ON THE MODULUS OF C. J. READ'S OPERATOR

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Abstract. Let $T\colon \ell_1 \to \ell_1$ be the quasinilpotent operator without an invariant subspace constructed by C. J. Read in [R3]. We prove that the modulus of this operator has an invariant subspace (and even an eigenvector). This answers a question posed by Y. Abramovich, C. Aliprantis and O. Burkinshaw in [AAB1, AAB3].

During the last several years there has been a noticeable increase of interest in the invariant subspace problem for positive operators on Banach lattices. A rather complete and comprehensive survey on this topic is presented in [AAB3], to which we refer the reader for details and for an extensive bibliography. In particular, the following theorem was proved in [AAB1].

Theorem 1 ([AAB1, AAB3]). *If the modulus of a continuous operator $T\colon \ell_p \to \ell_p$ $(1 \leqslant p < \infty)$ exists and is quasinilpotent, then T has a non-trivial closed invariant subspace which is an ideal.*

It follows that each positive quasinilpotent operator on ℓ_p $(1 \leqslant p < +\infty)$ has a nontrivial closed invariant subspace. In the same papers the authors posed the following problem.

Problem. *Does every positive operator on ℓ_1 have an invariant subspace?*

Keeping in mind that each operator on ℓ_1 has a modulus and that C. J. Read in [R1, R2, R3] has constructed several operators on ℓ_1 without invariant subspaces, it was suggested in [AAB1, AAB3] that the modulus of some of these operators might be a natural candidate for a counterexample to the above problem. Following this suggestion, we will be dealing in this paper with the modulus of the quasinilpotent operator T constructed in [R3]. It turns out, quite surprisingly, that not only does $|T|$ have an invariant subspace but it even has a positive eigenvector. This result increases the chances for an affirmative answer to the above problem.

The paper is organized as follows. After introducing some necessary notation and terminology we prove a general theorem on the existence of an invariant subspace for the modulus

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of a quasinilpotent operator. The rest of the paper will be devoted to the verification that C. J. Read's operator, constructed in [R3], satisfies all the hypotheses of this theorem and so its modulus does have an invariant subspace.

For terminology and notation regarding operators and Banach lattices we refer to [AB, Sch]. All operators considered in this work are linear and continuous. The space of all operators on a Banach space X is denoted by $\mathcal{B}(X)$, while $\mathcal{K}(X)$ stands for the subspace of all compact operators. A linear operator on a Banach lattice is said to be *positive* if it maps positive vectors to positive vectors. By an *invariant subspace* (*invariant ideal*) of an operator we mean a closed nontrivial subspace (resp. closed nontrivial ideal) which is invariant under the operator.

Together with the usual operator norm $\|S\|$ of an operator $S\colon X\to X$ on a Banach space, we will also consider the (essential) seminorm $\|S\|_e$ given by

$$\|S\|_e=\inf\{\,\|S-K\|\;:\;K\in\mathcal{K}(X)\,\}.$$

The *essential spectral radius* of S is computed via the formula $r_e(S)=\lim_n\sqrt[n]{\|S^n\|_e}$. This, of course, is an analogue of the familiar formula for the usual spectral radius $r(S)=\lim_n\sqrt[n]{\|S^n\|}$.

It is obvious that $\|S\|_e\leqslant\|S\|$ and $r_e(S)\leqslant r(S)$. It is known that if $r_e(S)=0$, then every nonzero point of $\sigma(S)$ is an eigenvalue. Further details on essential spectral radius can be found in [N1, CPY]. We will use the following important version of the Krein-Rutman theorem established by R. Nussbaum.

Theorem 2. [N2, Corollary 2.2] *Let S be a positive operator on a Banach lattice such that $r_e(S)<r(S)$, then $r(S)$ is an eigenvalue of S corresponding to a positive eigenvector.*[1]

We use this fact in the proof of the following simple but rather unexpected result.

Theorem 3. *Suppose that a quasinilpotent operator S on ℓ_p has no invariant ideals and S^- is compact. Then $r(|S|)$ is a positive eigenvalue of $|S|$ corresponding to a positive eigenvector. In particular, $|S|$ has an invariant subspace.*

Proof. First observe that the operator $|S|$ cannot be quasinilpotent. Indeed, if it were, then by Theorem 1 the operator S itself would have an invariant closed ideal contrary to our hypothesis. Thus, $r(|S|)>0$.

Next we claim that $r_e(|S|)=0$. To prove this, notice that $|S|=S+2S^-$, and so

$$|S|^n=(S+2S^-)^n=S^n+RS^-,$$

[1] In [N2] a more general form of this theorem is given which is valid for ordered Banach spaces.

where R is some polynomial in S and S^-. Hence RS^- is compact, whence

$$\||S|^n\|_e = \|S^n + RS^-\|_e \leqslant \|S^n\|,$$

and consequently

$$r_e(|S|) \leqslant \lim_{n\to\infty} \sqrt[n]{\|S^n\|} = r(S) = 0.$$

An application of Theorem 2 finishes the proof. □

Corollary 4. *Under the hypotheses of the above theorem the operator S^+ also has a nontrivial closed invariant subspace.*

Proof. There are two possibilities: either S^+ is quasinilpotent or it is not. If S^+ is quasinilpotent, then applying Theorem 1 again, we see that S^+ has an invariant ideal.

Assume that S^+ is not quasinilpotent. Since $S^+ = S + S^-$, the same argument as in the proof of Theorem 3 shows that $r_e(S^+) = 0$ and we can again apply Theorem 2. □

Theorem 3 is strong enough to enable us to prove Corollary 7 about the modulus of C. J. Read's operator. But first we would like to mention a nice generalization of Theorem 3. Recall that a positive operator on a Banach lattice is called *compact-friendly* if it commutes with another positive operator which dominates some non-zero operator which in turn is dominated by a positive compact operator. The class of compact-friendly operators was introduced and studied by Abramovich, Aliprantis, and Burkinshaw in [AAB2]. This class includes positive operators that dominate or are dominated by a non-zero compact operator, or commute with a non-zero compact operator. Also, every positive operator on any discrete Banach lattice and every positive kernel operator is compact-friendly. It follows from [AAB3, Theorem 11.2] that if a positive compact-friendly operator is quasinilpotent, then it has an invariant ideal. Mimicking the proof of Theorem 3 we can obtain the following theorem.

Theorem 5. *Let B be a compact-friendly operator with $r_e(B) = 0$. Then B has a nontrivial invariant subspace.*

To illustrate this theorem we mention the following result: *If S is a quasinilpotent kernel operator and S^- (resp. S^+) is compact, then $|S|$ and S^+ (resp. S^-) have invariant subspaces.*

Recall that an operator $S\colon X \to Y$ between two Banach spaces is called *nuclear* if it can be written in the form $S = \sum_{i=0}^{\infty} x_i^* \otimes y_i$ with x_i^* in X^*, y_i in Y and $\sum_{i=0}^{\infty}\|x_i^*\|\|y_i\| < \infty$. Here, as usual, the elementary tensor $x^* \otimes y : X \to Y$ is given by $(x^* \otimes y)(x) = x^*(x)y$. The *nuclear norm* $\nu(S)$ is defined by $\nu(S) = \inf \sum_{i=0}^{\infty}\|x_i^*\|\|y_i\|$, where the infimum is taken over all nuclear representations of S. For a nuclear operator S we have $\|S\| \leqslant \nu(S)$. This implies in particular that every nuclear operator can be approximated by finite-rank operators and, therefore, is compact.

Following [R1, R2, R3] we denote the standard unit vectors of ℓ_1 by $(f_i)_{i=0}^{\infty}$. It is well known that we can consider each $S \in \mathcal{B}(\ell_1)$ as an infinite matrix $S = (s_{ij})_{i,j=0}^{\infty}$. Let $S_{(i)}$ denote the i-th row of this matrix. If $x \in \ell_1$, then $(Sx)_i = \langle S_{(i)}, x\rangle$, so that $Sx = \sum_{i=0}^{\infty}\langle S_{(i)}, x\rangle f_i$. This gives a nuclear representation $S = \sum_{i=0}^{\infty} S_{(i)} \otimes f_i$, where the rows $S_{(i)}$ of S are considered as linear functionals on ℓ_1. It follows that

$$\nu(S) \leqslant \sum_{i=0}^{\infty} \|S_{(i)}\|_{\infty} \|f_i\|_1 = \sum_{i=0}^{\infty} \|S_{(i)}\|_{\infty},$$

so that S is nuclear if the last sum is finite.

In spite of the fact that the construction of a quasinilpotent operator on ℓ_1 without an invariant subspace in [R3] is far from being simple, the presentation in [R3] is very clearly structured. In Sections 2 and 3 of [R3] C. J. Read presents the construction of operator T, and in the rest of the paper he proves that T is bounded (Lemma 5.1), quasinilpotent (Theorem 6.5), and has no invariant subspaces (Theorem 7). In what follows we will restate (practically verbatim and retaining the notation) the definition of T. In the proof of [R3, Lemma 5.1] C. J. Read reveals a lot of information about the structure of the infinite matrix of T. Since we also are interested in this structure, we incorporate some fragments of the proof of [R3, Lemma 5.1] in our proof of Lemma 6.

The symbol F denotes the linear subspace of ℓ_1, spanned by f_i's, and thus, F is dense and consists of all eventually vanishing sequences. Let $\mathbf{d} = (a_1, b_1, a_2, b_2, \ldots)$ be a strictly increasing sequence of positive integers. Also let $a_0 = 1$, $v_0 = 0$, and $v_n = n(a_n + b_n)$ for $n \geqslant 1$. Then there is a unique sequence $(e_i)_{i=0}^{\infty} \subset F$ with the following properties:

0) $f_0 = e_0$;
A) if integers r, n, and i satisfy $0 < r \leqslant n$, $i \in [0, v_{n-r}] + ra_n$, then $f_i = (n^{ra_n} e_i - e_{i-ra_n})(n-r)^{i-ra_n} a_{n-r}$;
B) if integers r, n, and i satisfy $0 < r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$, (respectively, $1 \leqslant n$, $i \in (v_{n-1}, a_n)$), then $f_i = n^i 2^{(h-i)/\sqrt{a_n}} e_i$, where $h = (r + \frac{1}{2})a_n$ (respectively, $h = \frac{1}{2}a_n$);
C) if integers r, n, and i satisfy $0 < r \leqslant n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then $f_i = n^i e_i - b_n n^{i-b_n} e_{i-b_n}$;
D) if integers r, n, and i satisfy $0 \leqslant r < n$, $i \in (na_n + rb_n, (r+1)(a_n + b_n))$, then $f_i = n^i 2^{(h-i)/\sqrt{b_n}} e_i$, where $h = (r + \frac{1}{2})b_n$.

Indeed, since $f_i = \sum_{j=0}^{i} \lambda_{ij} e_j$ for each $i \geqslant 0$ and λ_{ii} is always nonzero, this linear relation is invertible. Further,

$$\mathrm{lin}\{e_i \,:\, i = 1, \ldots, n\} = \mathrm{lin}\{f_i \,:\, i = 1, \ldots, n\} \text{ for every } n \geqslant 0.$$

In particular all e_i are linearly independent and span F. Then C. J. Read defines $T\colon F \to F$ to be the unique linear map such that $Te_i = e_{i+1}$, and in Lemma 5.1 he proves that $\|Tf_i\| \leqslant 1$ for every $i \geqslant 0$ provided $\mathbf{d}$ increases sufficiently rapidly, i. e., satisfies several conditions of the form

$$\begin{aligned} a_n &\geqslant G(n, a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}), \text{ and} \\ b_n &\geqslant H(n, a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n), \end{aligned}$$

where G and H are some positive integer-valued functions. It follows that T can be extended to a bounded operator on ℓ_1. Finally, in Theorems 6.5 and 7, C. J. Read proves that this extension, which is also denoted by T, is quasinilpotent and has no invariant subspaces, provided $\mathbf{d}$ increases sufficiently rapidly.

Our plan is as follows: we will prove that the negative part of T is nuclear, hence compact. Then Theorem 3 will imply that $r(|T|)$ is a positive eigenvalue of $|T|$, corresponding to a positive eigenvector.

Lemma 6. *The operator T^- is nuclear, provided* $\mathbf{d}$ *increases sufficiently rapidly.*

Proof. Similarly to the proofs of [R3, Lemma 5.1] and [R2, Lemma 6.1] we study the matrices $(t_{ij})_{i,j=0}^{\infty}$ and $(t_{ij}^-)_{i,j=0}^{\infty}$ of T and T^- respectively. Recall that $t_{ki} = (Tf_i)_k$, so that it suffices to look at the images of the standard unit vectors under T. We will see that the matrix of T is quite sparse and has the following structure: every entry on the diagonal right under the main diagonal is strictly positive, there is no nonnegative entries below this diagonal, and there are some entries above it. We consider consecutively all the cases mentioned above.

0) $Tf_0 = e_1 = 2^{(1-a_1/2)/\sqrt{a_1}} f_1$, so that $T^- f_0 = 0$.

A) If $i < v_{n-r} + ra_n$, i. e. i is not the right end point of the interval $[ra_n, v_{n-r} + ra_n]$, then $Tf_i = (n-r)^{-1} f_{i+1}$, so that $T^- f_i = 0$. The only nontrivial case here is when i is the right end of the interval, i. e. $i = v_{n-r} + ra_n$. Then we have

$$\begin{aligned} Tf_i = a_{n-r} n^{ra_n} (n-r)^{v_{n-r}} e_{1+ra_n+v_{n-r}} - a_{n-r}(n-r)^{v_{n-r}} e_{1+v_{n-r}} \\ = \varepsilon_1 f_{1+v_{n-r}+ra_n} - \varepsilon_2 f_{1+v_{n-r}}, \end{aligned}$$

where $\varepsilon_1 > 0$ and ε_2 is given by

$$\varepsilon_2 = (n-r+1)^{-1-v_{n-r}} 2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}} a_{n-r}(n-r)^{v_{n-r}},$$

so that

$$T^{-}f_{v_{n-r}+ra_n}=\varepsilon_2 f_{1+v_{n-r}}. \tag{1}$$

B) Similarly, if $ra_n+v_{n-r}<i<(r+1)a_n-1$ or $v_{n-1}<i<a_n-1$, then $Tf_i=n^{-1}2^{1/\sqrt{a_n}}f_{i+1}$, so that $T^{-}f_i=0$. If $i=(r+1)a_n-1$, then $Tf_i=n^i 2^{(1-a_n/2)/\sqrt{a_n}}e_{(r+1)a_n}$. To express this in terms of the f_i's, we notice that $f_{(r+1)a_n}=a_{n-r-1}(n^{(r+1)a_n}e_{(r+1)a_n}-e_0)$, which implies

$$e_{(r+1)a_n}=n^{-(r+1)a_n}(a^{-1}_{n-r-1}f_{(r+1)a_n}+f_0), \tag{2}$$

In this case $T^{-}f_i=0$. Analogously, if $i=a_n-1$, then $Tf_i=n^i 2^{(1-a_n/2)/\sqrt{a_n}}e_{a_n}$. It follows from $f_{a_n}=a_{n-1}(n^{a_n}e_{a_n}-e_0)$ that

$$Tf_i=n^{-1}2^{(1-a_n/2)/\sqrt{a_n}}(a^{-1}_{n-1}f_{a_n}+f_0),$$

and again $T^{-}f_i=0$. Thus, case (B) produces no nontrivial entries in T^{-}.

C) If i is not the right end of the interval, i.e. $i<na_n+rb_n$, then $Tf_i=n^{-1}f_{i+1}$, so that $T^{-}f_i=0$. If $i=na_n+rb_n$, then

$$\begin{aligned}Tf_i=n^{na_n+rb_n}e_{1+na_n+rb_n}-b_n n^{na_n+(r-1)b_n}e_{1+na_n+(r-1)b_n}&\\ =\varepsilon_1 f_{1+na_n+rb_n}-\varepsilon_2 f_{1+na_n+(r-1)b_n},&\end{aligned}$$

where $\varepsilon_1>0$ and $\varepsilon_2=b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}$. It follows that

$$T^{-}f_{na_n+rb_n}=b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}f_{1+na_n+(r-1)b_n}. \tag{3}$$

D) If $i<(r+1)(a_n+b_n)-1$, then $Tf_i=n^{-1}2^{1/\sqrt{b_n}}f_{i+1}$, so that $T^{-}f_i=0$. If $i=(r+1)(a_n+b_n)-1$ then

$$Tf_i=n^i 2^{(-a_n/2-(r+1)a_n/2+1)/\sqrt{b_n}}e_{(r+1)(a_n+b_n)}.$$

Using (C) inductively we obtain the following identity:

$$\begin{aligned}e_{(r+1)(a_n+b_n)}=n^{-(r+1)(a_n+b_n)}\{f_{(r+1)(a_n+b_n)}+b_n f_{(r+1)a_n+rb_n}+\dots&\\ +n^r_n f_{(r+1)a_n+b_n}\}+b_n^{r+1}n^{-(r+1)b_n}e_{(r+1)a_n}.&\end{aligned}$$

Substitute $e_{(r+1)a_n}$ from (2) and notice that all the the coefficients are positive and, therefore, $T^{-}f_i=0$. Thus, case (D) does not produce any nontrivial entries in T^{-}.

Summarizing the calculations, the only nonzero entries of T^{-} are given by (1) and (3):

$$t^{-}_{1+v_{n-r},v_{n-r}+ra_n}=(n-r+1)^{-1-v_{n-r}}2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}}a_{n-r}(n-r)^{v_{n-r}}$$

and

$$t^{-}_{na_n+(r-1)b_n+1,na_n+rb_n}=b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}$$

for all $0 < r \leqslant n$. To show that T^- is nuclear it suffices to show $\sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty < +\infty$. Look at the rows of T^- containing non-zero entries. Notice that

$$t^-_{1+v_{n-r},v_{n-r}+ra_n} \leqslant a_{n-r}2^{(1+v_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}} \leqslant 2^{-(1+v_{n-r})}$$

for all $0 < r \leqslant n$ provided $\mathbf{d}$ increases sufficiently rapidly. It follows that $\|T^-_{(1+v_m)}\|_\infty \leqslant 2^{-(1+v_m)}$ for every $m \geqslant 0$ and $\sum\limits_{m=0}^{\infty}\|T^-_{(1+v_m)}\|_\infty \leqslant \sum\limits_{m=0}^{\infty} 2^{-1-v_m} < 1$.

Further, the entries $t^-_{na_n+(r-1)b_n+1,na_n+rb_n}$ do not depend on r, and their contribution to $\sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty$ does not exceed the sum of all of them, which can be easily estimated:

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n} b_n n^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant \sum_{n=1}^{\infty} b_n 2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant \sum_{n=1}^{\infty} 2^{-n} = 1,$$

because $b_n 2^{(1+na_n-b_n/2)/\sqrt{b_n}} \leqslant 2^{-n}$ for all $n \geqslant 1$ provided $\mathbf{d}$ increases sufficiently rapidly. Thus, $\nu(T^-) \leqslant \sum_{k=0}^{\infty}\|T^-_{(k)}\|_\infty < 2$ provided $\mathbf{d}$ increases sufficiently rapidly. □

Corollary 7. *C. J. Read's operator T satisfies the following properties, provided* $\mathbf{d}$ *increases sufficiently rapidly:*

(1) *$|T|$, T^+, and T^- have positive eigenvectors;*
(2) *Neither $|T|$ nor T^+ has an invariant ideal.*

Proof. It follows from Theorem 3 and Lemma 6 that $|T|$ has a positive eigenvector. It was noticed in the proof of Lemma 6 that $T^-f_0 = 0$, so that T^- also has a positive eigenvector.

To prove (2), assume that J is a closed ideal in ℓ_1 invariant under $|T|$ or T^+, and that $0 \neq x \in J$, then $x_k \neq 0$ for some $k \geqslant 0$, so that $f_k \in J$. It follows from the proof of Lemma 6 that both $|T|f_i$ and T^+f_i have nonzero $(i+1)$-th component, implying $f_{k+1} \in J$. Proceeding inductively, we see that $f_i \in J$ for all $i \geqslant k$. Further, the proof of Lemma 6 also shows that $(|T|f_i)_0 \neq 0$ and $(T^+f_i)_0 \neq 0$ for infinitely many i's, so that $f_0 \in J$. It follows that $f_i \in J$ for every $i \geqslant 0$, so that $J = \ell_1$. In fact, (2) is a manifestation of the fact that a positive operator S on ℓ_p $(1 \leqslant p < \infty)$ has no invariant ideals if and only if there is a path between every two columns of S (c.f. [AAB3, TV2]).

It follows from (2) and Theorem 1 that T^+ cannot be quasinilpotent. On the other hand, since $T^+ = T + T^-$ then, analogously to the proof of Theorem 3, we have $r_e(T^+) = 0$. Then by Theorem 2 we conclude that $r(T^+)$ is a positive eigenvalue of T^+, corresponding to a positive eigenvector. □

The last statement of Corollary 7 emphasizes that the hypothesis of not having invariant ideals in Theorem 3 is weaker than not having invariant subspaces. We do not know if the analogues of the results of this paper hold for the operators produced in [R1, R2].

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We remark in conclusion that in [TV1] we use one of C. J. Read's operators to solve one more problem related to invariant subspaces. Namely, we construct operators S_1, S_2, and K (not multiples of the identity) on ℓ_1 such that T commutes with S_1, S_1 commutes with S_2, S_2 commutes with K, and K is compact. This shows that the celebrated Lomonosov theorem cannot be extended to chains of four operators.

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