Math 334 A1 Homework 5 (Due Dec. 8 5pm)

- No "Advanced" or "Challenge" problems will appear in homeworks.

Basic Problems

Problem 1. (6.3 1) Sketch for $t \geqslant 0$

$$g(t) = u(t-1) + 2\,u(t-3) - 6\,u(t-4). \tag{1}$$

Solution. We see that $g(t) = \begin{cases} 0 & t<1 \\ 1 & 1\leqslant t<3 \\ 3 & 3\leqslant t<4 \\ -3 & t\geqslant 4 \end{cases}$.

Problem 2. (6.3 12) Express

$$f(t) = \begin{cases} t & 0\leqslant t<2 \\ 2 & 2\leqslant t<5 \\ 7-t & 5\leqslant t<7 \\ 0 & t\geqslant 7 \end{cases} \tag{2}$$

in terms of the unit step function.

Solution. We have

$$f(t) = t + (2-t)\,u(t-2) + [(7-t)-2]\,u(t-5) + [0-(7-t)]\,u(t-7) \tag{3}$$

which after simplification becomes

$$f(t) = t + (2-t)\,u(t-2) + (5-t)\,u(t-5) + (t-7)\,u(t-7). \tag{4}$$

Problem 3. (6.3 13) Find the Laplace transform for

$$f(t) = \begin{cases} 0 & t<2 \\ (t-2)^2 & t\geqslant 2 \end{cases}. \tag{5}$$

Solution. First we write

$$f(t) = (t-2)^2\,u(t-2). \tag{6}$$

To find its Laplace transform we use the formula

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}. \tag{7}$$

Here $g(t) = (t-2)^2$ and $a=2$, thus $g(t+a) = g(t+2) = ((t+2)-2)^2 = t^2$. Therefore

$$\mathcal{L}\{f\} = e^{-2s}\,\mathcal{L}\{t^2\} = \frac{2\,e^{-2s}}{s^3}. \tag{8}$$

Problem 4. (6.3 21) Find the inverse Laplace transform of

$$F(s) = \frac{2\,(s-1)\,e^{-2s}}{s^2-2\,s+2}. \tag{9}$$

Solution. Spotting e^{-2s} we know that the step function is involved. We use the formula

$$\mathcal{L}^{-1}\{e^{-as}\,F(s)\} = f(t-a)\,u(t-a). \tag{10}$$

Here $a=2$, $F(s) = \frac{2\,(s-1)}{(s^2-2\,s+2)}$. We compute

$$f(t) = \mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{2\,(s-1)}{(s-1)^2+1}\right\} = 2\,e^t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = 2\,e^t\cos t. \tag{11}$$

Therefore

$$f(t-2) = 2\,e^{t-2}\cos(t-2). \tag{12}$$

Finally

$$\mathcal{L}^{-1}\{F\} = 2\,e^{t-2}\cos(t-2)\,u(t-2). \tag{13}$$

Problem 5. (6.3 24) Find the inverse Laplace transform of

$$F(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}. \tag{14}$$

Solution. We have

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} = u(t-1) + u(t-2) - u(t-3) - u(t-4). \quad (15)$$

INTERMEDIATE PROBLEMS

Problem 6. (6.3 33) Find the Laplace transform of

$$f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u(t-k). \quad (16)$$

(You can assume that term-by-term integration is permissible)
Solution. We have

$$\begin{aligned}\mathcal{L}\{f\} &= \mathcal{L}\{1\} + \sum_{k=1}^{\infty} (-1)^k \mathcal{L}\{u(t-k)\}\\ &= \frac{1}{s} + \sum_{k=1}^{\infty} (-1)^k \mathcal{L}\{u(t-k)\cdot 1\}\\ &= \frac{1}{s} + \sum_{k=1}^{\infty} (-1)^k \frac{e^{-ks}}{s}\\ &= \frac{1}{s}\left[\sum_{k=0}^{\infty} (-1)^k e^{-ks}\right]\\ &= \frac{1}{s}\left[\sum_{k=0}^{\infty} (-e^{-s})^k\right]\\ &= \frac{1}{s}\frac{1}{1+e^{-s}}. \end{aligned} \quad (17)$$

Problem 7. (6.4 9) Solve

$$y'' + y = g(t) = \begin{cases} t/2 & 0 \leqslant t < 6 \\ 3 & t \geqslant 6 \end{cases}, \qquad y(0) = 0, \quad y'(0) = 1. \quad (18)$$

Solution. First write

$$g(t) = t/2 + (3 - t/2)\, u(t-6). \quad (19)$$

Now compute

$$\mathcal{L}\{y''\} = s^2 Y - s\, y(0) - y'(0) = s^2 Y - 1. \quad (20)$$

$$\mathcal{L}\{g\} = \mathcal{L}\{t/2\} + \mathcal{L}\{(3 - t/2)\, u(t-6)\} = \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left\{3 - \frac{t+6}{2}\right\} = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}. \quad (21)$$

The transformed equation is then

$$s^2 Y + Y = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2} + 1 \Longrightarrow Y = \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)} + \frac{1}{s^2+1}. \quad (22)$$

To compute y, we compute the inverse Laplace transform of the right hand side one by one.

- $\mathcal{L}^{-1}\left\{\frac{1}{2s^2(s^2+1)}\right\}$. We use partial fraction.[1] Write

$$\frac{1}{2s^2(s^2+1)} = \frac{1/2}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{A\,s\,(s^2+1) + B\,(s^2+1) + (Cs+D)\,s^2}{s^2(s^2+1).}. \quad (23)$$

  Thus we have

$$\frac{1}{2} = A\,s\,(s^2+1) + B\,(s^2+1) + (Cs+D)\,s^2. \quad (24)$$

  Setting $s = 0$ we have

$$B = 1/2. \quad (25)$$

  Setting $s = \pm i$ we have[2]

$$\frac{1}{2} = -(D + Ci); \qquad \frac{1}{2} = -(D - Ci) \Longrightarrow C = 0, D = -1/2. \quad (26)$$

  Finally comparing the coefficients of the highest order term s^3:

$$0 = A + C \Longrightarrow A = 0. \quad (27)$$

1. In this problem it is possible to skip the following calculation by realizing $1/s^2 - 1/(s^2+1) = 1/[s^2(s^2+1)]$.

2. $s = \pm i$ makes $s^2 + 1 = 0$. The disadvantage of setting s to be complex number is that the calculation is usually much more complicated than the real case. However, our case here is among the simplest so it doesn't cause much trouble.

So the partial fraction representation is

$$\frac{1}{2\,s^2\,(s^2+1)}=\frac{1}{2\,s^2}-\frac{1}{2\,(s^2+1)}. \tag{28}$$

Consequently

$$\mathcal{L}^{-1}\left\{\frac{1}{2\,s^2\,(s^2+1)}\right\}=\frac{t}{2}-\frac{1}{2}\sin t. \tag{29}$$

- $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{2\,s^2\,(s^2+1)}\right\}$. Recall the formula

$$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\}=f(t-a)\,u(t-a). \tag{30}$$

Here $a=6$, $F=\frac{1}{2\,s^2\,(s^2+1)}$. As we have just computed

$$f=\mathcal{L}^{-1}\left\{\frac{1}{2\,s^2\,(s^2+1)}\right\}=\frac{t}{2}-\frac{1}{2}\sin t \tag{31}$$

We can immediately write down

$$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{2\,s^2\,(s^2+1)}\right\}=\left[\frac{t}{2}-3-\frac{1}{2}\sin(t-6)\right]u(t-6). \tag{32}$$

- The last term is standard:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t. \tag{33}$$

Putting everything together we have

$$y=\frac{t}{2}+\frac{1}{2}\sin t-\left[\frac{t}{2}-3-\frac{1}{2}\sin(t-6)\right]u(t-6). \tag{34}$$

Problem 8. (6.5 10) Solve

$$2\,y''+y'+4\,y=\delta(t-\pi/6)\sin t;\qquad y(0)=0,\quad y'(0)=0. \tag{35}$$

Solution. We compute

$$\mathcal{L}\{y''\}=s^2\,Y-s\,y(0)-y'(0)=s^2\,Y;\qquad \mathcal{L}\{y'\}=s\,Y-y(0)=s\,Y; \tag{36}$$

$$\mathcal{L}\{\delta(t-\pi/6)\sin t\}=\int_0^\infty e^{-st}\sin t\,\delta(t-\pi/6)\,\mathrm{d}t=e^{-\frac{\pi}{6}s}\sin\frac{\pi}{6}=\frac{1}{2}\,e^{-\frac{\pi}{6}s}. \tag{37}$$

The transformed equation is then

$$\left(2\,s^2+s+4\right)Y=\frac{1}{2}\,e^{-\frac{\pi}{6}s} \tag{38}$$

leading to

$$Y=\frac{1}{2}\,e^{-\frac{\pi}{6}s}\,\frac{1}{2\,s^2+s+4}. \tag{39}$$

To performe the inverse transform, we use the formula

$$\mathcal{L}^{-1}\{e^{-a\,s}\,F(s)\}=f(t-a)\,u(t-a). \tag{40}$$

with $a=\pi/6$ and $F(s)=\frac{1}{2\,s^2+s+4}$.

First compute

$$\mathcal{L}^{-1}\left\{\frac{1}{2\,s^2+s+4}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{1}{(s+1/4)^2+\frac{31}{16}}\right\}=\frac{1}{2}\,\frac{4}{\sqrt{31}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{31}/4}{(s+1/4)^2+\frac{31}{16}}\right\}=\frac{2}{\sqrt{31}}\,e^{-t/4}\sin\left(\frac{\sqrt{31}}{4}\,t\right). \tag{41}$$

Now we have

$$y=\frac{1}{2}\,\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{6}s}\,\frac{1}{2\,s^2+s+4}\right\}=\frac{1}{\sqrt{31}}\,e^{-(t-\pi/6)/4}\sin\left(\frac{\sqrt{31}}{4}\,(t-\pi/y)\right)u(t-\pi/6). \tag{42}$$

Problem 9. (6.6 4) Find the Laplace transform of

$$f(t)=\int_0^t(t-\tau)^2\cos 2\,\tau\,\mathrm{d}\tau. \tag{43}$$

Solution. Recognize that

$$f(t)=t^2*\cos(2\,t). \tag{44}$$

Consequently

$$\mathcal{L}\{f\}=\mathcal{L}\{t^2\}\,\mathcal{L}\{\cos 2\,t\}=\frac{2}{s^3}\,\frac{s}{s^2+4}=\frac{2}{s^2\,(s^2+4)}. \tag{45}$$

Problem 10. (6.6 8) Find the inverse Laplace transform using the convolution theorem

$$F(s)=\frac{1}{s^4\,(s^2+1)}. \tag{46}$$

Solution. We have

$$\begin{aligned}
\mathcal{L}^{-1}\{F\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= \frac{1}{6} t^3 * \sin t \\
&= \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, \mathrm{d}\tau \\
&= \frac{1}{6}\left[-\int_0^t (t-\tau)^3 \,\mathrm{d}\cos\tau\right] \\
&= \frac{1}{6}\left[-(t-\tau)^3 \cos\tau\,|_0^t + \int_0^t \cos\tau \,\mathrm{d}(t-\tau)^3\right] \\
&= \frac{1}{6}\left[t^3 - 3\int_0^t (t-\tau)^2 \cos\tau \,\mathrm{d}\tau\right] \\
&= \frac{t^3}{6} - \frac{1}{2}\int_0^t (t-\tau)^2 \,\mathrm{d}\sin\tau \\
&= \frac{t^3}{6} - \frac{1}{2}\left[(t-\tau)^2 \sin\tau\,|_0^t - \int_0^t \sin\tau \,\mathrm{d}(t-\tau)^2\right] \\
&= \frac{t^3}{6} - \frac{1}{2}\left[2\int_0^t (t-\tau)\sin\tau \,\mathrm{d}\tau\right] \\
&= \frac{t^3}{6} + \int_0^t (t-\tau)\,\mathrm{d}\cos\tau \\
&= \frac{t^3}{6} + (t-\tau)\cos\tau\,|_0^t - \int_0^t \cos\tau \,\mathrm{d}(t-\tau) \\
&= \frac{t^3}{6} - t + \int_0^t \cos\tau \,\mathrm{d}\tau \\
&= \frac{t^3}{6} - t + \sin t. \qquad (47)
\end{aligned}$$

Problem 11. (7.1 6) Transform the given initial value problem into an initial value problem of two first order equations:

$$u'' + p(t)\,u' + q(t)\,u = g(t), \qquad u(0) = u_0, \quad u'(0) = u_0'. \qquad (48)$$

Solution. Let $v = u'$. We have

$$u'' = -p(t)\,u' - q(t)\,u + g(t) \Longrightarrow v' = -p(t)\,v - q(t)\,u + g(t). \qquad (49)$$

Thus the two first order equations are

$$u' = v \qquad (50)$$

$$v' = -p(t)\,v - q(t)\,u + g(t) \qquad (51)$$

with initial values

$$u(0) = u_0, \qquad v(0) = u_0'. \qquad (52)$$