

UNBOUNDED NORM TOPOLOGY BEYOND NORMED LATTICES

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ABSTRACT. In this paper, we generalize the concept of unbounded norm (un) convergence: let X be a normed lattice and Y a vector lattice such that X is an order dense ideal in Y ; we say that a net (y_α) un-converges to y in Y with respect to X if $\| |y_\alpha - y| \wedge x \| \rightarrow 0$ for every $x \in X_+$. We extend several known results about un-convergence and un-topology to this new setting. We consider the special case when Y is the universal completion of X . If $Y = L_0(\mu)$, the space of all μ -measurable functions, and X is an order continuous Banach function space in Y , then the un-convergence on Y agrees with the convergence in measure. If X is atomic and order complete and $Y = \mathbb{R}^A$ then the un-convergence on Y agrees with the coordinate-wise convergence.

1. INTRODUCTION AND PRELIMINARIES

All vector lattices in this paper are assumed to be Archimedean. A net (x_α) in a normed lattice X is said to **un-converge** to x if $|x_\alpha - x| \wedge u \rightarrow 0$ in norm for every $u \in X_+$. This convergence, as well as the corresponding **un-topology**, has been introduced and studied in [Tro04, DOT17, KMT17]. In particular, if $X = L_p(\mu)$ for a finite measure μ and $1 \leq p < \infty$, then un-convergence agrees with convergence in measure. Convergence in measure naturally extends to $L_0(\mu)$, the space of all measurable functions (as usual, we identify functions

Date: October 17, 2017.

2010 Mathematics Subject Classification. Primary: 46B42. Secondary: 46A40.

Key words and phrases. Banach lattice, normed lattice, vector lattice, un-convergence, un-topology.

The first author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0222). The third author was supported by an NSERC grant.

that are equal a.e.). However, $L_0(\mu)$ is not a normed lattice, so that the preceding definition of un-convergence does not apply to $L_0(\mu)$. Motivated by this example, we generalize un-convergence as follows.

Throughout the paper, X is a normed lattice and Y is a vector lattice such that X is an ideal in Y . For a net (y_α) in Y , we say that y_α **un-converges to $y \in Y$ with respect to X** if $\| |y_\alpha - y| \wedge x \| \rightarrow 0$ for every $x \in X_+$; we write $y_\alpha \xrightarrow{\text{un-}X} y$. In the case when (y_α) and y are in X , it is easy to see that $y_\alpha \xrightarrow{\text{un-}X} y$ iff $y_\alpha \xrightarrow{\text{un}} y$ in X . Therefore, the new convergence is an extension of the un-convergence on X . We will write $y_\alpha \xrightarrow{\text{un}} y$ instead of $y_\alpha \xrightarrow{\text{un-}X} y$ when there is no confusion.

In this paper, we study properties of extended un-convergence and un-topology. We show that many properties of the original un-convergence remain valid in this setting. We study when this convergence is independent of the choice of X . We consider several special cases: when Y is the universal completion of X , when X is atomic and $Y = \mathbb{R}^A$, and when $Y = L_0(\mu)$ and X is a Banach function space in $L_0(\mu)$.

Example 1.1. Let $X = L_p(\mu)$ where μ is a finite measure and $1 \leq p < \infty$, and $Y = L_0(\mu)$. A net in $L_0(\mu)$ un-converges to zero with respect to $L_p(\mu)$ iff it converges to zero in measure. The proof is analogous to Example 23 of [Tro04]. In particular, $L_p(\mu)$ spaces for all p in $[1, \infty)$ induce the same un-convergence on $L_0(\mu)$; namely, the convergence in measure.

Just as for the original un-convergence, we have $y_\alpha \xrightarrow{\text{un}} y$ in Y iff $|y_\alpha - y| \xrightarrow{\text{un}} 0$. This often allows one to reduce general un-convergence to the un-convergence of positive nets to zero. The proof of the following fact is similar to that of [DOT17, Lemma 2.1].

Proposition 1.2. *Un-convergence in Y preserves algebraic and lattice operations.*

In Section 7 of [DOT17], it was observed that in the case when $X = Y$ the un-convergence on a normed lattice is given by a topology, and a base of zero neighbourhoods for that topology was given. We will now proceed analogously. Given a positive real number ε and a

positive vector $x \in X$, define

$$U_{\varepsilon,x} = \left\{ y \in Y : \| |y| \wedge x \| < \varepsilon \right\}.$$

Similarly to Section 7 of [DOT17], one may verify that

- zero is contained in $U_{\varepsilon,x}$ for all $\varepsilon > 0$ and $x \in X_+$;
- For every $\varepsilon_1, \varepsilon_2 > 0$ and every $x_1, x_2 \in X_+$ there exists $\varepsilon > 0$ and $x \in X_+$ such that $U_{\varepsilon,x} \subseteq U_{\varepsilon_1,x_1} \cap U_{\varepsilon_2,x_2}$;
- Given $y \in U_{\varepsilon,x}$ for some $y \in Y$, $x \in X_+$, and $\varepsilon > 0$, we have $y + U_{\delta,x} \subseteq U_{\varepsilon,x}$ for some $\delta > 0$.

For every $y \in Y$, define the family \mathcal{N}_y of subsets of Y as follows: $W \in \mathcal{N}_y$ if $y + U_{\varepsilon,x} \subseteq W$ for some $\varepsilon > 0$ and $x \in X_+$. It follows from, e.g., Theorem 3.1.10 of [Run05] that there is a unique topology on Y such that \mathcal{N}_y is exactly the set of all neighbourhoods of y for every $y \in Y$. It is also easy to see that $y_\alpha \xrightarrow{\text{un}} y$ in Y iff for every $\varepsilon > 0$ and every $x \in X_+$, the set $U_{\varepsilon,x}$ contains a tail of the net $(y_\alpha - y)$; it follows that the un-convergence on Y with respect to X is exactly the convergence with respect to this topology. We call it the ***un-topology on Y induced by X*** .

There are, however, two important differences with [DOT17]. First, unlike in [DOT17], this topology need not be Hausdorff.

Example 1.3. Let $Y = L_p(\mu)$, where μ is a finite measure and $1 \leq p < \infty$; let X be a band in Y , i.e., $X = L_p(A, \mu)$, where A is a measurable set. Consider the un-convergence on Y with respect to X . In this case, for every net (y_α) in the disjoint complement X^d of X in Y and every $y \in X^d$ we have $y_\alpha \xrightarrow{\text{un}-X} y$. This shows that un-limits need not be unique, so that un-topology need not be Hausdorff. Note, also, that the un-convergence on Y induced by X is different from the “native” un-convergence of Y .

Recall that a sublattice F of a vector lattice E is

- ***order dense*** if for every non-zero $x \in E_+$ there exists $y \in F$ such that $0 < y \leq x$; and
- ***majorizing*** if for every $x \in E_+$ there exists $y \in F$ with $x \leq y$.

An ideal F of E is order dense iff $u = \sup\{u \wedge v : v \in F_+\}$ for every $u \in E_+$; see [AB03, Theorem 1.27].

Proposition 1.4. *The un-topology on Y induced by X is Hausdorff iff X is order dense in Y .*

Proof. Suppose that the topology is Hausdorff and let $0 < y \in Y$. Then $y \notin U_{\varepsilon, x}$ for some $\varepsilon > 0$ and some $x \in X_+$. It follows that $y \wedge x \neq 0$. Note that $y \wedge x \in X$ and $y \wedge x \leq y$. Therefore, X is order dense.

Conversely, suppose that X is order dense and take $0 \neq y \in Y$. Find $x \in X$ with $0 < x \leq |y|$. Then $y \notin U_{\varepsilon, x}$ where $\varepsilon = \|x\|$. \square

The second important difference between our setting and that of [DOT17] is linearity. It is shown in [DOT17] that the base zero neighbourhoods $V_{u, \varepsilon}$ are absorbing; then Theorem 5.1 of [KN76] is used to conclude that the resulting topology is linear. In our setting, however, the sets $U_{\varepsilon, x}$ need not be absorbing and the topology need not be linear.

Example 1.5. Let $X = \ell_\infty$ and $Y = \mathbb{R}^{\mathbb{N}}$. Put $x = \mathbb{1}$, the constant one sequence, $\varepsilon = 1$, and $z = (1, 2, 3, \dots)$. Then $U_{\varepsilon, x} = \{y \in Y : \sup |y_i| < 1\}$. It is easy to see that no scalar multiple of z is in $U_{\varepsilon, x}$, hence $U_{\varepsilon, x}$ is not absorbing. It also follows that the sequence $\frac{1}{n}z$ does not un-converge to zero as $n \rightarrow \infty$. This shows that the un-topology on Y induced by X is not linear.

However, it is easy to see that the un-topology on Y induced by X is translation invariant. Moreover, addition is jointly continuous by Proposition 1.2, so the problem is only with the continuity of scalar multiplication. It is easy to see that the un-topology on Y generated by X is linear when X is order continuous or when Y is a normed lattice and $\|\cdot\|_X$ and $\|\cdot\|_Y$ agree on X . Note also that, in general, the restriction of this topology to X agrees with the “native” un-topology of X , which is Hausdorff and linear.

We finish the introduction with the following two easy facts that will be used throughout the rest of the paper. The following is an analogue of [KMT17, Lemma 1.2].

Proposition 1.6. *Suppose that X is order dense in Y . If $y_\alpha \uparrow$ and $y_\alpha \xrightarrow{\text{un}} y$ in Y then $y_\alpha \uparrow y$.*

Proof. Without loss of generality, $y_\alpha \geq 0$ for each α ; otherwise, pass to a tail $(y_\alpha)_{\alpha \geq \alpha_0}$ and consider the net $(y_\alpha - y_{\alpha_0})_{\alpha \geq \alpha_0}$. Therefore, we may assume that $y \geq 0$ by Proposition 1.2. We claim that $y_\alpha \wedge x \uparrow y \wedge x$ for every $x \in X_+$. Indeed, it follows from $|y_\alpha - y| \wedge x \xrightarrow{\|\cdot\|} 0$ that $y_\alpha \wedge x \xrightarrow{\|\cdot\|} y \wedge x$. Since the net $(y_\alpha \wedge x)$ is increasing, it follows that $y_\alpha \wedge x \uparrow y \wedge x$. This proves the claim.

Since X is order dense in Y , we have

$$y = \sup_{x \in X_+} y \wedge x = \sup_{x \in X_+} \sup_{\alpha} y_\alpha \wedge x = \sup_{\alpha} \sup_{x \in X_+} y_\alpha \wedge x = \sup_{\alpha} y_\alpha.$$

□

For a vector lattice E , we write E^δ for the order (Dedekind) completion of E . Recall that E is order dense and majorizing in E^δ ; moreover, these properties characterize E^δ . Suppose that F is an ideal in E . Let Z be the ideal generated by F in E^δ . It is easy to see that F is order dense and majorizing in Z ; it follows that we may identify Z with F^δ . Therefore, if F is an ideal of E then F^δ may be viewed as an ideal in E^δ .

In particular, we view X^δ as an ideal in Y^δ . The norm on X admits an extension to a lattice norm on X^δ ; see, e.g., [Vul67, p. 179] or [AA02, p. 26]. Note that such an extension need not be unique; see [Sol66]. The following proposition is valid for any such extension; the proof is straightforward.

Proposition 1.7. *Let X be a normed lattice which is an ideal in a vector lattice Y , and (y_α) a net in Y . Then $y_\alpha \xrightarrow{\text{un-}X} 0$ in Y iff $y_\alpha \xrightarrow{\text{un-}X^\delta} 0$ in Y^δ .*

2. UNIQUENESS OF UN-TOPOLOGY

Example 1.3 shows that un-convergence on Y may depend on the choice of X . Here is another example of the same phenomenon.

Example 2.1. Let $Y = C[0, 1]$ equipped with the supremum norm and let X be the subspace of Y consisting of all the functions which vanish at 0. Then X is an order dense ideal in Y which is not norm dense. Let $f_n \in Y$ be such that $\|f_n\| = 1$ and $\text{supp } f_n = [\frac{1}{n+1}, \frac{1}{n}]$. Then (f_n)

un-converges to zero in Y with respect to X , but fails to un-converge in Y (with respect to Y). Therefore, the un-topology in Y induced by X does not agree with the “native” un-topology in Y .

In the rest of this section, we consider situations when different normed ideals of Y induce the same un-topology on Y .

Proposition 2.2. *Let Z be a norm dense ideal in X . Then Z and X induce the same un-topology on Y .*

Proof. It suffices to show that $y_\alpha \xrightarrow{\text{un-}X} 0$ iff $y_\alpha \xrightarrow{\text{un-}Z} 0$ for every net (y_α) in Y_+ . It is clear that $y_\alpha \xrightarrow{\text{un-}X} 0$ implies $y_\alpha \xrightarrow{\text{un-}Z} 0$. To prove the converse, suppose that $y_\alpha \xrightarrow{\text{un-}Z} 0$; fix $x \in X_+$ and $\varepsilon > 0$. Find $z \in Z_+$ such that $\|x - z\| < \varepsilon$. By assumption, $y_\alpha \wedge z \rightarrow 0$. This implies that there exists α_0 such that $\|y_\alpha \wedge z\| < \varepsilon$ whenever $\alpha \geq \alpha_0$. It follows that

$$\|y_\alpha \wedge x\| \leq \|y_\alpha \wedge z\| + \|x - z\| < 2\varepsilon,$$

so that $y_\alpha \xrightarrow{\text{un-}X} 0$. □

Example 2.3. Let $X = c_0$ and $Y = \ell_\infty$. Since Y has a strong unit, the “native” un-topology on Y agrees with its norm topology. We claim that the un-convergence induced on Y by X is the coordinate-wise convergence. Indeed, if $y_\alpha \xrightarrow{\text{un-}X} 0$ in Y then $|y_\alpha| \wedge e_i \rightarrow 0$ for every i , where e_i is the i -th unit vector in c_0 ; it follows that y_α converges to zero coordinate-wise. Conversely, if y_α converges to zero coordinate-wise in Y then $|y_\alpha| \wedge x \rightarrow 0$ for every $x \in c_{00}$, so that $y_\alpha \xrightarrow{\text{un-}c_{00}} 0$. Proposition 2.2 now yields that $y_\alpha \xrightarrow{\text{un-}c_0} 0$.

In Proposition 2.2, the norm on Z was the restriction of the norm of X . We would now like to consider situations where X and Z have different norms, e.g., $X = L_p(\mu)$ and $Z = L_q(\mu)$, where μ is a finite measure and $1 \leq p \leq q < \infty$. We need the following lemma, which is a variant of Amemiya’s Theorem (see, e.g., Theorem 2.4.8 in [MN91]). We provide a proof for completeness.

Lemma 2.4. *Let X be a Banach lattice and Z an order continuous normed lattice such that Z continuously embeds into X as an ideal. Then the norm topologies of X and Z agree on order intervals of Z .*

Proof. Fix $u \in Z_+$ and let (z_n) be a sequence in $[-u, u]$. Since the embedding of Z into X is continuous, if $z_n \rightarrow z$ in Z then $z_n \rightarrow z$ in X . Suppose now that $z_n \rightarrow z$ in X . Then $z \in [-u, u]$, hence $z \in Z$. Without loss of generality, $z = 0$. For the sake of contradiction, suppose that (z_n) does not converge to zero in Z . Passing to a subsequence, we can find $\varepsilon > 0$ such that $\|z_n\|_Z > \varepsilon$ for every n . Since $\|z_n\|_X \rightarrow 0$, passing to a further subsequence we may assume that $z_n \xrightarrow{\circ} 0$ in X . Since $z_n \in [-u, u]$ and Z is an ideal in X , we conclude that $z_n \xrightarrow{\circ} 0$ in Z . Since Z is order continuous, this yields $\|z_n\|_Z \rightarrow 0$; a contradiction. \square

Theorem 2.5. *Suppose that X is a Banach lattice and Z is an order continuous normed lattice such that Z continuously embeds into X as a norm dense ideal. Then X and Z induce the same un-topology on Y .*

Proof. Again, it suffices to show that $y_\alpha \xrightarrow{\text{un-}X} 0$ iff $y_\alpha \xrightarrow{\text{un-}Z} 0$ for every net (y_α) in Y_+ . Suppose that $y_\alpha \xrightarrow{\text{un-}X} 0$. Fix $z \in Z_+$. Then $\|y_\alpha \wedge z\|_X \rightarrow 0$. Since this net is contained in $[0, z]$, Lemma 2.4 yields that $\|y_\alpha \wedge z\|_Z \rightarrow 0$. Thus, $y_\alpha \xrightarrow{\text{un-}Z} 0$.

Suppose now that $y_\alpha \xrightarrow{\text{un-}Z} 0$. Since the inclusion of Z into X is continuous, we have $\|y_\alpha \wedge z\|_X \rightarrow 0$ for every $z \in Z_+$. It follows that (y_α) un-converges to zero with respect to $(Z, \|\cdot\|_X)$. It follows now from Proposition 2.2 that $y_\alpha \xrightarrow{\text{un-}X} 0$. \square

Theorem 2.6. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two order continuous Banach lattices which are both order dense ideals in a vector lattice Y . Then X_1 and X_2 induce the same un-topology on Y .*

Proof. Let $Z = X_1 \cap X_2$. It is easy to see that Z is an order dense ideal of Y . In particular, Z is an order (and, therefore, norm) dense ideal in both X_1 and X_2 . For $z \in Z$, define $\|z\| = \max\{\|z\|_{X_1}, \|z\|_{X_2}\}$. Then Z is an order continuous normed lattice and the inclusions of Z into X_1 and X_2 are continuous. Applying Theorem 2.5 to pairs (X_1, Z) and (X_2, Z) , we get the desired result. \square

3. UN-TOPOLOGY AND WEAK UNITS

In this section, we assume that X is a Banach lattice, though most of the results extend to the case when X is only a normed lattice. As

before, we assume that X is also an ideal in a vector lattice Y . It was shown in Lemma 2.11 of [DOT17] that a net (x_α) in a Banach lattice with a quasi-interior point u un-converges to x iff $|x_\alpha - x| \wedge u \rightarrow 0$. We now extend this result.

Proposition 3.1. *Suppose that $u \geq 0$ is a quasi-interior point of X and (y_α) and y are in Y . Then y_α un-converges to y with respect to X iff $|y_\alpha - y| \wedge u \xrightarrow{\|\cdot\|} 0$ in X .*

Proof. The forward implication is trivial. Suppose that $|y_\alpha - y| \wedge u \xrightarrow{\|\cdot\|} 0$ in X . Then, clearly, $|y_\alpha - y| \wedge x \xrightarrow{\|\cdot\|} 0$ for every positive x in the principal ideal I_u . Now apply Proposition 2.2. \square

Corollary 3.2. *Suppose that X has a quasi-interior point and $y_\alpha \xrightarrow{\text{un}} 0$ in Y . There exist $\alpha_1 < \alpha_2 < \dots$ such that $y_{\alpha_n} \xrightarrow{\text{un}} 0$.*

It was shown in Theorem 3.2 of [KMT17] that un-topology on X is metrizable iff X has a quasi-interior point. We now extend this result to Y .

Theorem 3.3. *The following are equivalent:*

- (i) *The un-topology on Y is metrizable;*
- (ii) *The un-topology on X is metrizable and X is order dense in Y ;*
- (iii) *X contains a quasi-interior point which is also a weak unit in Y .*

Proof. (i) \Rightarrow (ii) Suppose that the un-topology on Y is metrizable. It follows immediately that its restriction to X is metrizable. Furthermore, being metrizable, the un-topology on Y is Hausdorff, so that X is order dense in Y by Proposition 1.4.

(ii) \Rightarrow (iii) By Theorem 3.2 in [KMT17], X has a quasi-interior point, say u . It follows that u is a weak unit in X . Since X is order dense in Y , u is a weak unit in Y .

(iii) \Rightarrow (i) Let u be as in (iii). For $y_1, y_2 \in Y$, define $d(y_1, y_2) = \||y_1 - y_2| \wedge u\|$. Since u is a weak unit in Y , d is a metric on Y ; the proof is similar to that of [KMT17, Theorem 3.2]. Note that $d(y_\alpha, y) \rightarrow 0$ iff $|y_\alpha - y| \wedge u \xrightarrow{\|\cdot\|} 0$ in X . By Proposition 3.1, this is equivalent to $y_\alpha \xrightarrow{\text{un}} y$. Therefore, d is a metric for un-topology. \square

4. ATOMIC BANACH LATTICES

Recall that a positive non-zero vector a in a vector lattice E is an **atom** if the principal ideal I_a equals the span of a . In this case, I_a is a projection band. The corresponding band projection P_a has the form $P_a x = \varphi_a(x)a$, where φ is the **coordinate functional** of a . We say that E is atomic if it equals the band generated by all the atoms of E .

Suppose that E is atomic. Let A be a maximal collection of pair-wise disjoint atoms in E . A net (x_α) in E converges to zero **coordinate-wise** if $\varphi_a(x_\alpha) \rightarrow 0$ for every atom a (or, equivalently, for every $a \in A$). For every $x \in E_+$, one has $x = \sup\{\varphi_a(x)a : a \in A\}$. This allows one to identify x with the function $a \in A \mapsto \varphi_a(x)$ in \mathbb{R}^A . Extending this map to E , one produces a lattice isomorphism from E onto an order dense sublattice of \mathbb{R}^A . Thus, every atomic vector lattice can be identified with an order dense sublattice of \mathbb{R}^A . Furthermore, E is order complete iff it is an ideal in \mathbb{R}^A . For details, we refer the reader to [Sch74, p. 143]. With a minor abuse of notation, we identify every $x \in E$ with the function $a \mapsto \varphi_a(x)$ in \mathbb{R}^A ; in particular, we identify $a \in A$ with the characteristic function of $\{a\}$.

It was shown in Corollary 4.14 of [KMT17] that if X is an atomic order continuous Banach lattice then un-convergence in X coincides with coordinate-wise convergence. Taking $X = \ell_\infty$ shows that this may fail when X is not order continuous.

Let X be an order complete atomic Banach lattice, represented as an order dense ideal in \mathbb{R}^A . The coordinate-wise convergence on X is then the restriction of the point-wise convergence on \mathbb{R}^A . We can now define un-convergence on \mathbb{R}^A induced by X .

Proposition 4.1. *Let X be an order complete atomic Banach lattice represented as an order dense ideal in \mathbb{R}^A . For a net (y_α) in \mathbb{R}^A , if $y_\alpha \xrightarrow{\text{un}} 0$ then $y_\alpha \rightarrow 0$ point-wise. The converse is true iff X is order continuous.*

Proof. Suppose that $y_\alpha \xrightarrow{\text{un}} 0$ in \mathbb{R}^A . For every $a \in A$, we have $|y_\alpha| \wedge a \xrightarrow{\|\cdot\|} 0$; it follows easily that $y_\alpha(a) \rightarrow 0$.

Suppose that X is order continuous and (y_α) is a net in \mathbb{R}^A which converges to zero point-wise. Let $Z = \text{span } A$ in X . Clearly, Z is an

ideal in X ; it is norm dense because X is order continuous. It is easy to see that $|y_\alpha| \wedge z \rightarrow 0$ in norm whenever $z \in Z_+$. Therefore, $y_\alpha \xrightarrow{\text{un-}Z} 0$. By Proposition 2.2, we have $y_\alpha \xrightarrow{\text{un-}X} 0$.

Suppose now that if $y_\alpha \rightarrow 0$ point-wise then $y_\alpha \xrightarrow{\text{un}} 0$ for every net (y_α) in \mathbb{R}^A . To prove that X is order continuous, it suffices to show that every disjoint order bounded sequence (x_n) in X is norm null. Clearly, such a sequence converges to zero coordinate-wise, hence, by assumption, $x_n \xrightarrow{\text{un}} 0$. Since (x_n) is order bounded, we have $x_n \xrightarrow{\|\cdot\|} 0$. \square

What happens when X is order complete but not order continuous? Recall that in this case, X contains a lattice copy of ℓ_∞ ; see, e.g., [MN91, Corollary 2.4.3]. So the following example is, in some sense, representative.

Example 4.2. Let $X = \ell_\infty(\Omega)$ for some set Ω . In this case, X is atomic and may be viewed as an order dense ideal of \mathbb{R}^Ω . Note that $u = \mathbb{1}$ is a strong unit and, therefore, a quasi-interior point in X . It can now be easily deduced from Proposition 3.1 that the un-convergence induced on \mathbb{R}^Ω by X coincides with uniform convergence.

Example 4.3. Both ℓ_1 and ℓ_∞ may be viewed as order dense ideals in $\mathbb{R}^{\mathbb{N}}$. Let (e_n) be the standard unit vector sequence. Then e_n un-converges to zero in $\mathbb{R}^{\mathbb{N}}$ with respect to ℓ_1 but not to ℓ_∞ .

5. BANACH FUNCTION SPACES

We are going to extend Example 1.1 to Banach function spaces. Throughout this section, we say that X is a **Banach function space** if it is a Banach lattice which is an order dense ideal in $L_0(\mu)$ for some measure space $(\Omega, \mathcal{F}, \mu)$. Throughout this section, we assume that μ is σ -finite. Note that by [AA02, Corollary 5.22], X has a weak unit. Theorem 3.3 yields the following.

Corollary 5.1. *Let X be an order continuous Banach function space over $(\Omega, \mathcal{F}, \mu)$. The un-topology induced by X on $L_0(\mu)$ is metrizable.*

Next, we are going to show the un-convergence induced by X on $L_0(\mu)$ agrees with “local” convergence in measure. Let $A \in \mathcal{F}$. For

$x \in L_0(\mu)$ we write $x|_A$ for the restriction of x to A . With some abuse of notation, we identify it with $x \cdot \chi_A$. For $p \in [0, \infty]$, we write $L_p(A)$ for the set $\{x|_A : x \in L_p(\mu)\}$.

Theorem 5.2. *Let X be an order continuous Banach function space over $(\Omega, \mathcal{F}, \mu)$, and (y_α) is a net in $L_0(\mu)$. Then $y_\alpha \xrightarrow{\text{un-}X} 0$ iff $y_\alpha|_A$ converges to zero in measure whenever $\mu(A) < \infty$.*

Proof. By Theorem 2.6, we may assume without loss of generality that $X = L_1(\mu)$. Suppose that $y_\alpha \xrightarrow{\text{un-}L_1(\mu)} 0$, and let A be a measurable set of finite measure. For every positive $x \in L_1(A)$, we have $|y_\alpha| \wedge x \rightarrow 0$ in norm. It follows that the net $(y_\alpha|_A)$ in $L_0(A)$ is un-null with respect to $L_1(A)$. By Example 1.1, we conclude that $(y_\alpha|_A)$ converges to zero in measure.

Conversely, suppose that $y_\alpha|_A$ converges to zero in measure whenever $\mu(A) < \infty$. Let Z be the set of all functions in $L_1(\mu)$ which vanish outside of a set of finite measure. It is easy to see that Z is a norm dense ideal in $L_1(\mu)$. Fix $z \in Z_+$. Find $A \in \mathcal{F}$ such that $\mu(A) < \infty$ and z vanishes outside of A . By assumption, $y_\alpha|_A$ converges to zero in measure. It follows by Example 1.1 that $y_\alpha|_A$ un-converges in $L_0(A)$ with respect to $L_1(A)$. In particular, we have $|y_\alpha| \wedge z \rightarrow 0$ in $L_1(A)$. It follows that $y_\alpha \xrightarrow{\text{un-}Z} 0$ in $L_0(\mu)$. Proposition 2.2 yields $y_\alpha \xrightarrow{\text{un-}L_1(\mu)} 0$. \square

6. UNIVERSAL COMPLETION

In this section, we consider the special case when $Y = X^u$. Recall that every vector lattice E may be identified with an order dense sublattice of its *universal completion* E^u , see, e.g., [AB03, Theorem 7.21]. If, in addition, E is order complete, then E is an ideal in E^u by, e.g., [AB03, Theorem 1.40]. It follows from [AB03, Theorem 7.23] that if F is an order dense sublattice of E then $F^u = E^u$.

Suppose that X is an order complete Banach lattice. The norm need not extend to X^u ; the latter is only a vector lattice. However, using our construction, the un-topology of X admits an extension to X^u . Combining results of the preceding sections, we get the following.

Theorem 6.1. *Let X be an order complete Banach lattice. Un-topology on X^u is Hausdorff; it is metrizable iff X has a quasi-interior point.*

The following examples underline that much of what we did in previous section fits into this general framework.

Example 6.2. In Section 4, we discussed the un-convergence induced by an order complete atomic Banach lattice X onto \mathbb{R}^A , where A is a maximal pair-wise disjoint collection of atoms. It is easy to see that, in this case, X^u may be identified with \mathbb{R}^A .

Example 6.3. In Example 1.1, we discussed the case where $X = L_p(\mu)$ and $Y = L_0(\mu)$, where μ is a finite measure and $0 \leq p < \infty$. By [AB03, Theorem 7.73], $Y = X^u$ (even when μ is only assumed to be σ -finite).

Example 6.4. In Section 5, we considered the case when X is an order complete Banach function space over a σ -finite measure μ and $Y = L_0(\mu)$. Since X is an order dense ideal in $L_0(\mu)$ and the latter is universally complete, we have $X^u = L_0(\mu)$.

Remark 6.5. General un-convergence often reduces to un-convergence on X^u as follows. Suppose that X is an order complete Banach lattice which is an order dense ideal in a vector lattice Y . We are interested in the un-convergence on Y induced by X . Note that X is an order dense ideal in X^u and that $X^u = Y^u$, so we may assume that $X \subseteq Y \subseteq X^u$. It follows that the un-convergence on Y is the restriction of un-convergence on X^u .

Remark 6.6. We now extend our definition of un-topology on X^u to the case when X is not order complete (and, therefore, need not be an ideal in X^u). In this case, we consider the un-topology on X^u induced by X^δ . In this setting, Remark 6.5 remain valid provided that Y is order complete. Since X is majorizing in X^δ , the “native” un-topology of X still agrees with the restriction to X of the un-topology on X^u induced by X^δ .

In [KMT17, Proposition 6.2], it was shown that an order continuous Banach lattice is un-complete iff it is finite-dimensional. We will show

next that the situation is completely different when we consider un-topology on X^u instead of X .

Theorem 6.7. *If X is an order continuous Banach lattice then the un-topology on X^u is complete.*

Proof. The proof is similar to that of [KMT17, Theorem 6.4]. Suppose first that X has a weak unit. Then X may be identified with an order dense ideal of $L_1(\mu)$ for some finite measure μ ; see [LT79, Theorem 1.b.14] or [GTX17, Section 4]. It follows that $X^u = L_1(\mu)^u = L_0(\mu)$. By Corollary 5.1, the un-topology on $L_0(\mu)$ induced by X is metrizable. Therefore, it suffices to show that it is sequentially complete. Theorem 5.2 yields that this topology is the topology of convergence in measure, which is sequentially complete by [Fol99, Theorem 2.30].

Now we consider the general case. Let (y_α) be a un-Cauchy net in X^u . Without loss of generality, $y_\alpha \geq 0$; otherwise we separately consider the nets (y_α^+) and (y_α^-) . By [LT79, Proposition 1.a.9] (see also Section 4 in [KMT17]), X can be written as the closure of a direct sum of a family \mathcal{B} of pair-wise disjoint principal bands.

Fix $B \in \mathcal{B}$; let \tilde{B} be the band of X^u generated by B . Being a band in X^u , \tilde{B} is universally complete. Since B is order dense in \tilde{B} , we may identify \tilde{B} with B^u . Let $P_{\tilde{B}}: X^u \rightarrow \tilde{B}$ be the band projection for \tilde{B} . It is easy to see that the net $(P_{\tilde{B}}y_\alpha)$ is un-Cauchy in \tilde{B} with respect to B . Since B has a weak unit, the first part of the proof yields that there exists $y_B \in \tilde{B}$ such that $P_{\tilde{B}}y_\alpha$ un-converges to y_B in \tilde{B} (and, therefore, in X^u) with respect to B . It follows from Proposition 1.2 that $y_B \geq 0$.

Put $y = \sup\{y_B : B \in \mathcal{B}\}$. This supremum exists in X^u because X^u is universally complete. We claim that $P_{\tilde{B}}y = y_B$ for each $B \in \mathcal{B}$. Indeed, let γ be a finite subset of \mathcal{B} . Define $z_\gamma = \bigvee_{B \in \gamma} y_B$. Then (z_γ) may be viewed as a net, and $z_\gamma \uparrow y$. Let $B \in \mathcal{B}$. For every γ such that $B \in \gamma$, the order continuity of $P_{\tilde{B}}$ yields $y_B = P_{\tilde{B}}z_\gamma \uparrow P_{\tilde{B}}y$, so that $P_{\tilde{B}}y = y_B$ and, therefore, $P_{\tilde{B}}y_\alpha \xrightarrow{\text{un-}B} P_{\tilde{B}}y$ for every B .

It is left to show that $y_\alpha \xrightarrow{\text{un-}X} y$ in X^u . The argument is similar to the proof of [KMT17, Theorem 4.12] and we leave it as an exercise. \square

7. UN-COMPACT INTERVALS

Let X be a Banach lattice. It is well known that order intervals in X are norm compact iff X is atomic and order continuous; see, e.g., Theorem 6.2 in [Wnuk99]. Since un-convergence on order intervals of X agrees with norm convergence, we immediately conclude that order intervals in X are un-compact iff X is atomic and order continuous. Recall that $[a, b] = a + [0, b - a]$ whenever $a \leq b$; therefore, when dealing with order intervals, it often suffices to consider order intervals of the form $[0, u]$.

Suppose now that X is an order dense ideal in a vector lattice Y and consider order intervals in Y . It is easy to see that they are un-closed. Indeed, suppose that $y_\alpha \xrightarrow{\text{un}} y$ in Y and $u \in Y_+$ such that $0 \leq y_\alpha \leq u$ for every α ; it follows from Proposition 1.2 that $0 \leq y \leq u$.

Proposition 7.1. *Let X be a Banach lattice. TFAE:*

- (i) X is atomic and order continuous;
- (ii) Order intervals in X^u are un-compact.

Proof. (ii) \Rightarrow (i) Suppose that order intervals in X^u are un-compact. Since X^δ is an ideal in X^u , every order interval in X^δ is an order interval in X^u . Since the un-topology on X^δ is the restriction to X^δ of the un-topology on X^u , order intervals in X^δ are un-compact and, therefore, norm compact. Furthermore, since X is a closed sublattice of X^δ , order intervals of X are closed subsets of order intervals of X^δ and, therefore, are compact in X . Therefore, X has compact intervals; it follows that X is atomic and order continuous.

(i) \Rightarrow (ii) As in Proposition 4.1 and Example 6.2, we may assume that $X^u = \mathbb{R}^A$ for some set A and the un-convergence on \mathbb{R}^A agrees with the point-wise convergence. It is left to observe that order intervals in \mathbb{R}^A are compact with respect to the topology of point-wise convergence. Indeed, given $u \in \mathbb{R}_+^A$, we have $[0, u] = \prod_{a \in A} [0, u(a)]$. Since each of the intervals $[0, u(a)]$ is compact in \mathbb{R} , the set $[0, u]$ is compact by Tychonoff's Theorem. \square

Theorem 7.2. *Let X be a Banach lattice such that X is an order dense ideal in a vector lattice Y . Order intervals in Y are un-compact iff X is atomic and order continuous and Y is order complete.*

Proof. Suppose that X is atomic and order continuous and Y is order complete. By Proposition 7.1, order intervals of X^u are un-compact. Since X is order dense in Y , we have $Y^u = X^u$, so we may assume that Y is a sublattice of X^u . Since Y is order complete, Y is an ideal of X^u . Therefore, order intervals of Y are also order intervals in X^u and the un-topology on Y is the restriction of the un-topology on X^u to Y . It follows that order intervals of Y are un-compact.

Conversely, suppose that order intervals of Y are un-compact. It follows, in particular, that order intervals of X are un-compact, hence norm compact and, therefore, X is atomic and order continuous. To show that Y is order complete, suppose that $0 \leq y_\alpha \uparrow \leq u$ in Y . Since $[0, u]$ is un-compact, there is a subnet (z_γ) of (y_α) such that $z_\gamma \xrightarrow{\text{un}} z$ for some $z \in Y$. Since order intervals are un-closed, we have $z \in [0, u]$. Since the net (y_α) is increasing, so is (z_γ) and, therefore, $z_\gamma \uparrow z$ by Proposition 1.6. It follows that $y_\alpha \uparrow z$. \square

8. SPACES WITH A STRONG UNIT

In this section, we consider the case when X is a Banach lattice with a strong unit, such that X is an order dense ideal in a vector lattice Y . Cf. Example 4.2.

We start by recalling some preliminaries. Let E be a vector lattice and $e \in E_+$. For $x \in E$, we define

$$\|x\|_e = \inf\{\lambda > 0 : |x| \leq \lambda e\}.$$

This expression is finite iff x belongs to the principal ideal I_e ; it defines a lattice norm on I_e . For a net (y_α) and a vector y in Y , we say that y_α **converges to y uniformly with respect to e** if $\|y_\alpha - y\|_e \rightarrow 0$. Note that this implies that $y_\alpha - y \in I_e$ for all sufficiently large α . E is said to be **uniformly complete** if $(I_e, \|\cdot\|_e)$ is complete for every $e \in E_+$. Every σ -order complete vector lattice and every Banach lattice is uniformly complete; see [LZ71, §42] and [AB06, Theorem 4.21]. If $(I_e, \|\cdot\|_e)$ is complete then it is lattice isometric to $C(K)$ for some

compact Hausdorff space K with e corresponding to the constant one function $\mathbb{1}$. In particular, every Banach lattice with a strong unit e is lattice isomorphic to $C(K)$ with e corresponding to $\mathbb{1}$. We refer the reader to Section 3.1 in [AA02] for further details. It was observed in [KMT17] that if X is a Banach lattice with a strong unit e then un-convergence in X agrees with norm convergence, which, in turn, agrees with the uniform convergence with respect to e .

Proposition 8.1. *Let X be a Banach lattice with a strong unit e , such that X is an order dense ideal in a vector lattice Y . For a net (y_α) in Y , $y_\alpha \xrightarrow{\text{un}} 0$ iff (y_α) converges to zero uniformly with respect to e (in particular, a tail of (y_α) is contained in X).*

Proof. First, we consider the special case when Y is uniformly complete. Note that X equals the principal ideal generated by e in Y . It follows that the original norm of X is equivalent to $\|\cdot\|_e$. By Proposition 3.1, $y_\alpha \xrightarrow{\text{un-}X} 0$ iff $\| |y_\alpha| \wedge e \|_X \rightarrow 0$, which is equivalent to $\| |y_\alpha| \wedge e \|_e \rightarrow 0$. Note that if $\| |y_\alpha| \wedge e \|_e < 1$ then $|y_\alpha| \leq e$, and, therefore, $y_\alpha \in X$ and $\| |y_\alpha| \wedge e \|_e = \| y_\alpha \|_e$. It is now easy to see that $\| |y_\alpha| \wedge e \|_e \rightarrow 0$ iff y_α converges to 0 uniformly with respect to e .

Now we consider the general case. By Proposition 1.7, $y_\alpha \xrightarrow{\text{un-}X} 0$ in Y iff $y_\alpha \xrightarrow{\text{un-}X^\delta} 0$ in Y^δ . Observe that Y^δ is order complete, hence uniformly complete. Furthermore, e is a strong unit in X^δ . By the special case, $y_\alpha \xrightarrow{\text{un-}X^\delta} 0$ in Y^δ iff a tail of (y_α) is contained in X^δ and converges to zero uniformly with respect to e . It follows that $|y_\alpha| \leq e$ for all sufficiently large α . Since X is an ideal in Y , we conclude that $y_\alpha \in X$. \square

Suppose that Y is a uniformly complete vector lattice and $e \in Y_+$ is a weak unit. Put $X = (I_e, \|\cdot\|_e)$. Then X is a Banach lattice with a strong unit, and an order dense ideal in Y . Consider the un-topology induced by X on Y . The following result is an immediate corollary of Proposition 8.1.

Corollary 8.2. *Let Y be a uniformly complete vector lattice, $e \in Y_+$ a weak unit, and $X = (I_e, \|\cdot\|_e)$. For a net (y_α) in Y , $y_\alpha \xrightarrow{\text{un-}X} 0$ iff y_α converges to zero uniformly with respect to e .*

Let X be a Banach lattice with a strong unit e . As in Remark 6.6, we consider the un-topology on X^u induced by X^δ . Note that $X^u = (X^\delta)^u$. By Proposition 8.1, for a net (y_α) in X^u , $y_\alpha \xrightarrow{\text{un}} 0$ iff a tail of (y_α) is contained in X^δ and converges to zero uniformly with respect to e .

Since X is a Banach lattice with a strong unit, up to a lattice isomorphism we may identify X with $C(K)$ for some compact space K . Furthermore, X^δ is an order complete Banach lattice with strong unit, so we can identify it with $C(Q)$ for some extremally disconnected compact space Q . Since $X^u = (X^\delta)^u$, by [AB03, Theorem 7.29], we can identify X^u with $C_\infty(Q)$. Therefore, for a net (f_α) in $C_\infty(Q)$, $f_\alpha \xrightarrow{\text{un}} 0$ iff a tail of (f_α) is contained in $C(Q)$ and converges to zero uniformly on Q .

9. UN-CONVERGENCE VERSUS UO-CONVERGENCE

Let Y be a vector lattice. Recall that a net (y_α) in Y is said to **converge in order** to y if there exists a net (z_γ) in Y , which may, generally, have a different index set, such that $z_\gamma \downarrow 0$ and $\forall \gamma \exists \alpha_0 \forall \alpha \geq \alpha_0 |y_\alpha - y| \leq z_\gamma$. In this case, we write $y_\alpha \xrightarrow{o} y$. We say that y_α **uo-converges** to y and write $y_\alpha \xrightarrow{\text{uo}} y$ if $|y_\alpha - y| \wedge u \xrightarrow{o} 0$ for every $u \in Y_+$. We refer the reader to [GTX17] for a review of order and uo-convergence. We will only mention three facts here. First, it was observed in [GTX17, Corollary 3.6] that every disjoint sequence is uo-null. Second, for sequences in $L_0(\mu)$, uo-convergence coincides with almost everywhere (a.e.) convergence. For the third fact, we need the concept of a regular sublattice. Recall that a sublattice Z of Y is **regular** if $z_\alpha \downarrow 0$ in Z implies $z_\alpha \downarrow 0$ in Y . In this case, Theorem 3.2 of [GTX17] asserts that $z_\alpha \xrightarrow{\text{uo}} 0$ in Z iff $z_\alpha \xrightarrow{\text{uo}} 0$ in Y for every net (z_α) in Z . In particular, if Y is a regular sublattice of $L_0(\mu)$ then uo-convergence coincides with a.e. convergence for sequences in Y . Therefore, uo-convergence may be viewed as a generalization of a.e. convergence.

Suppose, as before, that X is a normed lattice which is an ideal in Y . Our goal is to compare the un-convergence on Y induced by X with the uo-convergence on Y .

The following is an extension of Proposition 3.5 in [KMT17].

Proposition 9.1. *The following are equivalent.*

- (i) X is order continuous;
- (ii) Every disjoint sequence in Y is un-null;
- (iii) Every disjoint net in Y is un-null.

Proof. To prove (i) \Rightarrow (ii), observe that if (y_n) is a disjoint sequence in Y then it is uo-null in Y . In particular, for every $x \in X_+$, the sequence $|y_n| \wedge x$ converges to zero in order and, therefore, in norm. The proof that (i) \Leftarrow (ii) \Leftrightarrow (iii) is straightforward, cf. [KMT17, Proposition 3.5]. \square

Proposition 9.2. *The following are equivalent.*

- (i) X is order continuous;
- (ii) $x_\alpha \xrightarrow{\text{uo}} 0$ in X implies $x_\alpha \xrightarrow{\text{un}} 0$ in X for every net (x_α) in X ;
- (iii) $y_\alpha \xrightarrow{\text{uo}} 0$ in Y implies $y_\alpha \xrightarrow{\text{un}} 0$ in Y for every net (y_α) in Y .

Proof. Note that being an ideal in Y , X is a regular sublattice and, therefore, $x_\alpha \xrightarrow{\text{uo}} 0$ in X iff $x_\alpha \xrightarrow{\text{uo}} 0$ in Y for every net (x_α) in X .

(i) \Rightarrow (iii) Suppose that $y_\alpha \xrightarrow{\text{uo}} 0$ in Y . Fix $x \in X_+$. Then $|y_\alpha| \wedge x \xrightarrow{o} 0$ in X , hence $|y_\alpha| \wedge x \xrightarrow{\|\cdot\|} 0$ and, therefore, $y_\alpha \xrightarrow{\text{un}} 0$.

(iii) \Rightarrow (ii) Suppose $x_\alpha \xrightarrow{\text{uo}} 0$ in X . Then $x_\alpha \xrightarrow{\text{uo}} 0$ in Y and, therefore, $x_\alpha \xrightarrow{\text{un}} 0$ in Y and in X .

(ii) \Rightarrow (i) We will apply Proposition 9.1 with $X = Y$. Every disjoint sequence in X is uo-null, hence it is un-null by the assumption. \square

Recall the following standard fact; see, e.g., [Fol99, Theorem 2.30].

Theorem 9.3. *Let μ be a finite measure. Every sequence in $L_0(\mu)$ which converges in measure has a subsequence which converges a.e.*

It is a natural question whether this result can be generalized with a.e. convergence and convergence in measure replaced with uo- and un-convergences, respectively. A partial advance in this direction was made in Proposition 4.1 of [DOT17]:

Proposition 9.4. [DOT17] *If X is a Banach lattice and $x_n \xrightarrow{\text{un}} 0$ in X then $x_{n_k} \xrightarrow{\text{uo}} 0$ for some subsequence (x_{n_k}) .*

However, formally speaking, Proposition 9.4 is not a generalization of Theorem 9.3 because $L_0(\mu)$ is not a Banach lattice. Using the framework of this paper, we are now ready to produce an appropriate extension of Theorem 9.3.

Theorem 9.5. *Let X be an order continuous Banach lattice with a weak unit, such that X is an order dense ideal in a vector lattice Y ; let (y_n) be a sequence in Y such that $y_n \xrightarrow{\text{un-}X} 0$. Then there is a subsequence (y_{n_k}) such that $y_{n_k} \xrightarrow{\text{uo}} 0$ in Y .*

Proof. Special case: $Y = X^u$. Represent X as an order dense ideal in $L_1(\mu)$ for some finite measure μ . We may then identify X^u with $L_0(\mu)$. By Theorem 5.2, $y_n \xrightarrow{\text{un-}X} 0$ yields $y_n \xrightarrow{\mu} 0$ in $L_0(\mu)$. By Theorem 9.3, there exists a subsequence (y_{n_k}) such that $y_{n_k} \xrightarrow{\text{a.e.}} 0$ and, therefore, $y_{n_k} \xrightarrow{\text{uo}} 0$ in $L_0(\mu)$.

General case. Since X is order dense in Y , we have $X^u = Y^u$. Therefore, we may assume without loss of generality, that Y is an order dense sublattice of X^u . Since $y_n \xrightarrow{\text{un-}X} 0$ in Y , it is trivial that $y_n \xrightarrow{\text{un-}X} 0$ in X^u . The special case yields $y_{n_k} \xrightarrow{\text{uo}} 0$ in X^u for some subsequence (y_{n_k}) . Since Y is an order dense sublattice of X^u , it is regular by [AB03, Theorem 1.23]. It follows that $y_{n_k} \xrightarrow{\text{uo}} 0$ in Y . \square

Note that in Proposition 9.4, X need not have a weak unit. Therefore, it is natural to ask whether a weak unit is really needed in Theorem 9.5. The following example shows that the weak unit assumption in Theorem 9.5 cannot be removed.

Example 9.6. We are going to construct an order continuous Banach lattice X with no weak units and a vector lattice Y such that X is an order dense ideal in Y (actually, $Y = X^u$) and a sequence (y_n) in Y such that $y_n \xrightarrow{\text{un-}X} 0$ in Y and yet no subsequence of (y_n) is uo-null.

Let Γ be an infinite set (we will choose a specific Γ later). Let X be the ℓ_1 -sum of infinitely many copies of $L_1[0, 1]$ indexed by Γ . That is, X is the space of functions $x: \Gamma \rightarrow L_1[0, 1]$ such that $\|x\| := \sum_{\gamma \in \Gamma} \|x(\gamma)\|_{L_1[0,1]}$ is finite. We may also view x as a function on the union of $|\Gamma|$ many copies of $[0, 1]$ or, equivalently, on $[0, 1] \times \Gamma$. We write x^γ instead of $x(\gamma)$ and call it a component of x . It is easy to

see that X is an AL-space; in particular, it is order continuous. It is also easy to see that each $x \in X$ has at most countably many non-zero components. Let Z be the subset of X consisting of those x which have finitely many non-zero components. It can be easily verified that Z is a norm dense ideal in X .

Let Y be the direct sum of infinitely many copies of $L_0[0, 1]$ indexed by Γ . That is, $Y = (L_0[0, 1])^\Gamma$, the space of all functions from Γ to $L_0[0, 1]$. Again, we write y^γ instead of $y(\gamma)$ and may view y as a real-valued function on $[0, 1] \times \Gamma$. It is easy to see that Y is a vector lattice under component-wise lattice operations, and X is an order dense ideal in Y . Furthermore, $Y = X^u$. We equip Y with the un-topology induced by X .

Let (y_n) be an arbitrary sequence in Y_+ . We claim that in order for it to be un-null, it suffices that the sequence y_n^γ converges to zero in measure in for every γ . Indeed, the latter implies that $\|y_n^\gamma \wedge u\|_{L_1[0,1]} \rightarrow 0$ for every positive $u \in L_1[0, 1]$. It follows easily that $\|y_n \wedge x\|_X \rightarrow 0$ for every $x \in Z_+$. Proposition 2.2 now yields that $y_n \xrightarrow{\text{un-}X} 0$.

Furthermore, it can be easily verified that if a sequence (y_n) in Y is uo-null then for every $\gamma \in \Gamma$ one has $y_n^\gamma \xrightarrow{\text{uo}} 0$ in $L_0[0, 1]$, hence $y_n^\gamma \xrightarrow{\text{a.e.}} 0$.

We now specify Γ : let it be the set of all strictly increasing sequences of natural numbers. Fix a sequence (f_k) in $L_0[0, 1]$ such that f_k converges to zero in measure but not a.e. For each n , define y_n in Y as follows: for each $\gamma \in \Gamma$, let $\gamma = (n_k)$ and put $y_n^\gamma = f_k$ if $n = n_k$ and $y_n^\gamma = 0$ if n is not in γ . It follows that $(y_{n_k}^\gamma)$ as a sequence in k is exactly (f_k) , while the rest of the terms of (y_n^γ) are zeros. Therefore, the sequence (y_n^γ) converges in measure to zero for every γ , hence $y_n \xrightarrow{\text{un}} 0$. On the other hand, (y_n) has no uo-null subsequences. Indeed, suppose that $y_{n_k} \xrightarrow{\text{uo}} 0$ for some subsequence $(n_k) = \gamma$. Then $y_{n_k}^\gamma \xrightarrow{\text{a.e.}} 0$ in $L_0[0, 1]$, so that $f_k \xrightarrow{\text{a.e.}} 0$; a contradiction.

Acknowledgement and further remarks. We would like to thank Niushan Gao and Mitchell Taylor for valuable discussions. After this project was completed, M. Taylor has generalized some of the results to the setting of locally solid vector lattices; see [Tay].

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