ON QUASI-AFFINE TRANSFORMS OF READ'S OPERATOR

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Abstract. We show that C. J. Read's example [Read85, Read86] of an operator T on ℓ_1 which does not have any non-trivial invariant subspaces is not the adjoint of an operator on a predual of ℓ_1. Furthermore, we present a bounded diagonal operator D such that even though D^{-1} is unbounded but $D^{-1}TD$ is a bounded operator on ℓ_1 with invariant subspaces, and is adjoint to an operator on c_0.

1. Introduction

In this note we deal with the Invariant Subspace Problem, the problem of the existence of a closed non-trivial invariant subspace for a given bounded operator on a Banach space. The problem was solved in the positive for certain classes of operators (see [RR73, AAB98] for details), however in the mid-seventies P. Enflo [Enf76, Enf87] constructed an example of a continuous operator on a Banach space with no invariant subspaces, thus answering the Invariant Subspace Problem for general Banach spaces in the negative. In [Read85] C. J. Read presented an example of a bounded operator T on ℓ_1 with no invariant subspace. Recently V. Lomonosov suggested that every adjoint operator has an invariant subspace. In the first part of this note we show that the Read operator T is not an adjoint of any bounded operator defined on some predual of ℓ_1.

Suppose that A has a non-trivial invariant (or a hyperinvariant) subspace, and suppose that B is similar to A, that is, $B = CAC^{-1}$ for some invertible operator C. Clearly, B also has a non-trivial invariant (respectively hyperinvariant) subspace. Moreover, it is known (see [RR73, Theorem 6.19]) that if A has a hyperinvariant subspace and B is quasi-similar to A (that is, $CA = BC$ and $AD = DB$, where C and D are two bounded one-to-one operators with dense range), then B also has a hyperinvariant subspace. To our knowledge it is still unknown whether or not A has a non-trivial invariant subspace if and only if B has a non-trivial invariant subspace, assuming A and B are quasi-similar.

Recall (cf. [Sz-NF68]) that an operator A is said to be a *a quasi-affine transform of* B if $CA = BC$, for some injective operator C with dense range. In the second part of this paper we construct an injective diagonal operator D on ℓ_1 such that even though D^{-1} is unbounded, the operator $S = D^{-1}TD$ (T being Read's operator) is bounded and has an invariant subspace. Thus, we show that a quasi-affine transform of an operator with no non-trivial invariant subspace might have a non-trivial invariant subspace. Furthermore, S is the adjoint of a bounded operator on c_0.

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Although we prove our statement for a specific choice of D, it is true for a much more general choice, and it seems to be true for any diagonal operator D that $S = D^{-1}TD$ has a non-trivial invariant subspace, whenever S is an adjoint of an operator on c_0. More generally, the following question is of interest in view of the above-mentioned conjecture by V. Lomonosov.

Question. Does every quasi-affine transform of Read's operator, which is an adjoint of an operator on c_0, have a non-trivial invariant subspace?

We introduce the following notations. Following [Read86] we denote by F the vector space of all eventually vanishing scalar sequences, and by (f_i) the standard unit vector basis of F. For an $x = \sum a_i f_i \in F$, we define the *support of* x to be the set $\{i \in \mathbb{N} : a_i \neq 0\}$ and denote it by $\operatorname{supp}(x)$. The linear span of some subset A of a vector space is denoted by $\operatorname{lin} A$.

2. Read's operator is not adjoint

We begin by reminding the reader of the construction of the operator T in [Read85, Read86]. It depends on a strictly increasing sequence $\mathbf{d} = (a_1, b_1, a_2, b_2, \ldots)$ of positive integers which has to be chosen to be *sufficiently rapidly increasing*. Also let $a_0 = 1$, $v_0 = 0$, and $v_n = n(a_n + b_n)$ for $n \geqslant 1$.

Read's operator T is defined by prescribing the orbit $(e_i)_{i\geqslant 0}$ of the first basis element f_0.

Definition 2.1. There is a unique sequence $(e_i)_{i=0}^{\infty} \subset F$ with the following properties:

(0) $f_0 = e_0$;

(A) if integers r, n, and i satisfy $0 < r \leqslant n$, $i \in [0, v_{n-r}] + ra_n$, we have
$$f_i = a_{n-r}(e_i - e_{i-ra_n});$$

(B) if integers r, n, and i satisfy $1 \leqslant r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$, (respectively, $1 \leqslant n$, $i \in (v_{n-1}, a_n)$), then
$$f_i = 2^{(h-i)/\sqrt{a_n}} e_i, \text{ where } h = (r + \tfrac{1}{2})a_n \text{ (respectively, } h = \tfrac{1}{2}a_n);$$

(C) if integers r, n, and i satisfy $1 \leqslant r \leqslant n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then
$$f_i = e_i - b_n e_{i-b_n};$$

(D) if integers r, n, and i satisfy $0 \leqslant r < n$, $i \in (na_n + rb_n, (r+1)(a_n + b_n))$, then
$$f_i = 2^{(h-i)/\sqrt{b_n}} e_i, \text{ where } h = (r + \tfrac{1}{2})b_n.$$

Indeed, since $f_i = \sum_{j=0}^{i} \lambda_{ij} e_j$ for each $i \geqslant 0$ and λ_{ii} is always nonzero, this linear relation is invertible. Further,
$$\operatorname{lin}\{e_i \mid i = 1, \ldots, n\} = \operatorname{lin}\{f_i \mid i = 1, \ldots, n\} \text{ for every } n \geqslant 0.$$
In particular, all e_i are linearly independent and also span F. Then Read defines $T\colon F \to F$ to be the unique linear map such that $Te_i = e_{i+1}$. Read proves that T can be extended to a bounded operator on ℓ_1 with no invariant subspaces provided $\mathbf{d}$ increases sufficiently rapidly.

Proposition 2.2. *T is not the adjoint of an operator $S : X \to X$ where X is a Banach space whose dual is isometric to ℓ_1.*

Proof. Assume that our claim is not true. Then there is a local convex topology τ on ℓ_1 so that

(a) τ is weaker than the norm topology of ℓ_1;
(b) $\mathrm{B}(\ell_1)$ is sequentially compact with respect to τ;
(c) if $(x_n) \subset \ell_1$ converges with respect to τ to x, then $\liminf_{n\to\infty}\|x_n\| \geqslant \|x\|$;
(d) T is continuous with respect to τ.

Note that with respect to any predual X of ℓ_1 the weak* topology has properties (a)–(d). Let $s \in \mathbb{N}$ be fixed, and $n > s$. Then $f_{(n-s)a_n} = a_s(e_{(n-s)a_n} - e_0)$ by (A) above. It follows that $T^{v_s+1}f_{(n-s)a_n} = a_s(e_{(n-s)a_n+v_s+1} - e_{v_s+1})$. Further, it follows from (B) that $e_{(n-s)a_n+v_s+1}$ equals $2^{(1+v_s-\frac{1}{2}a_n)/\sqrt{a_n}} f_{(n-s)a_n+v_s+1}$ and converges to zero in norm (and, hence, in τ) as $n \to \infty$. Therefore

$$\tau\text{-}\lim_{n\to\infty} T^{v_s+1}f_{(n-s)a_n} = -a_s e_{v_s+1} = T^{v_s+1}(-a_s e_0). \tag{1}$$

Notice that T^{v_s+1} is τ-continuous and one-to-one because its null space is T-invariant. By sequential compactness of $B(\ell_1)$, the sequence $f_{(n-s)a_n}$ must have a τ-convergent subsequence. Then, by (1), the limit point has to be $-a_s e_0$. Since that argument applies to any subsequence, we deduce that

$$\tau\text{-}\lim_{n\to\infty} f_{(n-s)a_n} = -a_s e_0. \tag{2}$$

Since $\|f_{(n-s)a_n}\| = 1$ for each n and s while $\|a_s e_0\| = a_s > 1$, this contradicts (2). ☐

Remark. The statement of the theorem remains valid if we consider an equivalent norm on ℓ_1. Indeed, suppose $\frac{1}{K}|\!|\!|\cdot|\!|\!| \leqslant \|\cdot\| \leqslant K|\!|\!|\cdot|\!|\!|$. Then $|\!|\!|f_{(n-s)a_n}|\!|\!| \leqslant K$ for each n and s, but since $\lim_{n\to\infty} a_n = \infty$, we can choose a_s in (2) so that $|\!|\!|a_s e_0|\!|\!| > K$.

3. An adjoint operator with invariant subspaces of the form $D^{-1}TD$

Define a sequence of positive reals (d_i) as follows:

$$d_i = \begin{cases} \frac{1}{r} & \text{if } ra_m \leqslant i \leqslant ra_m + v_{m-r} \text{ for some } 0 < r \leqslant m, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

Let D be the diagonal operator with diagonal (d_i), that is, $Df_i = d_i f_i$ for every i. Define $S = D^{-1}TD$. Clearly, S is defined on F. Once we write S in matrix form it will be clear that it is bounded on F and, therefore, can be extended to ℓ_1. Let $\hat{e}_i = D^{-1}e_i$, in particular $\hat{e}_0 = e_0$. Then $S\hat{e}_i = D^{-1}Te_i = \hat{e}_{i+1}$, so that the sequence $(\hat{e}_i)$ is the orbit of e_0 under S.

Next, we examine Definition 2.1 to represent the f_i's in terms of $\hat{e}_i$'s.

$(\widehat{0})$ $f_0 = e_0 = \hat{e}_0$;

$(\widehat{\mathrm{A}})$ if i satisfies $i \in [0, v_{n-r}] + ra_n$ for some $0 < r \leqslant n$, then

$$f_i = d_i D^{-1} f_i = d_i D^{-1}\big(a_{n-r}(e_i - e_{i-ra_n})\big) = \tfrac{a_{n-r}}{r}(\hat{e}_i - \hat{e}_{i-ra_n});$$

$(\widehat{\mathrm{B}})$ if integers r, n, and i satisfy $1 \leqslant r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$, (respectively, $1 \leqslant n$, $i \in (v_{n-1}, a_n)$), then

$$f_i = d_i D^{-1} f_i = 2^{(h-i)/\sqrt{a_n}}\hat{e}_i, \text{ where } h = (r + \tfrac{1}{2})a_n \text{ (respectively, } h = \tfrac{1}{2}a_n\text{);}$$

($\widehat{\mathrm{C}}$) if integers r, n, and i satisfy $1 \leqslant r \leqslant n$, $i \in [r(a_n+b_n), na_n + rb_n]$, then
$$f_i = d_i D^{-1} f_i = \hat{e}_i - b_n \hat{e}_{i-b_n};$$

($\widehat{\mathrm{D}}$) if integers r, n, and i satisfy $0 \leqslant r < n$, $i \in (na_n + rb_n, (r+1)(a_n+b_n))$, then
$$f_i = d_i D^{-1} f_i = 2^{(h-i)/\sqrt{b_n}} \hat{e}_i, \text{ where } h = (r + \tfrac{1}{2})b_n.$$

We see that it differs from Definition 2.1 only in case ($\widehat{\mathrm{A}}$). Now we can actually write the matrix of S:

$$Sf_i = \begin{cases}
2^{(1-\frac{1}{2}a_1)/\sqrt{a_1}} f_1 & \text{if } i = 0 \\
f_{i+1} & \text{if } i \in [0, v_{n-r}) + ra_n, \\
 & \quad \text{with } r = 1, 2, \dots, n \\
f_{i+1} & \text{if } i \in [r(a_n+b_n), na_n + rb_n), \\
 & \quad \text{with } r = 1, 2, \dots, n \\
2^{1/\sqrt{a_n}} f_{i+1} & \text{if } i \in (ra_n + v_{n-r}, (r+1)a_n - 1), \\
 & \quad \text{with } r = 1, 2, \dots, n-1 \\
 & \quad \text{or } i \in (v_{n-1}, a_n - 1) \\
2^{1/\sqrt{b_n}} f_{i+1} & \text{if } i \in (na_n + rb_n, (r+1)(a_n+b_n) - 1) \\
 & \quad \text{with } r = 0, 1, \dots, n-1 \\
\frac{a_{n-r}}{r}(\varepsilon_1 f_{i+1} - \varepsilon_2 f_{v_{n-r}+1}) & \text{if } i = ra_n + v_{n-r}, \\
\quad \text{where} & \quad \text{with } r = 1, 2, \dots, n \\
\qquad \varepsilon_2 = 2^{(1+v_{n-r}-\frac{1}{2}a_{n-r+1})/\sqrt{a_{n-r+1}}} & \\
\qquad \varepsilon_1 = 2^{(1+v_{n-r}-\frac{1}{2}a_n)/\sqrt{a_n}} & \text{if } r < n \text{ and} \\
\qquad \varepsilon_1 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \text{if } r = n, \\
2^{(1-\frac{1}{2}a_n)/\sqrt{a_n}} \big[f_0 + \frac{(r+1)f_{i+1}}{a_{n-r-1}}\big] & \text{if } i = (r+1)a_n - 1 \\
 & \quad \text{with } r = 0, 1, \dots, n-1 \\
\varepsilon_1 f_{i+1} - b_n \varepsilon_2 f_{i+1-b_n} & \text{if } i = na_n + rb_n \\
\quad \text{where} & \quad \text{with } r = 1, 2, \dots, n \\
\qquad \varepsilon_2 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \\
\qquad \varepsilon_1 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \text{if } r < n, \text{ and} \\
\qquad \varepsilon_1 = 2^{(v_n+1-\frac{1}{2}a_{n+1})/\sqrt{a_{n+1}}} & \text{if } r = n \\
2^{-((r+1)a_n + \frac{1}{2}b_n - 1)/\sqrt{b_n}} & \text{if } i = (r+1)(a_n+b_n) - 1 \\
\quad \cdot \Big[\sum_{j=0}^{r} b_n^j f_{i-jb_n+1} & \\
\qquad + b_n^{r+1}\big(f_0 + \frac{(r+1)f_{(r+1)a_n}}{a_{n-r-1}}\big)\Big] & \quad \text{with } r = 0, 1, \dots, n-1
\end{cases}$$

Inspecting the matrix line by line we observe that, assuming (a_n) and (b_n) are increasing sufficiently rapidly, it follows that $\|S\| \leqslant 2$. Again by inspecting each line of the matrix, we deduce that if f_j^* is the j-th coordinate functional on ℓ_1, $j \geqslant 0$, it follows that $\lim_{i\to\infty} f_j^*(S(f_i)) = 0$. In other words, the rows of the matrix converge to zero. Therefore S is the adjoint of a linear bounded operator on c_0.

Theorem 3.1. *S has a non-trivial closed invariant subspace.*

We shall show that S has an invariant subspace by producing a vector x_∞ such that the linear span of the orbit of x_∞ stays away from e_0, hence its closure is a non-trivial S-invariant subspace.

We will introduce the following notations.

First we choose two sequences of positive integers (m_i) and (r_i) as follows. Let $m_0 \geqslant 2$ be arbitrary, put $r_0 = 1$. Once m_i and r_i are defined, choose $r_{i+1} \in \mathbb{N}$ so that

$$r_{i+1} \in [a_{m_i-1} \cdot \max_{\ell \leqslant v_{m_i-1}} \|\hat{e}_\ell\|, 1 + a_{m_i-1} \cdot \max_{\ell \leqslant v_{m_i-1}} \|\hat{e}_\ell\|] \tag{4}$$

and let

$$m_{i+1} = m_i + r_{i+1}. \tag{5}$$

Define an increasing sequence (j_i) of positive integers inductively: pick any

$$j_0 \in [r_0 a_{m_0}, r_0 a_{m_0} + v_{m_0 - r_0}], \tag{6}$$

and once j_i is defined, put

$$j_{i+1} = j_i + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}. \tag{7}$$

Finally, for each $i \geqslant 0$ define

$$p_i = \prod_{k=0}^{i} b_{m_k}^{-r_k}, \tag{8}$$

$$z_i = f_{j_i + r_i b_{m_i}} + b_{m_i} f_{j_i + (r_i - 1) b_{m_i}} + \cdots + b_{m_i}^{r_i - 1} f_{j_i + b_{m_i}} + \frac{r_{i+1} f_{j_{i+1}}}{a_{m_i}}, \tag{9}$$

$$x_i = p_{i-1} \hat{e}_{j_i}. \tag{10}$$

We note the following easy-to-prove properties for our choices.

Proposition 3.2. *For each $i \geqslant 0$ the following statements hold:*

(a) $j_i \in [r_i a_{m_i}, r_i a_{m_i} + v_{m_i - r_i}]$*;*
(b) $x_{i+1} = x_i + p_i z_i$*, and thus* $x_i = \hat{e}_{j_0} + \sum_{k=0}^{i-1} p_k z_k$*;*
(c) *if i and $i+\ell$ both belong to $[r a_n, r a_n + v_{n-r}]$ or both belong to $[r(a_n + b_n), n a_n + r b_n]$, then $S^\ell f_i = f_{i+\ell}$;*
(d) *if $\ell < m_i a_{m_i} - j_i$, then* $\min \operatorname{supp} S^\ell z_k \geqslant j_i + b_{m_i}$ *whenever $k \geqslant i$.*

Proof. (a) The proof is by induction. For $i = 0$ the required inclusion follows from the choice of j_0, and if this condition holds for j_i, then

$$\begin{aligned} j_{i+1} &= j_i + r_i b_{m_i} + r_{i+1} a_{m_{i+1}} \\ &\in [r_i a_{m_i} + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}, r_i a_{m_i} + v_{m_i - r_i} + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}] \\ &\subseteq [r_{i+1} a_{m_{i+1}}, r_{i+1} a_{m_{i+1}} + m_i (a_{m_i} + b_{m_i})] = [r_{i+1} a_{m_{i+1}}, r_{i+1} a_{m_{i+1}} + v_{m_i}]. \end{aligned}$$

(b) First note that by using $(\widehat{\mathrm{D}})$ we obtain for a $i \in [r(a_n+b_n), na_n+rb_n]$, with $1 \leqslant r \leqslant n$ in $\mathbb{N}$, that

$$\begin{aligned} \hat{e}_i &= b_n \hat{e}_{i-b_n} + f_i \\ &= b_n^2 \hat{e}_{i-2b_n} + b_n f_{i-b_n} + f_i \\ &\vdots \\ &= b_n^r \hat{e}_{i-rb_n} + b_n^{r-1} f_{i-(r-1)b_n} + \ldots + b_n f_{i-b_n} + f_i. \end{aligned} \tag{11}$$

Note that $j_i + r_i b_{m_i} \in [r_i(a_{m_i}+b_{m_i}), m_i a_{m_i} + r_i b_{m_i}]$. By using first $(\widehat{\mathrm{A}})$ and then (11) we obtain

$$\begin{aligned} \hat{e}_{j_{i+1}} &= \hat{e}_{j_i + r_i b_{m_i} + r_{i+1} a_{m_i}} \\ &= \hat{e}_{j_i + r_i b_{m_i}} + \frac{r_{i+1}}{a_{m_i}} f_{j_i + r_i b_{m_i} + r_{i+1} a_{m_i}} \\ &= b_{m_i}^{r_i} \hat{e}_{j_i} + b_{m_i}^{r_i - 1} f_{j_i + b_{m_i}} + \ldots + b_{m_i} f_{j_i + (r_i - 1) b_{m_i}} + \frac{r_{i+1}}{a_{m_i}} f_{j_i + r_i b_{m_i} + r_{i+1} a_{m_i}} \\ &= b_{m_i}^{r_i} \hat{e}_{j_i} + z_i. \end{aligned}$$

Thus, $x_{i+1} = p_i \hat{e}_{j_{i+1}} = p_{i-1} \hat{e}_{j_i} + p_i z_i = x_i + p_i z_i$.
(c) If i and $i+\ell$ are both in $[ra_n, ra_n + v_{n-r}]$, it follows from $(\widehat{\mathrm{A}})$ that

$$S^\ell(f_i) = \frac{a_{n-r}}{r} S^\ell(\hat{e}_i - \hat{e}_{i-ra_n}) = \frac{a_{n-r}}{r}(\hat{e}_{i+\ell} - \hat{e}_{i-ra_n+\ell}) = f_{i+\ell}.$$

The second part of (c) can be deduced in a similar way using $(\widehat{\mathrm{C}})$.
(d) First note that for $k \geqslant i$ it follows that (recall that $m_k \geqslant m_0 \geqslant 2$)

$$m_k a_{m_k} - j_k > (m_k - r_k - 1) a_{m_k} = (m_{k-1} - 1) a_{m_k} \geqslant m_{k-1} a_{m_{k-1}} - j_{k-1}.$$

We can therefore assume that $k = i$. Furthermore, note that for any $1 \leqslant r \leqslant r_i$ it follows that

$$r(a_{m_i} + b_{m_i}) \leqslant j_i + rb_{m_i} \leqslant j_i + rb_{m_i} + \ell \leqslant m_i a_{m_i} + rb_{m_i}$$

and

$$\begin{aligned} r_{i+1} a_{m_{i+1}} \leqslant j_{i+1} \leqslant j_{i+1} + \ell &\leqslant j_{i+1} + m_i a_{m_i} - j_i \\ &= r_{i+1} a_{m_{i+1}} + r_i b_{m_i} + m_i a_{m_i} \\ &\leqslant r_{i+1} a_{m_{i+1}} + v_{m_i} \\ &= r_{i+1} a_{m_{i+1}} + v_{m_{i+1} - r_{i+1}}. \end{aligned}$$

Therefore the claim follows from the definition of z_i, (9) and part (c). $\square$

Notice that

$$\|z_i\| = 1 + b_{m_i} + b_{m_i}^2 + \cdots + b_{m_i}^{r_i - 1} + \frac{r_{i+1}}{a_{m_i}} \leqslant m_i b_{m_i}^{r_i - 1} + \frac{r_{i+1}}{a_{m_i}}.$$

Further, since $p_i \leqslant \frac{1}{b_{m_i}^{r_i}}$, we have

$$\|p_i z_i\| \leqslant \frac{m_i}{b_{m_i}} + \frac{r_{i+1}}{a_{m_i} b_{m_i}^{r_i}}.$$

The series $\sum_{i=0}^{\infty}\frac{m_i}{b_{m_i}}$ converges because (b_i) increases sufficiently rapidly. Secondly, it follows from the definition of (r_i) that

$$a_{m_i}^{-1}r_{i+1}\leqslant a_{m_i}^{-1}[1+a_{m_i-1}\cdot\max_{\ell\leqslant v_{m_i-1}}\|\hat{e}_\ell\|].$$

Thus, again since (b_i) is increasing fast enough, it follows that the series

$$\sum_{i=0}^{\infty}\frac{r_{i+1}}{a_{m_i}b_{m_i}^{r_i}}$$

converges. Therefore the $\sum_{i=0}^{\infty}p_iz_i$ converges, and the following definition is justified.

Definition 3.3. Define $x_\infty=\lim_i x_i=\lim_i p_{i-1}\hat{e}_{j_i}=\hat{e}_{j_0}+\sum_{i=0}^{\infty}p_iz_i$.

Now we can state and prove the key result for proving Theorem 3.1.

Lemma 3.4. *There exists a constant $C>0$ such that $\operatorname{dist}(y,e_0)\geqslant C$ for every i and every vector of the form $y=\sum_{j=j_i}^{m_ia_{m_i}}\gamma_j\hat{e}_j$.*

Proof. Let $C=\inf\Big\{\operatorname{dist}(y,e_0)\mid y=\sum_{j=j_0}^{m_0a_{m_0}}\gamma_j\hat{e}_j\Big\}$. Since the infimum is taken over a finite-dimensional set, it must be attained at some y_0. However since all $\hat{e}_j$ are linear independent, it follows that $C=\operatorname{dist}(y_0,e_0)>0$.

We shall prove the statement of the lemma by induction on i. The way we defined C guarantees that the base of the induction holds. Suppose $y=\sum_{j=j_i}^{m_ia_{m_i}}\gamma_j\hat{e}_j$. Write $y=y_1+y_2+y_3$, where

$$y_1=\sum_{j=j_i}^{r_ia_{m_i}+v_{m_{i-1}}}\gamma_j\hat{e}_j,\quad y_2=\sum_{r=r_i+1}^{m_i}\sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}}\gamma_j\hat{e}_j,\quad\text{and}\quad y_3=\sum_{r=r_i}^{m_i-1}\sum_{j=ra_{m_i}+v_{m_i-r}+1}^{(r+1)a_{m_i}-1}\gamma_j\hat{e}_j.$$

Notice that by $(\widehat{\mathrm{B}})$

$$y_3=\sum_{r=r_i}^{m_i-1}\sum_{j=ra_{m_i}+v_{m_i-r}+1}^{(r+1)a_{m_i}-1}\gamma_j2^{-(r+\frac{1}{2}-j)/\sqrt{a_{m_i}}}f_j,$$

so that $\operatorname{supp}y_3\subseteq\bigcup_{r=r_i}^{m_i-1}(ra_{m_i}+v_{m_i-r},(r+1)a_{m_i})$. Furthermore, using $(\widehat{\mathrm{A}})$, we write $y_2=y_2'+y_2''$ where

$$y_2'=\sum_{r=r_i+1}^{m_i}\sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}}\gamma_j\hat{e}_{j-ra_{m_i}}=\sum_{r=r_i+1}^{m_i}\sum_{j=0}^{v_{m_i-r}}\gamma_{j+ra_{m_i}}\hat{e}_j$$

$$\text{and}\quad y_2''=\sum_{r=r_i+1}^{m_i}\sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}}\frac{\gamma_jr}{a_{m_i-r}}f_j.$$

Therefore,

$$\operatorname{supp}(y_1+y_2)\subseteq[0,r_ia_{m_i}+v_{m_{i-1}}]\cup\bigcup_{r=r_i+1}^{m_i}[ra_{m_i},ra_{m_i}+v_{m_i-r_i}].$$

One observes that the vectors y_1+y_2 and y_3 have disjoint supports; it follows that $\operatorname{dist}(y,e_0)\geqslant\operatorname{dist}(y_1+y_2,e_0)$.

Furthermore,

$$\|y_2'\| = \Big\| \sum_{r=r_i+1}^{m_i} \sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}} \gamma_j \hat{e}_{j-ra_{m_i}} \Big\| \leqslant \sum_{r=r_i+1}^{m_i} \sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}} |\gamma_j| \cdot \max_{k\leqslant v_{m_{i-1}-1}} \|\hat{e}_k\|.$$

By choice of (r_i) (4), we have $\max_{k\leqslant v_{m_{i-1}-1}} \|\hat{e}_k\| \leqslant \frac{r_i}{a_{m_i-r_i-1}} \leqslant \frac{r}{a_{m_i-r}}$ when $r_i < r \leqslant m_i$. This yields

$$\|y_2'\| \leqslant \Big\| \sum_{r=r_i+1}^{m_i} \sum_{j=ra_{m_i}}^{ra_{m_i}+v_{m_i-r}} \frac{\gamma_j r}{a_{m_i-r}} f_j \Big\| = \|y_2''\|.$$

Since the support of y_2'' is disjoint from that of $y_1 + y_2'$ and doesn't contain 0, we have

$$\begin{aligned}\operatorname{dist}(y_1, e_0) &\leqslant \operatorname{dist}(y_1 + y_2', e_0) + \|y_2'\| \\ &= \operatorname{dist}(y_1 + y_2' + y_2'', e_0) - \|y_2''\| + \|y_2'\| \\ &\leqslant \operatorname{dist}(y_1 + y_2, e_0) \leqslant \operatorname{dist}(y, e_0).\end{aligned}$$

It is left to show that $\operatorname{dist}(y_1, e_0) \geqslant C$. Since $j_i \geqslant r_i a_{m_i}$, it follows from $(\widehat{\mathrm{A}})$ that $y_1 = y_1' + y_1''$ where

$$y_1' = \sum_{j=j_i}^{r_i a_{m_i}+v_{m_{i-1}}} \gamma_j \hat{e}_{j-r_i a_{m_i}} \quad \text{and} \quad y_1'' = \sum_{j=j_i}^{r_i a_{m_i}+v_{m_{i-1}}} \frac{\gamma_j r}{a_{m_i-r_i}} f_j.$$

Since $j_i = j_{i-1} + r_{i-1} b_{m_{i-1}} + r_i a_{m_i}$, we have $y_1' = \sum_{j=j_{i-1}+r_{i-1}b_{m_{i-1}}}^{v_{m_{i-1}}} \beta_j \hat{e}_j$, where $\beta_j = \gamma_{j+r_i a_{m_i}}$. In particular this means, that $\operatorname{supp} y_1' \subseteq [0, v_{m_{i-1}}]$, while $\min\operatorname{supp} y_1'' \geqslant j_i \geqslant r_i a_{m_i}$. Thus, the supports are disjoint, which yields $\operatorname{dist}(y_1, e_0) \geqslant \operatorname{dist}(y_1', e_0)$.

Split the index set of y_1' into two disjoint subsets: let

$$A = [j_{i-1} + r_{i-1} b_{m_{i-1}}, v_{m_{i-1}}] \cap \bigcup_{r=r_{i-1}}^{m_{i-1}} \big(m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}, (r+1)(a_{m_{i-1}} + b_{m_{i-1}})\big),$$

$$B = [j_{i-1} + r_{i-1} b_{m_{i-1}}, v_{m_{i-1}}] \cap \bigcup_{r=r_{i-1}}^{m_{i-1}} \big[r(a_{m_{i-1}} + b_{m_{i-1}}), m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}\big].$$

Write $y_1' = z_a + z_b$ where $z_a = \sum_{j\in A} \beta_j \hat{e}_j$ and $z_b = \sum_{j\in B} \beta_j \hat{e}_j$. For $j \in A$ we have $\hat{e}_j = 2^{((r+1/2)b_{m_{i-1}}-j)/\sqrt{b_{m_{i-1}}}} f_j$, so that $\operatorname{supp} z_a \subseteq A$. In view of (11) we can write $z_b = z_b' + z_b''$, where

$$z_b' = \sum_{j\in B} \sum_{k=0}^{r-1} \beta_j b_{m_{i-1}}^k f_{j-kb_{m_{i-1}}} \quad \text{and} \quad z_b'' = \sum_{j\in B} \beta_j b_{m_{i-1}}^r \hat{e}_{j-rb_{m_{i-1}}}.$$

We first note that $\operatorname{supp} z_b' \subseteq B$ and determine the support of z_b'' as follows. If $j \in B$, then $j \geqslant j_{i-1} + r_{i-1} b_{m_{i-1}}$ and $j \in \big[r(a_{m_{i-1}} + b_{m_{i-1}}), m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}\big]$ for some $r \in [r_{i-1}, m_{i-1}]$. If $r = r_{i-1}$, then $j - r b_{m_{i-1}} \geqslant j_{i-1}$. If $r > r_{i-1}$, then $j - r b_{m_{i-1}} \geqslant r a_{m_{i-1}} > r_{i-1} a_{m_{i-1}} + v_{m_{i-2}} \geqslant j_{i-1}$ by (7). We see that z_b'' is a linear combination of $\hat{e}_j$'s with $j_{i-1} \leqslant j \leqslant m_{i-1} a_{m_{i-1}}$. Hence its support is contained in $[0, m_{i-1} a_{m_{i-1}}]$ and, therefore, is disjoint from that of z_a and z_b'. It follows that $\operatorname{dist}(y, e_0) \geqslant \operatorname{dist}(y_1', e_0) \geqslant \operatorname{dist}(z_b'', e_0)$. Finally, $\operatorname{dist}(z_b'', e_0) \geqslant C$ by the induction hypothesis. ☐

Proof of Theorem 3.1. We will prove that the linear span of the orbit of x_∞ under S is at least distance C from e_0, hence its closure is a non-trivial invariant subspace for S. Consider a linear combination $\sum_{\ell=0}^{N}\alpha_\ell S^\ell x_\infty$. It follows from (7) that the sequence $(m_i a_{m_i} - j_i)$ is unbounded, so that $N < m_i a_{m_i} - j_i$ for some $i \geqslant 0$. Recall that $x_\infty = x_i + \sum_{k=i}^{\infty} p_k z_k$; then

$$\sum_{\ell=0}^{N}\alpha_\ell S^\ell x_\infty = \sum_{s=0}^{N}\alpha_\ell S^\ell x_i + \sum_{\ell=0}^{N}\sum_{k=i}^{\infty}\alpha_\ell S^\ell(p_k z_k).$$

Notice that the two sums have disjoint supports, and the support of the second one does not contain 0. Indeed, since $x_i = p_{i-1}\hat{e}_{j_i}$ then $S^\ell x_i = p_{i-1}\hat{e}_{j_i+\ell}$ for $\ell = 1, \ldots, N$. Furthermore,

$$j_i \leqslant j_i + \ell \leqslant j_i + N < j_i + (m_i a_{m_i} - j_i) = m_i a_{m_i}.$$

It follows that $\sum_{\ell=0}^{N} S^\ell x_i$ is a linear combination of $\hat{e}_j$'s with $j_i \leqslant j \leqslant m_i a_{m_i}$. In particular, its support is contained in $[0, m_i a_{m_i}]$. On the other hand, Proposition 3.2 (d) implies that

$$\min\operatorname{supp}\Bigl(\sum_{\ell=0}^{N}\sum_{k=i}^{\infty} S^\ell(p_k z_k)\Bigr) \geqslant j_i + b_{m_i}.$$

Therefore, by Lemma 3.4

$$\operatorname{dist}\Bigl(\sum_{\ell=0}^{N} S^\ell x_\infty, e_0\Bigr) \geqslant \operatorname{dist}\Bigl(\sum_{\ell=0}^{N} S^\ell x_i, e_0\Bigr) \geqslant C.$$

□

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