. If not explicitly written down, the 1 pt is combined into the final answer. That is the final answer would worth 2 pts.

**Problem 6 (4.1 25; 5 pts)** Consider $f(t) = t^2 |t|$ and $g(t) = t^3$.

a) (1 pts) Show that $f, g$ are linearly dependent on $0 < t < 1$ and $-1 < t < 0$.

b) (2 pts) Show that $f$ and $g$ are linearly independent on $-1 < t < 1$.

c) (2 pt) Show that the Wronskian $W(f, g)(t)$ is zero for all $t$ in $-1 < t < 1$.

**Solution.**

a) $f - g = 0$ for $0 < t < 1$ and $f + g = 0$ for $-1 < t < 0$.

b) Let $C_1, C_2$ be constants such that $C_1 f + C_2 g = 0$ for all $-1 < t < 1$. All we need to show is $C_1 = C_2 = 0$.

The requirement is the same as

$$C_1 t^3 + C_2 t^3 = 0 \qquad 0 < t < 1 \tag{45}$$
$$-C_1 t^3 + C_2 t^3 = 0 \qquad -1 < t < 0 \tag{46}$$

which gives $C_1 + C_2 = 0$ and at the same time $-C_1 + C_2 = 0$. Therefore $C_1 = C_2 = 0$.

c) Recall

$$W(f, g) = \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} \tag{47}$$

is 0 whenever $f, g$ are linearly dependent. Therefore $W(f, g) = 0$ for $t > 0$ and $t < 0$. At $t = 0$, compute $g'(0) = 0$, $f'(0) = 0$,[4] it's clear that

$$W(f, g)(0) = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{49}$$

4. $f'$ is easy to compute for $t > 0$ and $t < 0$. At $t = 0$, use definition of derivatives:

$$f'(0) = \lim_{\delta \to 0} \frac{f(\delta) - f(0)}{\delta} = \lim_{\delta \to 0} \frac{\pm \delta^3 - 0}{\delta} = 0. \tag{48}$$