(Oct. 18, 12:00pm – 12:50pm, DP6069)

Math 334 2010 Midterm 1

Name ________________________

ID# ________________________

Signature ________________________

- Only pen/pencil/eraser are allowed. Scratch papers will be provided.
- Please **write clearly**, with intermediate steps to **show sufficient work** even if you can solve the problem in "one go". Otherwise you may not receive full credit.
- Please box, underline, or highlight the most important parts of your answers.

Problem	Points	Score
1	25	
2	25	
3	15	
4	20	
5	15	
6	5 (extra)	
Total	100+5	

Problem 1. (25 pts) Solve the initial value problem

$$y'' + 4\,y' + 4\,y = 0, \qquad y(-1) = 2,\ y'(-1) = 1. \tag{1}$$

Solution. This is problem 3.4 14.

- (5 pts) Characteristic equation:

$$r^2 + 4\,r + 4 = 0. \tag{2}$$

- (5 pts) Solutions:

$$r_1 = r_2 = -2. \tag{3}$$

- (4 pts) General solution

$$y = C_1\,e^{-2t} + C_2\,t\,e^{-2t}. \tag{4}$$

- (6 pts) Find C_1, C_2 using initial conditions:

  - (2 pts) $y(-1) = 2$ gives

$$C_1\,e^2 - C_2\,e^2 = 2; \tag{5}$$

  - (2 pts) Compute

$$y' = -2\,C_1\,e^{-2t} - 2\,C_2\,t\,e^{-2t} + C_2\,e^{-2t}. \tag{6}$$

  - (2 pts) $y'(-1) = 1$ gives

$$-2\,C_1\,e^2 + 3\,C_2\,e^2 = 1; \tag{7}$$

- (2 pts) Solve the equations. Multiply the first by 2 and add to the 2nd, we get $C_2 = 5\ e^{-2}$. then $C_1 = 7\,e^{-2}$.

- (3 pts) Solution is given by

$$y = 7\,e^{-2(t+1)} + 5\,t\,e^{-2(t+1)}. \tag{8}$$

Problem 2 (25 pts) Find the general solution

$$y' + 3\,y = t + e^{-2t}. \tag{9}$$

Solution. This is Problem 2.1 1.

It is a linear first order equation.

- (5 pts). $p(t) = 3$, so $e^{\int p} = e^{3t}$.
- (2 pts). We have

$$\left(e^{3t}\,y\right)' = e^{3t}\,t + e^{t}. \tag{10}$$

- (3 pts). This gives

$$e^{3t}\,y = \int e^{3t}\,t\,\mathrm{d}t + \int e^{t}\,\mathrm{d}t + C. \tag{11}$$

- (2 pts) Calculate

$$\int e^{t}\,\mathrm{d}t = e^{t}. \tag{12}$$

- (9 pts) Calculate $\int\, e^{3t}\,t$.

$$\begin{aligned} \int e^{3t}\,t\,\mathrm{d}t &= \int t\,\mathrm{d}\left(\frac{1}{3}\,e^{3t}\right) \qquad (3\,\mathrm{pts}) & (13)\\ &= \frac{t}{3}\,e^{3t} - \int\,\frac{1}{3}\,e^{3t}\,\mathrm{d}t \qquad (3\,\mathrm{pts}) & (14)\\ &= \frac{t}{3}\,e^{3t} - \frac{1}{9}\,e^{3t}. \qquad (3\,\mathrm{pts}) & (15) \end{aligned}$$

- (2 pts) We reach

$$e^{3t}\,y = \frac{1}{3}\,t\,e^{3t} - \frac{1}{9}\,e^{3t} + e^{t} + C. \tag{16}$$

- (2 pts) The solution then is

$$y = \frac{1}{3}t - \frac{1}{9} + e^{-2t} + C e^{-3t}. \tag{17}$$

Solution 2. As the equation is linear, we can solve it by solving

$$y' + 3y = 0 \Longrightarrow \tilde{y}; \quad y' + 3y = t \Longrightarrow y_{p1}; \quad y' + 3y = e^{-2t} \Longrightarrow y_{p2} \tag{18}$$

and have

$$y = \tilde{y} + y_{p1} + y_{p2}. \tag{19}$$

- (9 pts) Solve $y' + 3y = 0$.
  - (2 pts) Realize the need to solve $y' + 3y = 0$.
  - (5 pts) Find the correct integrating factor e^{3t}.
  - (2 pts) Write down the general solution

$$\tilde{y} = C e^{-3t}. \tag{20}$$

- (7 pts) Solve $y' + 3y = t$.
  - (2 pts) Realize the need to solve $y' + 3y = t$.
  - (2 pts) Guess $y = A t + B$.
  - (2 pts) Finding out $A = 1/3, B = -1/9$.
  - (1 pt) Write down

$$y_{p1} = \frac{t}{3} - \frac{1}{9}. \tag{21}$$

- (7 pts) Solve $y' + 3y = e^{-2t}$.
  - (2 pts) Realize the need to solve $y' + 3y = e^{-2t}$.
  - (2 pts) Guess $y = A e^{-2t}$.
  - (2 pts) Finding out $A = 1$.
  - (1 pt) Write down

$$y_{p2} = e^{-2t}. \tag{22}$$

- (2 pts) Write down the solution

$$y = C e^{-3t} + \frac{t}{3} - \frac{1}{9} + e^{-2t}. \tag{23}$$

Problem 3. (15 pts) Find the general solution for

$$t^2\, y'' + 4\, t\, y' + 2\, y = 0. \qquad t > 0. \tag{24}$$

Solution. This is 3.3 36.

- (2 pts) This is Euler equation.
- (5 pts) Guess $y = t^r$ to get the characteristic equation
$$r\,(r-1) + 4\, r + 2 = 0. \tag{25}$$
- (4 pts) Solve
$$r_1 = -1, \quad r_2 = -2. \tag{26}$$
- (4 pts) Write down
$$y = C_1\, t^{-1} + C_2\, t^{-2}. \tag{27}$$

Solution 2.

- (3 pts) Guess one solution t^{-1} or t^{-2}.
- (8 pts) Carry out reduction of order correctly. Note that you need to write the equation into
$$y'' + \frac{4}{t}\, y' + \frac{2}{t^2}\, y = 0 \tag{28}$$
before using the formula
$$y_2 = y_1 \int \frac{e^{-\int p}}{y_1^2}. \tag{29}$$
- (4 pts) Write down the solution.

Problem 4 (20 pts) Find the general solution:

$$y\,\mathrm{d}x + \left(2\,x\,y - e^{-2\,y}\right)\mathrm{d}y = 0. \tag{30}$$

Solution. This is 2.6 28.

- (2 pts) $M = y, N = 2\,x\,y - e^{-2\,y}$.
- (2 pts) Check
$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 2\,y \tag{31}$$
  $1 \neq 2\,y$ so not exact!
- (2 pts). Need μ such that
$$\frac{\partial(\mu\,M)}{\partial y} = \frac{\partial(\mu\,N)}{\partial x} \tag{32}$$
  which leads to
$$M\,\frac{\partial \mu}{\partial y} - N\,\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu. \tag{33}$$
- (1 pt) This is
$$y\,\frac{\partial \mu}{\partial y} - \left(2\,x\,y - e^{-2\,y}\right)\frac{\partial \mu}{\partial x} = (2\,y - 1)\,\mu. \tag{34}$$
- (1 pts). Guess $\mu(x)$, not working. Guess $\mu(y)$.
- We have
$$y\,\mu' = (2\,y - 1)\,\mu \qquad (1\,\mathrm{pt}) \tag{35}$$
$$\frac{\mu'}{\mu} = 2 - \frac{1}{y} \qquad (1\,\mathrm{pt}) \tag{36}$$
$$(\ln\mu)' = 2\,y - \ln y \qquad (1\,\mathrm{pt}) \tag{37}$$
$$\mu = e^{2\,y}/y. \qquad (1\,\mathrm{pt}) \tag{38}$$
- (2 pts) Multiply the equation
$$e^{2\,y}\,\mathrm{d}x + \left(2\,x\,e^{2\,y} - 1/y\right)\mathrm{d}y = 0. \tag{39}$$
- (2 pts)
$$\frac{\partial u}{\partial x} = e^{2\,y} \Longrightarrow u(x,y) = x\,e^{2\,y} + g(y). \tag{40}$$
- (2 pts)
$$\frac{\partial u}{\partial y} = 2\,x\,e^{2\,y} + g'(y) = 2\,x\,e^{2\,y} - 1/y \Longrightarrow g'(y) = -1/y \Longrightarrow g = -\ln|y|. \tag{41}$$
- (1 pt) The solution is
$$x\,e^{2\,y} - \ln|y| = C. \tag{42}$$
- (1 pt) Finally check $y = 0$ as our multiplier is $e^{2\,y}/y$. Rigorously speaking $y = 0$ is a solution. But it's OK for this exam if you conclude $-1\,\mathrm{d}y = 0$ is not satisfied by $y = 0$.

Problem 5 (15 pts) Can $y=\sin\left(t^2\right)$ be a solution on an interval containing $t=0$ of an equation $y''+p(t)\,y'+q(t)\,y=0$ with continuous coefficients? Explain your answer.

Solution. This is 3.2 16.

- (1 pt) Plug $y=\sin\left(t^2\right)$ into the equation.
- Compute

$$\left(\sin t^2\right)' = 2\,t\cos t^2 \qquad (2\,\text{pts}) \tag{43}$$
$$\left(\sin t^2\right)'' = 2\cos t^2 - 4\,t^2\sin t^2 \qquad (2\,\text{pts}) \tag{44}$$

- (3 pts) Thus

$$2\cos t^2 - 4\,t^2\sin t^2 + 2\,p(t)\,t\cos t^2 + q(t)\sin t^2 = 0. \tag{45}$$

- (5 pts) As p, q are continuous on an interval containing 0, $p(0)$, $q(0)$ are finite. Taking $t=0$ in the above we get

$$2-0+0+0=0. \tag{46}$$

  A contradiction!
- (2 pts) So the answer is No.

Problem 6 (5 pts) Assume that p and q are continuous and that the functions y_1, y_2 are solutions of the differential equation $y'' + p(t)\ y' + q(t)\ y = 0$ on an open interval I. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be linearly independent over I.

Solution. This is 3.2 39.

- (2 pts) Let x_0 be the point. Then as y_1 reaches maximum/minimum there, $y_1'(x_0) = 0$; Similarly $y_2'(x_0)$=0.
- (2 pts) Compute the Wronskian:

$$W(y_1, y_2)(x_0) = y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0) = 0. \tag{47}$$

- (1 pt) Therefore y_1, y_2 cannot be linearly independent.