SPECTRAL RADII OF BOUNDED OPERATORS ON TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper we develop a version of spectral theory for bounded linear operators on topological vector spaces. We show that Gelfand's formula for the spectral radius and Neumann series can be naturally interpreted for operators on topological vector spaces. Of course, the resulting theory has many similarities to the conventional spectral theory of bounded operators on Banach spaces, though there are several important differences. The main difference is that an operator on a topological vector space has several spectra and several spectral radii, which fit a well-organized pattern.

Key words: *spectrum, spectral radius, bounded operator, topological vector space, locally convex space, closed operator.*

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Contents

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0. Introduction

The spectral radius of a bounded linear operator T on a Banach space is defined by Gelfand's formula $r(T) = \lim_n \sqrt[n]{\|T^n\|}$. It is well known that $r(T)$ equals the actual radius of the spectrum $\bigl|\sigma(T)\bigr| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Furthermore, it is known that the resolvent $R_\lambda = (\lambda I - T)^{-1}$ is given by the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$ whenever $|\lambda| > r(T)$. It is natural to ask if similar results are valid in a more general setting, e.g., for a bounded linear operator on an arbitrary topological vector space. The author arrived at these questions when generalizing some results on the Invariant Subspace Problem from Banach lattices to ordered topological vector spaces. A major difficulty is the lack of a readily available developed spectral theory. The spectral theory of bounded operators on Banach spaces has been thoroughly studied for a long time, and is extensively used. Unfortunately, little has been known about the spectral theory of bounded operators on general topological vector spaces, and many techniques used in Banach spaces cannot be applied for operators on topological vector spaces. In particular, the spectrum, the spectral radius, and the Neumann series are the tools which are widely used in the study of the Invariant Subspace Problem in Banach spaces, but which have not been sufficiently studied for general topological vector spaces. To overcome this obstacle we have developed a version of the spectral theory of bounded operators on general topological vector spaces and on locally convex spaces. Some results in this direction have also been obtained by B. Gramsch [10], and by F. Garibay and R. Vera [8, 9, 20].

We consider the following classification of bounded operators on a topological vector space. We call a linear operator T

- ***nb-bounded*** if T maps some neighborhood of zero into a bounded set,
- ***nn-bounded*** if there is a base of neighborhoods of zero such that T maps every neighborhood in this base into a multiple of itself, and
- ***bb-bounded*** if T maps bounded sets into bounded sets.

The classes of all linear operators, of all bb-bounded operators, of all continuous operators, of all nn-bounded operators, and of all nb-bounded operators form nested algebras. The spectrum of an operator T in each of these algebras is defined as usual, i.e., the set of λ's for which $\lambda I - T$ is not invertible in this algebra. We show in Section 4 that the well known Gelfand formula for the spectral radius of an operator on a Banach space, $r(T) = \lim_n \sqrt[n]{\|T^n\|}$, can be generalized to each of these five classes of operators on topological vector spaces, and then we use this formula to define the spectral radius of an operator with respect to each of the classes. Then in Section 5 we show that if T is a continuous operator on a sequentially complete locally convex space and $|\lambda|$ is greater than the spectral radius of T, in any of the five classes, then the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges in the topology of the class, and λ does not belong to the corresponding spectrum of T. In particular, the spectral radius of T with respect to each class is greater than or equal to the geometrical radius of the spectrum of T in this class. In Sections 6 and 7 we show that the radii are equal for nb-bounded and compact operators.

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1. Preliminaries and notation

The symbols X and Y shall always denote topological vector spaces. A ***neighborhood*** of a point $x \in X$ is any subset of X containing an open set which contains x. Neighborhoods of zero will often be referred to as ***zero neighborhoods***. Every zero neighborhood V is ***absorbing***, i.e., $\bigcup_{n=1}^{\infty} nV = X$. In every topological vector space (over $\mathbb{R}$ or $\mathbb{C}$) there exists a base $\mathcal{N}_0$ of zero neighborhoods with the following properties:

(i) Every $V \in \mathcal{N}_0$ is ***balanced***, i.e., $\lambda V \subseteq V$ whenever $|\lambda| \leqslant 1$;
(ii) For every $V_1, V_2 \in \mathcal{N}_0$ there exists $V \in \mathcal{N}_0$ such that $V \subseteq V_1 \cap V_2$;
(iii) For every $V \in \mathcal{N}_0$ there exists $U \in \mathcal{N}_0$ such that $U + U \subseteq V$;
(iv) For every $V \in \mathcal{N}_0$ and every scalar λ the set λV is in $\mathcal{N}_0$.

Whenever we mention a basic neighborhood of zero, we always assume that it belongs to a base which satisfies these properties.

A topological vector space is called ***normed*** if the topology is given by a norm. In this case the collection of all balls centered at zero is a base of zero neighborhoods. A complete normed space is referred to as a ***Banach space***. See [6] for a detailed study of normed and Banach spaces.

A set A in a topological vector space is called ***bounded*** if it is absorbed by every zero neighborhood, i.e., for every zero neighborhood V one can find $\alpha > 0$ such that $A \subseteq \alpha V$. A set A in a topological vector space is said to be ***pseudo-convex*** or ***semi-convex*** if $A + A \subseteq \alpha A$ for some number α (for convex sets $\alpha = 2$). If U is a zero neighborhood, (x_γ) is a net in X, and $x \in X$, we write $x_\gamma \xrightarrow{U} x$ if for every $\varepsilon > 0$ there exists an index γ_0 such that $x_\gamma - x \in \varepsilon U$ whenever $\gamma \geqslant \gamma_0$. It is easy to see that when U is pseudo-convex, this convergence determines a topology on X, and the set of all scalar multiples of U forms a base of the topology. We denote X when equipped with this topology by (X, U). Clearly, (X, U) is Hausdorff if and only if $\bigcap_{n=1}^{\infty} \frac{1}{n} U = \{0\}$.

A topological vector space is said to be ***locally bounded*** if there exists a bounded zero neighborhood. Notice that if U is a bounded zero neighborhood then it is pseudo-convex. Conversely, if U is a pseudo-convex zero neighborhood, then (X, U) is locally bounded. Recall that a ***quasinorm*** is a real-valued function on a vector space which satisfies all the axioms of norm except the triangle inequality, which is replaced by $\|x+y\| \leqslant k\left(\|x\|+\|y\|\right)$ for some fixed positive constant k. It is known (see, e.g., [13]) that a topological vector space is quasinormable if and only if it is locally bounded and Hausdorff. A complete quasinormed space is called ***quasi-Banach***.

If the topology of a topological vector space X is given by a ***seminorm*** p, we say that $X=(X,p)$ is a ***seminormed*** space. Clearly, in this case $X=(X,U)$ where the convex set U is the unit ball of p and, conversely, p is the Minkowski functional of U. A Hausdorff topological vector space is called ***locally convex*** if there is a base of convex zero neighborhoods or, equivalently, if the topology is generated by a family of seminorms (namely, the Minkowski functionals of the convex zero neighborhoods). When dealing with a base of zero neighborhoods in a locally convex space we will always assume that it consists of convex sets. Similarly, a Hausdorff topological vector space is said to be ***locally pseudo-convex*** if it has a base of pseudo-convex zero neighborhoods. A complete metrizable topological vector space is usually referred to as a ***Fréchet space***.

By an ***operator*** we always mean a ***linear operator*** between vector spaces. We will usually use the symbols S and T to denote operators. Recall that an operator T between normed spaces is said to be ***bounded*** if its operator norm defined by $\|T\|=\sup\{\|Tx\| : \|x\|\leqslant 1\}$ is finite. It is well known that an operator between normed spaces is bounded if and only if it is continuous. An operator between two vector spaces is said to be of ***finite rank*** if the range of T is finite dimensional.

If $\mathcal{A}$ is a unital algebra and $a\in\mathcal{A}$, then the ***resolvent set*** of a is the set $\rho(a)$ of all $\lambda\in\mathbb{C}$ such that $e-\lambda a$ is invertible in $\mathcal{A}$. The resolvent set of an element a in a non-unital algebra $\mathcal{A}$ is defined as the set of all $\lambda\in\mathbb{C}$ for which $e-\lambda a$ is invertible in the unitalization $\mathcal{A}_\times$ of $\mathcal{A}$. The ***spectrum*** of an element of an algebra is defined via $\sigma(a)=\mathbb{C}\setminus\rho(a)$. It is well-known that whenever $\mathcal{A}$ is a unital Banach algebra then $\sigma(a)$ is compact and nonempty for every $a\in\mathcal{A}$. In this case the ***spectral radius*** $r(a)$ is defined via ***Gelfand's formula***: $r(a)=\lim_n\sqrt[n]{\|a^n\|}$. It is also well known that $r(a)=\bigl|\sigma(a)\bigr|$, where $\bigl|\sigma(a)\bigr|$ is the ***geometric radius*** of $\sigma(a)$, i.e., $\bigl|\sigma(a)\bigr|=\sup\{|\lambda| : \lambda\in\sigma(a)\}$. An element $a\in A$ is said to be ***quasinilpotent*** if $\sigma(a)=\{0\}$.

If T is a bounded operator on a Banach space X then we consider the spectrum $\sigma(T)$ and the resolvent set $\rho(T)$ in the sense of the Banach algebra of bounded operators on X. If $\lambda\in\rho(T)$ then the inverse $(I-\lambda T)^{-1}$ is called the ***resolvent operator*** and is denoted by $R(T;\lambda)$ or just R_λ. If $\lambda\in\mathbb{C}$ satisfies $|\lambda|>r(T)$ then the ***Neumann series*** $\sum_{i=0}^\infty\frac{T^i}{\lambda^{i+1}}$ converges to R_λ in operator norm. Following [1, 2] we say that T is ***locally quasinilpotent*** at $x\in X$ if $\lim_n\sqrt[n]{\|T^nx\|}=0$.

Further details on topological vector spaces can be found in [6, 13, 17, 7, 18, 12]. For details on locally bounded and quasinormed topological vector spaces we refer the reader to [13, 14, 16].

2. Bounded operators

Operator Theory on Banach spaces deals with the class of bounded operators, which coincides with the class of continuous operators. The operator zoo in general topological vector spaces is much richer. In this paper we will deal with the following types of bounded operators. To avoid confusion, we give different names to different types of boundedness.

Definition 2.1. *Let X and Y be topological vector spaces. An operator $T\colon X \to Y$ is said to be*

(i) ***bb-bounded*** *if it maps every bounded set into a bounded set;*
(ii) ***nb-bounded*** *if it maps some neighborhood into a bounded set;*

Further, if $X = Y$ we will say that $T\colon X \to X$ is ***nn-bounded*** *if there exists a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ there is a positive scalar α such that $T(U) \subseteq \alpha U$.*

Remark 2.2. [7] and [12] present (i) as the definition of a bounded operator on a topological vector space, while [17] and [18] use (ii) for the same purpose. As we will see, these definitions are far from being equivalent.

Proposition 2.3. *Let X and Y be topological vector spaces. For an operator $T\colon X \to Y$ consider the following statements:*

(i) *T is bb-bounded;*
(ii) *T is continuous;*
(iii) *T is nn-bounded;*
(iv) *T is nb-bounded.*

Then (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i). *Furthermore, if $X = Y$ then* (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Proof. The implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. To show (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) assume that $X = Y$ and fix a base $\mathcal{N}_0$ of zero neighborhoods. If T is nb-bounded then $T(U)$ is bounded for some $U \in \mathcal{N}_0$. Note that $\mathcal{N}_0' = \{\, \lambda U \cap V \;:\; \lambda > 0,\ V \in \mathcal{N}_0 \,\}$ is another base of zero neighborhoods. For each $W = \lambda U \cap V$ in $\mathcal{N}_0'$ we have $T(W) \subseteq \lambda T(U)$. But $T(U)$ is bounded and so $T(W) \subseteq \lambda T(U) \subseteq \lambda\alpha W$ for some positive α, i.e., T is nn-bounded.[1]

Finally, if T is nn-bounded, then there is a base $\mathcal{N}_0$ such that for every zero neighborhood $U \in \mathcal{N}_0$ there is a positive scalar α such that $T(U) \subseteq \alpha U$. Let V be an arbitrary zero neighborhood. Then there exists $U \in \mathcal{N}_0$ such that $U \subseteq V$, so that $T(U) \subseteq \alpha U \subseteq \alpha V$ for some $\alpha > 0$. Taking $W = \frac{1}{\alpha} U$ we get $T(W) \subseteq V$, hence T is continuous. $\square$

2.4. It can be easily verified that all the statements in Proposition 2.3 are equivalent if X is locally bounded. In general, however, these notions are not equivalent. Obviously, the identity operator I is always nn-bounded, continuous, and bb-bounded, but I is nb-bounded if and only if the space is locally bounded. Every bb-bounded operator between two locally convex spaces is continuous if and only if the domain space is bornological. (Recall that a locally convex space is ***bornological*** if every balanced convex set which absorbs every bounded set is a zero neighborhood; for details see [18, 17].)

Example 2.5. *A continuous but not nn-bounded operator.* Let T be the left shift on the space of all real sequences $\mathbb{R}^{\mathbb{N}}$ with the topology of coordinate-wise convergence, i.e., $T\colon (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$. Clearly T is continuous. We will show that T is

[1] Note that if the topology is locally convex, then we can assume that U is convex and $\mathcal{N}_0$ consists of convex neighborhoods. In this case $\mathcal{N}_0'$ also consists of convex neighborhoods.

not nn-bounded. Assume that for every zero neighborhood U in some base $\mathcal{N}_0$ there is a positive scalar α such that $T(U) \subseteq \alpha U$. Since the set $\bigl\{x = (x_k) : |x_0| < 1\bigr\}$ is a zero neighborhood, there must be a neighborhood $U \in \mathcal{N}_0$ such that $U \subseteq \{x : |x_0| < 1\}$. Since $T(U) \subseteq \alpha U$ for some positive α then $T^n(U) \subseteq \alpha^n U$, so that if $x = (x_k) \in U$ then $T^n x \in \alpha^n U$, so that $|x_n| = \bigl|(T^n x)_0\bigr| < \alpha^n$. Hence $U \subseteq \bigl\{x : |x_n| < \alpha^n \text{ for each } n > 0\bigr\}$. But this set is bounded, while the space is not locally bounded, a contradiction.

2.6. *Algebraic properties of bounded operators.* The sum of two bb-bounded operators is bb-bounded because the sum of two bounded sets in a topological vector space is bounded. Clearly the product of two bb-bounded operators is bb-bounded. It is well known that sums and products of continuous operators are continuous. Obviously, the product of two nn-bounded operators is nn-bounded, and it can be easily verified that the sum of two nn-bounded operators on a locally convex (or locally pseudo-convex) space is again nn-bounded. It is not difficult to see that the sum of two nb-bounded operator is nb-bounded. Indeed, suppose that T_1 and T_2 are two nb-bounded operators, then the sets $T_1(U_1)$ and $T_2(U_2)$ are bounded for some zero neighborhoods U_1 and U_2. There exists another zero neighborhood U so that $U \subseteq U_1 \cap U_2$. Then the sets $T_1(U)$ and $T_2(U)$ are bounded, so that $(T_1 + T_2)(U) \subseteq T_1(U) + T_2(U)$ is bounded. Finally, it is not difficult to see that the product of two nb-bounded operators is again nb-bounded. This follows immediately from Proposition 2.3 and the following simple observation: if we multiply an nb-bounded operator by a bb-bounded operator on the left or by an nn-bounded operator on the right, the product is nb-bounded.

Thus, the class of all bb-bounded operators, the class of all continuous operators, and the class of all nb-bounded operators are subalgebras of the algebra of all linear operators. The class of nn-bounded operators is an algebra provided that the space is locally convex or locally pseudo-convex.

Boundedness in terms of convergence. Suppose $T\colon X \to Y$ is an operator between two topological vector spaces. It is well known that T is continuous if and only if it maps convergent nets to convergent nets.

Notice that a subset of a topological vector space is unbounded if and only if it contains an unbounded sequence. Therefore, an operator is bb-bounded if and only if it maps bounded sequences (nets) to bounded sequences (respectively, nets).

It is easy to see that T is nn-bounded if and only if T maps U-bounded (U-convergent to zero) sequences to U-bounded (respectively, U-convergent to zero) sequences for every U in some base of zero neighborhoods. If U is a zero neighborhood, we say that a net (x_γ) is ***U-bounded*** if it is contained in αU for some $\alpha > 0$, we say that $x_\gamma \xrightarrow{U} 0$ if for every $\alpha > 0$ there exits γ_0 such that $x_\gamma \in \alpha U$ whenever $\gamma > \gamma_0$.

2.7. Suppose T is nb-bounded, then $T(U)$ is bounded for some zero neighborhood U. Obviously $x_\gamma \xrightarrow{U} 0$ implies $Tx_\gamma \to 0$. The converse implication is also valid: if T maps U-convergent sequences to convergent sequences, then $T(U)$ is bounded, hence T is nb-bounded. Indeed, suppose that $T(U)$ is unbounded, then there is a zero neighborhood V

in Y such that V does not absorb $T(U)$. Then for every $n \geqslant 1$ there exists $y_n \in T(U)\backslash nV$. Further, if $y_n = Tx_n$ for some $x_n \in U$, then $\frac{x_n}{n} \xrightarrow{U} 0$, but $T(\frac{x_n}{n}) = \frac{y_n}{n} \notin V$, so that $T(\frac{x_n}{n})$ does not converge to zero.

Normed, quasinormed, and seminormed spaces. Next, we discuss bounded operators in some particular topologies. Notice that every normed, seminormed, or quasinormed vector space is locally bounded. Therefore bb-boundedness, continuity, nn-boundedness and nb-boundedness coincide for operators on such spaces.

Locally convex topology. Similarly to the norm of an operator on a Banach space, we introduce the seminorm of an operator on a seminormed space.

Definition 2.8. *Let T be an operator on a seminormed vector space (X, p). As in the case with normed spaces, p generates an **operator seminorm** $p(T)$ defined by*

$$p(T) = \sup_{p(x)\neq 0} \frac{p(Tx)}{p(x)}.$$

*More generally, let $S\colon X \to Y$ be a linear operator between two seminormed spaces (X, p) and (Y, q). Then we define a **mixed operator seminorm** associated with p and q via*

$$\mathfrak{m}_{pq}(S) = \sup_{p(x)\neq 0} \frac{q(Sx)}{p(x)}.$$

The seminorm $\mathfrak{m}_{pq}(S)$ is a measure of how far in the seminorm q the points of the p-unit ball can go under S. Notice, that $p(T)$ and $\mathfrak{m}_{pq}(S)$ may be infinite. Clearly, if T is an operator on a seminormed space (X, p), then $\mathfrak{m}_{pp}(T) = p(T)$.

Lemma 2.9. *If $S\colon X \to Y$ is an operator between two seminormed spaces (X, p) and (Y, q), then*

(i) $\mathfrak{m}_{pq}(S) = \sup\limits_{p(x)=1} q(Sx) = \sup\limits_{p(x)\leqslant 1} q(Sx)$;

(ii) $q(Sx) \leqslant \mathfrak{m}_{pq}(S)p(x)$ *whenever* $\mathfrak{m}_{pq}(S) < \infty$.

Proof. The first equality in (i) follows immediately from the definition of $p(T)$. We obviously have

$$\sup_{p(x)=1} q(Sx) \leqslant \sup_{p(x)\leqslant 1} q(Sx).$$

In order to prove the opposite inequality, notice that if $0 < p(x) \leqslant 1$, then $q(Sx) \leqslant \frac{q(Sx)}{p(x)} \leqslant \mathfrak{m}_{pq}(S)$. Thus, it is left to show that $p(x) = 0$ implies $q(Sx) \leqslant \mathfrak{m}_{pq}(S)$. Pick any z with $p(z) > 0$, then

$$p(\tfrac{z}{n}) = p(x + \tfrac{z}{n} - x) \leqslant p(x + \tfrac{z}{n}) + p(x) = p(x + \tfrac{z}{n}) \leqslant p(x) + p(\tfrac{z}{n}) = \tfrac{p(z)}{n},$$

so that $p(x + \frac{z}{n}) = p(\frac{z}{n}) \in (0, 1)$ for $n > p(z)$. Further, since $Sx + \frac{Sz}{n}$ converges to Sx we have

$$q(Sx) = \lim_{n\to\infty} q(Sx + \tfrac{Sz}{n}) \leqslant \lim_{n\to\infty} \frac{q\left(S(x + \frac{z}{n})\right)}{p(x + \frac{z}{n})} \leqslant \mathfrak{m}_{pq}(S).$$

Finally, (ii) follows directly from the definition if $p(x) \neq 0$. In the case when $p(x) = 0$, again pick any z with $p(z) > 0$. Then $p(x + \frac{z}{n}) \neq 0$ and

$$q(Sx) = \lim_{n\to\infty} q(Sx + \tfrac{Sz}{n}) = \lim_{n\to\infty} q\left(S(x + \tfrac{z}{n})\right) \leqslant \lim_{n\to\infty} \mathfrak{m}_{pq}(S)p(x + \tfrac{z}{n}) = 0.$$

□

Corollary 2.10. *If T is an operator on a seminormed space (X,p), then*

(i) $p(T) = \sup\limits_{p(x)=1} p(Tx) = \sup\limits_{p(x)\leqslant 1} p(Tx)$;

(ii) $p(Tx) \leqslant p(T)p(x)$ *whenever* $p(T) < \infty$.

The following propositions characterize continuity and boundedness of an operator on a locally convex space in terms of operator seminorms. We assume that X and Y are locally convex spaces with generating families of seminorms $\mathcal{P}$ and $\mathcal{Q}$ respectively.

Proposition 2.11. *Let S be an operator from X to Y, then S is continuous if and only if for every $q \in \mathcal{Q}$ there exists $p \in \mathcal{P}$ such that $\mathfrak{m}_{pq}(S)$ is finite.*

Proposition 2.12. *An operator T on X is nn-bounded if and only if $p(T)$ is finite for every $p \in \mathcal{P}$, or, equivalently, if T maps p-bounded sets to p-bounded sets for every p in some generating family $\mathcal{P}$ of seminorms.*

Proposition 2.13. *Let $S\colon X \to Y$ be a linear operator, then the following are equivalent:*

(i) *S is nb-bounded;*

(ii) *S maps p-bounded sets into bounded sets for some $p \in \mathcal{P}$;*

(iii) *There exists $p \in \mathcal{P}$ such that $\mathfrak{m}_{pq}(S) < \infty$ for every $q \in \mathcal{Q}$.*

Since the balanced convex hull of a bounded set in a locally convex space is again bounded, we also have the following characterization.

Proposition 2.14. *An operator $S\colon X \to Y$ is bb-bounded if and only if $\mathfrak{m}_{pq}(S) < \infty$ whenever $q \in \mathcal{Q}$ and p is the Minkowski functional of a convex balanced bounded set.*

Operator topologies. For each of the five classes of operators, we introduce an appropriate natural operator topology.

2.15. The class of all linear operators between two topological vector spaces will be usually equipped with the ***strong*** operator topology. Recall, that a sequence (S_n) of operators from X to Y is said to ***converge strongly*** or ***pointwise*** to a map S if $S_nx \to Sx$ for every $x \in X$. Clearly, S will also be a linear operator.

2.16. The class of all bb-bounded operators will usually be equipped with the topology of ***uniform convergence on bounded sets***. Recall, that a sequence (S_n) of operators is said to converge to zero ***uniformly*** on A if for each zero neighborhood V in Y there exists an index n_0 such that $S_n(A) \subseteq V$ for all $n > n_0$. We say that (S_n) converges to S uniformly on bounded sets if $(S_n - S)$ converges to zero uniformly on bounded sets.

Recall also that a family $\mathcal{G}$ of operators is called ***uniformly bounded*** on a set $A \subseteq X$ if the set $\bigcup_{S\in\mathcal{G}} S(A)$ is bounded in Y.

Lemma 2.17. *If a sequence (S_n) of bb-bounded operators converges uniformly on bounded sets to an operator S, then S is also bb-bounded.*

Proof. Fix a bounded set A and a zero neighborhood V. There exists a zero neighborhood U such that $U + U \subseteq V$. Since $S - S_n$ converges to zero uniformly on bounded sets then there exists an index n_0 such that $(S_n - S)(A) \subseteq U$ whenever $n \geqslant n_0$. This yields $S(A) \subseteq S_n(A) + U$. The set $S_n(A)$ is bounded, so that $S_n(A) \subseteq \gamma U$ for some $\gamma \geqslant 1$. Thus, $S(A) \subseteq \gamma U + U \subseteq \gamma V$. Hence $S(A)$ is bounded for every bounded set A, and so S is bb-bounded. □

2.18. The class of all continuous operators will be usually equipped with the topology of ***equicontinuous convergence***. Recall, that a family $\mathcal{G}$ of operators from X to Y is called ***equicontinuous*** if for each zero neighborhood V in Y there is a zero neighborhood U in X such that $S(U) \subseteq V$ for every $S \in \mathcal{G}$. We say that a sequence (S_n) converges to zero ***equicontinuously*** if for each zero neighborhood V in Y there is a zero neighborhood U in X such that for every $\varepsilon > 0$ there exists an index n_0 such that $S_n(U) \subseteq \varepsilon V$ for all $n > n_0$.

Lemma 2.19. *If a sequence S_n of continuous operators converges equicontinuously to S, then S is also continuous.*

Proof. Fix a zero neighborhood V. There exist zero neighborhoods V_1 and U and an index n_0 such that $V_1 + V_1 \subseteq V$ and $(S_n - S)(U) \subseteq V_1$ whenever $n > n_0$. Fix $n > n_0$. The continuity of S_n guarantees that there exists a zero neighborhood $W \subseteq U$ such that $S_n(W) \subseteq V_1$. Since $(S_n - S)(W) \subseteq V_1$, we get $S(W) \subseteq S_n(W) + V_1 \subseteq V_1 + V_1 \subseteq V$, which shows that S is continuous. □

2.20. The class of all nn-bounded operators will be usually equipped with the topology of ***nn-convergence***, defined as follows. We will call a collection $\mathcal{G}$ of operators ***uniformly nn-bounded*** if there exists a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ there exists a positive real β such that $S(U) \subseteq \beta U$ for each $S \in \mathcal{G}$. We say that a sequence (S_n) ***nn-converges*** to zero if there is a base $\mathcal{N}_0$ of zero neighborhoods such that for every $U \in \mathcal{N}_0$ and every $\varepsilon > 0$ we have $S_n(U) \subseteq \varepsilon U$ for all sufficiently large n.

Question. *Is the class of all nn-bounded operators closed relative to nn-convergence?*

2.21. Finally, the class of all nb-bounded operators will be usually equipped with the topology of uniform convergence on a zero neighborhood.

Example 2.22. *The class of nb-bounded operators is not closed in the topology of uniform convergence on a zero neighborhood.* Let $X = \mathbb{R}^{\mathbb{N}}$, the space of all real sequences with the topology of coordinate-wise convergence. Let T_n be the projection on the first n components. Clearly, every T_n is nb-bounded because it maps the zero neighborhood

$U_n = \{(x_i)_{i=1}^\infty : |x_i| < 1 \text{ for } i = 1, \dots, n\}$ to a bounded set. On the other hand, (T_n) converges uniformly on X to the identity operator, while the identity operator on X is not nb-bounded.

3. Spectra of an operator

Recall that if T is a continuous operator on a Banach space, then its ***resolvent set*** $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ such that the ***resolvent operator*** $R_\lambda = (\lambda I - T)^{-1}$ exists, while the ***spectrum*** of T is defined by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The Open Mapping Theorem guarantees that if R_λ exists then it is automatically continuous. Now, if T is an operator on an arbitrary topological vector space and $\lambda \in \mathbb{C}$ then the algebraic inverse $R_\lambda = (\lambda I - T)^{-1}$ may exist but it need not be continuous, or it may be continuous but not nb-bounded, etc. In order to treat all these cases properly we introduce the following definitions.

Definition 3.1. *Let T be a linear operator on a topological vector space. We denote the set of all scalars $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is invertible in the algebra of linear operators by $\rho^l(T)$. We say that $\lambda \in \rho^{bb}(T)$ (respectively $\rho^c(T)$ or $\rho^{nn}(T)$) if the inverse of $\lambda I - T$ is bb-bounded (respectively continuous or nn-bounded). Finally, we say that $\lambda \in \rho^{nb}(T)$ if the inverse of $\lambda I - T$ belongs to the unitalization of the algebra of nb-bounded operators, i.e., when $(\lambda I - T)^{-1} = \alpha I + S$ for a scalar α and an nb-bounded operator S.*

The spectral sets $\sigma^l(T)$, $\sigma^{bb}(T)$, $\sigma^c(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$ are defined to be the complements of the resolvent sets $\rho^l(T)$, $\rho^{bb}(T)$, $\rho^c(T)$, $\rho^{nn}(T)$, and $\rho^{nb}(T)$, respectively.[2] *We will denote the (left and right) inverse of $\lambda I - T$ whenever it exists by R_λ.*

3.2. It follows immediately from Proposition 2.3 that

$$\sigma^l(T) \subseteq \sigma^{bb}(T) \subseteq \sigma^c(T) \subseteq \sigma^{nn}(T) \subseteq \sigma^{nb}(T).$$

It follows from the Open Mapping Theorem that for a continuous operator T on a Banach space all the introduced spectra coincide with the usual spectrum $\sigma(T)$. Since the Open Mapping Theorem is still valid on Fréchet spaces, we have $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T)$ for a continuous operator T on a Fréchet space.

3.3. If T is an operator on a locally bounded space (X, U), then by 2.4, bb-boundedness of T is equivalent to nb-boundedness, so that $\sigma^{bb}(T) = \sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T)$. We will denote this set by $\sigma_U(T)$ to avoid ambiguity. A spectral theory of continuous operators on quasi-Banach spaces was developed in [10].

3.4. There are several reasons why we define σ^{nb} in a slightly different fashion than the other spectra. Namely, for λ to be in $\rho^{nb}(T)$ we require $(\lambda I - T)^{-1}$ be not just nb-bounded, but be nb-bounded up to a multiple of the identity operator. On one hand, this is the standard way to define the spectrum of an element in a non-unital algebra, and we know that the algebra of nb-bounded operators is unital only when the space is locally bounded.

[2] We use superscripts in order to avoid confusion with $\sigma_c(T)$, which is commonly used for continuous spectrum.

On the other hand, if we defined $\rho^{nb}(T)$ as the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ is nb-bounded, then we wouldn't have obtained a reasonable theory because $(\lambda I - T)^{-1}$ is almost never nb-bounded unless the space is locally bounded.

Indeed, suppose that X is not locally bounded, T is a bb-bounded operator on X and $\lambda \in \mathbb{C}$. Then $R_\lambda = (\lambda I - T)^{-1}$ cannot be nb-bounded, because in this case $I = (\lambda I - T)R_\lambda$ would be nb-bounded by 2.6 as a product of a bb-bounded and an nb-bounded operator. But we know that I is not nb-bounded because X is not locally bounded.

We will see in Proposition 6.2 that in a locally convex but non locally bounded space nb-bounded operators are never invertible, which implies that in such spaces $(\lambda I - T)^{-1}$ is not nb-bounded for any linear operator T.

3.5. Next, let T be a (norm) continuous operator on a Banach space, $\sigma(T)$ the usual spectrum of T, and let $\sigma^l(T)$, $\sigma^{bb}(T)$, $\sigma^c(T)$ be computed with respect to the weak topology. It is known that an operator on a Banach space is weak-to-weak continuous if and only if it is norm-to-norm continuous; therefore it follows that $\sigma^c(T) = \sigma(T)$. Furthermore, $\sigma^l(T)$ does not depend on the topology, so that it also coincides with $\sigma(T)$. Thus $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T) = \sigma(T)$.

4. Spectral radii of an operator

The spectral radius of a bounded linear operator T on a Banach space is usually defined via Gelfand's formula, $r(T) = \lim\limits_{n\to\infty} \sqrt[n]{\|T^n\|}$. The formula involves a norm and so makes no sense in a general topological vector space. Fortunately, this formula can be rewritten without using a norm, and then generalized to topological vector spaces. Similarly to the situation with spectra, this generalization can be done in several ways, so that we will obtain various types of spectral radii for an operator on a topological vector space. We will show later that, as in the Banach space case, there are certain relations between the spectral radii, the radii of the spectra, and the convergence of the Neumann series of an operator on a locally convex topological vector space. This section may appear technical at the beginning, but later on the reader will see that all the facts lead to a simple and natural classification. We start with an almost obvious numerical lemma.

Lemma 4.1. *If (t_n) is a sequence in $\mathbb{R}^+ \cup \{\infty\}$, then*

$$\limsup_{n\to\infty} \sqrt[n]{t_n} = \inf\{\, \nu > 0 \;:\; \lim_{n\to\infty} \tfrac{t_n}{\nu^n} = 0 \,\} = \inf\{\, \nu > 0 \;:\; \limsup_{n\to\infty} \tfrac{t_n}{\nu^n} < \infty \,\}.$$

Proof. Suppose $\limsup_n \sqrt[n]{t_n} = r$. If $0 < \nu < r$, then $\sqrt[n_k]{t_{n_k}} > \mu > \nu$ for some μ and some subsequence (t_{n_k}), so that $\frac{t_{n_k}}{\nu^{n_k}} > \frac{\mu^{n_k}}{\nu^{n_k}} \to \infty$ as k goes to infinity. It follows that $\limsup_n \frac{t_n}{\nu^n} = \infty$. On the other hand, if r is finite and $\nu > r$ then $\sqrt[n]{t_n} < \mu < \nu$ for some μ and for all sufficiently large n. Then $\lim\limits_{n\to\infty} \frac{t_n}{\nu^n} \leqslant \lim\limits_{n\to\infty} \frac{\mu^n}{\nu^n} = 0$. $\square$

This lemma implies that the spectral radius $r(T)$ of a (norm) continuous operator T on a Banach space equals the infimum of all positive real scalars ν such that the sequence $\left(\frac{T^n}{\nu^n}\right)$ converges to zero (or just is bounded) in operator norm topology. This can be considered as an alternative definition of the spectral radius, and can be generalized to any topological vector space with an operator topology. In view of 2.15-2.21, we arrive to the following definition.

Definition 4.2. *Given a linear operator T on a topological vector space X, we define the following spectral radii of T:*

$$\begin{aligned} r_l(T) &= \inf\{\,\nu>0 \;:\; \textit{the sequence } \big(\tfrac{T^n}{\nu^n}\big) \textit{ converges strongly to zero}\,\};\\ r_{bb}(T) &= \inf\{\,\nu>0 \;:\; \tfrac{T^n}{\nu^n}\to 0 \textit{ uniformly on every bounded set}\,\};\\ r_c(T) &= \inf\{\,\nu>0 \;:\; \tfrac{T^n}{\nu^n}\to 0 \textit{ equicontinuously }\};\\ r_{nn}(T) &= \inf\{\,\nu>0 \;:\; \big(\tfrac{T^n}{\nu^n}\big) \textit{ nn-converges to zero}\};\\ r_{nb}(T) &= \inf\{\,\nu>0 \;:\; \tfrac{T^n}{\nu^n}\to 0 \textit{ uniformly on some 0-neighborhood}\,\}.\end{aligned}$$

The following proposition explains the relations between the introduced radii.

Proposition 4.3. *If T is a linear operator on a topological vector space X, then*

$$r_l(T)\leqslant r_{bb}(T)\leqslant r_c(T)\leqslant r_{nn}(T)\leqslant r_{nb}(T).$$

Proof. Let T be a linear operator on a topological vector space X. Since every singleton is bounded, $r_l(T)\leqslant r_{bb}(T)$. Next, assume $\nu>r_c(T)$ and fix μ such that $r_c(T)<\mu<\nu$. Then the sequence $(\frac{T^n}{\mu^n})$ converges to zero equicontinuously. Take a bounded set A and a zero neighborhood U. There exists a zero neighborhood V and a positive integer N such that $\frac{T^n}{\mu^n}(V)\subseteq U$ whenever $n\geqslant N$. Also, $A\subseteq\alpha V$ for some $\alpha>0$, so that $\frac{T^n}{\nu^n}(A)\subseteq\frac{\mu^n}{\nu^n}\frac{T^n}{\mu^n}(\alpha V)\subseteq\frac{\mu^n\alpha}{\nu^n}U\subseteq U$ for all sufficiently large n. It follows that the sequence $(\frac{T^n}{\nu^n})$ converges to zero uniformly on A and, therefore, $\nu\geqslant r_{bb}(T)$. Thus, $r_{bb}(T)\leqslant r_c(T)$.

To prove the inequality $r_c(T)\leqslant r_{nn}(T)$ we let $\nu>r_{nn}(T)$. Then for some base $\mathcal{N}_0$ of zero neighborhoods and for every $V\in\mathcal{N}_0$ and $\varepsilon>0$ there exists a positive integer N such that $\frac{T^n}{\nu^n}(V)\subseteq\varepsilon V$ for every $n\geqslant N$. Given a zero neighborhood U, we can find $V\in\mathcal{N}_0$ such that $V\subseteq U$. Then $\frac{T^n}{\nu^n}(V)\subseteq\varepsilon V\subseteq\varepsilon U$ for every $n\geqslant N$, so that the sequence $(\frac{T^n}{\nu^n})$ converges to zero equicontinuously, and therefore $\nu\geqslant r_c(T)$.

Finally, we must show that $r_{nn}(T)\leqslant r_{nb}(T)$. Suppose that $\nu>r_{nb}(T)$. We claim that $\nu\geqslant r_{nn}(T)$. Take μ so that $\nu>\mu>r_{nb}(T)$. One can find a zero neighborhood U such that for every zero neighborhood V there is a positive integer N such that $\frac{T^n}{\mu^n}(U)\subseteq V$ for every $n\geqslant N$. Fix a base $\mathcal{N}_0$ of zero neighborhoods, and define a new base $\mathcal{N}_0'$ of zero neighborhoods via $\mathcal{N}_0'=\{mU\cap W \;:\; m\in\mathbb{N},\, W\in\mathcal{N}_0\}$. Let $V\in\mathcal{N}_0'$ and $\varepsilon>0$. Then $V=mU\cap W$ for some positive integer m and $W\in\mathcal{N}_0$. Then $\frac{T^n}{\mu^n}(V)\subseteq m\frac{T^n}{\mu^n}(U)\subseteq mV$ for every sufficiently large n. Thus $\frac{T^n}{\nu^n}(V)\subseteq\frac{\mu^n}{\nu^n}mV\subseteq\varepsilon V$, for each sufficiently large n, which implies $\nu\geqslant r_{nn}(T)$. ☐

The following lemma shows that, similarly to the case of Banach spaces, one can use boundedness instead of convergence when defining the spectral radii of an operator on a topological vector space.

Lemma 4.4. *Let T be a linear operator on a topological vector space, then*

(i) $r_l(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\frac{T^n x}{\nu^n}\bigr)$ is bounded for every $x \,\bigr\}$;

(ii) *if T is bb-bounded then*
$r_{bb}(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\frac{T^n}{\nu^n}\bigr)$ is uniformly bounded on every bounded set $\bigr\}$;

(iii) *if T is continuous then*
$r_c(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\frac{T^n}{\nu^n}\bigr)$ is equicontinuous $\bigr\}$;

(iv) *if T is nn-bounded then*
$r_{nn}(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\frac{T^n}{\nu^n}\bigr)$ is uniformly nn-bounded $\bigr\}$;

(v) *if T is nb-bounded then*
$r_{nb}(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\frac{T^n}{\nu^n}\bigr)$ is uniformly bounded on some 0-neighborhood $\bigr\}$.

Moreover, in each of these cases it suffices to consider any tail of the sequence $\bigl(\frac{T^n}{\nu^n}\bigr)$ instead of the whole sequence.

Proof. To prove (i) let

$$r_l'(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\tfrac{T^n}{\nu^n}x\bigr) \text{ is bounded for every } x \,\bigr\}.$$

Since every convergent sequence is bounded, we certainly have $r_l(T) \geqslant r_l'(T)$. Conversely, suppose $\nu > r_l'(T)$, and take any positive scalar μ such that $\nu > \mu > r_l'(T)$. Then for every $x \in X$ the sequence $\frac{T^n}{\mu^n}x$ is bounded, and it follows that the sequence $\frac{T^n x}{\nu^n} = \frac{\mu^n}{\nu^n}\frac{T^n x}{\mu^n}$ converges to zero, so that $\nu \geqslant r_l(T)$ and, therefore $r_l'(T) \geqslant r_l(T)$.

To prove (ii), suppose T is bb-bounded and let

$$r_{bb}'(T) = \inf\bigl\{\, \nu > 0 \;:\; \bigl(\tfrac{T^n}{\nu^n}\bigr) \text{ is uniformly bounded on every bounded set } \bigr\}.$$

We'll show that $r_{bb}'(T) = r_{bb}(T)$. If $\bigl(\frac{T^n}{\nu^n}\bigr)$ converges to zero uniformly on every bounded set, then for each bounded set A and for each zero neighborhood U there exists a positive integer N such that $\frac{T^n}{\nu^n}(A) \subseteq U$ whenever $n \geqslant N$. Also, since T is bb-bounded, for every $n < N$ we have $\frac{T^n}{\nu^n}(A) \subseteq \alpha_n U$ for some $\alpha_n > 0$. Therefore, if $\alpha = \max\{\alpha_1, \dots, \alpha_{N-1}, 1\}$, then $\frac{T^n}{\nu^n}(A) \subseteq \alpha U$ for every n, so that the sequence $\frac{T^n}{\nu^n}$ is uniformly bounded on A. Thus $\nu \geqslant r_{bb}'(T)$, so that $r_{bb}'(T) \leqslant r_{bb}(T)$.

Now suppose $\nu > r_{bb}'(T)$. There exists μ such that $\nu > \mu > r_{bb}'(T)$. The set $\bigcup_{n=1}^{\infty} \frac{T^n}{\mu^n}(A)$ is bounded for every bounded set A, so that for every zero neighborhood U there exists a scalar α such that $\frac{T^n}{\mu^n}(A) \subseteq \alpha U$ for every $n \in \mathbb{N}$. Then $\frac{T^n}{\nu^n}(A) \subseteq \frac{\mu^n \alpha}{\nu^n}U \subseteq U$ for all sufficiently large n. This means that the sequence $\bigl(\frac{T^n}{\nu^n}\bigr)$ converges to zero uniformly on A, and it follows that $\nu \geqslant r_{bb}(T)$.

Further, if T is bb-bounded, then any finite initial segment $(\frac{T^n}{\nu^n})_{n=0}^{N}$ is always uniformly bounded on every bounded set, so that a tail $(\frac{T^n}{\nu^n})_{n=N}^{\infty}$ is uniformly bounded on every bounded set if and only if the whole sequence $(\frac{T^n}{\nu^n})_{n=0}^{\infty}$ is uniformly bounded on every bounded set.

The statements (iii), (iv), and (v) can be proved in a similar way. □

4.5. *Locally bounded spaces.* If T is a linear operator on a locally bounded topological vector space (X, U), then it follows directly from Definition 4.2 that $r_{bb}(T) = r_c(T) =$

$r_{nn}(T)=r_{nb}(T)$, because the corresponding convergences are equivalent. In this case we would denote each of these radii by $r_U(T)$.

Spectral radii via seminorms. The following proposition provides formulas for computing spectral radii of an operator on a locally convex space in terms of seminorms.

Proposition 4.6. *If T is an operator on a locally convex space X with a generating family of seminorms $\mathcal{P}$, then*

(i) $r_l(T)=\sup\limits_{p\in\mathcal{P},\ x\in X}\limsup\limits_{n\to\infty}\sqrt[n]{p(T^nx)}$;

(ii) $r_{bb}(T)=\sup\limits_{p\in\mathcal{B},\,q\in\mathcal{P}}\limsup\limits_{n\to\infty}\sqrt[n]{\mathfrak{m}_{pq}(T^n)}$, *where $\mathcal{B}$ is the collection of Minkowski functionals of all balanced convex bounded sets in X;*

(iii) $r_c(T)=\sup\limits_{q\in\mathcal{P}}\inf\limits_{p\in\mathcal{P}}\limsup\limits_{n\to\infty}\sqrt[n]{\mathfrak{m}_{pq}(T^n)}$;

(iv) $r_{nn}(T)=\inf\limits_{\mathcal{Q}}\sup\limits_{p\in\mathcal{Q}}\limsup\limits_{n\to\infty}\sqrt[n]{p(T^n)}$, *where the infimum is taken over all generating families of seminorms;*

(v) $r_{nb}(T)=\inf\limits_{p\in\mathcal{P}}\sup\limits_{q\in\mathcal{P}}\limsup\limits_{n\to\infty}\sqrt[n]{\mathfrak{m}_{pq}(T^n)}$;

Proof. It follows from the definition of $r_l(T)$ and Lemma 4.1 that

$$\begin{aligned}r_l(T)&=\inf\bigl\{\,\nu>0\ :\ \lim_{n\to\infty}p\left(\tfrac{T^nx}{\nu^n}\right)=0\text{ for every }x\in X,p\in\mathcal{P}\,\bigr\}\\&=\sup_{x\in X,\ p\in\mathcal{P}}\inf\bigl\{\,\nu>0\ :\ \lim_{n\to\infty}\tfrac{p(T^nx)}{\nu^n}=0\,\bigr\}=\sup_{x\in X,\ p\in\mathcal{P}}\limsup_{n\to\infty}\sqrt[n]{p(T^nx)}.\end{aligned}$$

Similarly, since the balanced convex hull of a bounded set is bounded,

$$\begin{aligned}r_{bb}(T)&=\inf\bigl\{\,\nu>0\ :\ \forall\text{ bounded }A\quad\forall V\in\mathcal{N}_0\quad\exists N\in\mathbb{N}\quad\forall n\geqslant N\quad\tfrac{T^n}{\nu^n}(A)\subseteq V\,\bigr\}\\&=\inf\bigl\{\,\nu>0\ :\ \forall p\in\mathcal{B}\quad\forall q\in\mathcal{P}\quad\exists N\in\mathbb{N}\quad\forall n\geqslant N\quad\mathfrak{m}_{pq}\bigl(\tfrac{T^n}{\nu^n}\bigr)\leqslant1\,\bigr\}\\&=\sup_{p\in\mathcal{B},\,q\in\mathcal{P}}\inf\bigl\{\,\nu>0\ :\ \lim_{n\to\infty}\tfrac{\mathfrak{m}_{pq}(T^n)}{\nu^n}\leqslant1\,\bigr\}=\sup_{p\in\mathcal{B},\,q\in\mathcal{P}}\limsup_{n\to\infty}\sqrt[n]{\mathfrak{m}_{pq}(T^n)}.\end{aligned}$$

Let $U_p=\{\,x\in X\ :\ p(x)<1\,\}$ for every $p\in\mathcal{P}$. Then, rephrasing the definition of $r_c(T)$ and applying Lemma 4.1, we have

$$\begin{aligned}r_c(T)&=\inf\bigl\{\,\nu>0\ :\ \forall q\in\mathcal{P}\ \exists p\in\mathcal{P}\ \forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geqslant N\ \tfrac{T^n}{\nu^n}(U_p)\subseteq\varepsilon U_q\,\bigr\}\\&=\sup_{q\in\mathcal{P}}\inf_{p\in\mathcal{P}}\inf\bigl\{\,\nu>0\ :\ \forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geqslant N\ \mathfrak{m}_{pq}\bigl(\tfrac{T^n}{\nu^n}\bigr)<\varepsilon\,\bigr\}\\&=\sup_{q\in\mathcal{P}}\inf_{p\in\mathcal{P}}\inf\bigl\{\,\nu>0\ :\ \lim_{n\to\infty}\tfrac{\mathfrak{m}_{pq}(T^n)}{\nu^n}=0\,\bigr\}=\sup_{q\in\mathcal{P}}\inf_{p\in\mathcal{P}}\limsup_{n\to\infty}\sqrt[n]{\mathfrak{m}_{pq}(T^n)}.\end{aligned}$$

Similarly,

$$r_{nn}(T) = \inf\big\{\ \nu > 0\ :\ \exists\mathcal{Q}\ \forall p \in \mathcal{Q}\ \forall\varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \tfrac{T^n}{\nu^n}(U_p) \subseteq \varepsilon U_p\ \big\}$$
$$= \inf_{\mathcal{Q}} \sup_{p\in\mathcal{Q}} \inf\big\{\ \nu > 0\ :\ \lim_{n\to\infty} \tfrac{p(T^n)}{\nu^n} = 0\ \big\} = \inf_{\mathcal{Q}} \sup_{p\in\mathcal{Q}} \limsup_{n\to\infty} \sqrt[n]{p(T^n)}.$$

Finally,

$$r_{nb}(T) = \inf\big\{\ \nu > 0\ :\ \exists p \in \mathcal{P}\ \forall q \in \mathcal{P}\ \exists N \in \mathbb{N}\ \forall n > N\ \tfrac{T^n}{\nu^n}(U_p) \subseteq U_q\ \big\}$$
$$= \inf_{p\in\mathcal{P}} \sup_{q\in\mathcal{P}} \inf\big\{\ \nu > 0\ :\ \limsup_{n\to\infty} \tfrac{\mathfrak{m}_{pq}(T^n)}{\nu^n} \leqslant 1\ \big\} = \inf_{p\in\mathcal{P}} \sup_{q\in\mathcal{P}} \limsup_{n\to\infty} \sqrt[n]{\mathfrak{m}_{pq}(T^n)}.$$

□

Some special properties of $r_c(T)$. Continuity of an operator can be characterized in terms of neighborhoods (the pre-image of every neighborhood contains a neighborhood) or, alternatively, in terms of convergence (every convergent net is mapped to a convergent net). Analogously, though defined in terms of neighborhoods, $r_c(T)$ can also be characterized in terms of convergent nets. This approach was used by F. Garibay and R. Vera in a series of papers [8, 9, 20]. Recall that a net (x_α) in a topological vector space is said to be ***ultimately bounded*** if every zero neighborhood absorbs some tail of the net, i.e., for every zero neighborhood V one can find an index α_0 and a positive real $\delta > 0$ such that $x_\alpha \in \delta V$ whenever $\alpha > \alpha_0$. As far as we know, ultimately bounded sequences were first studied in [5] for certain locally-convex topologies. The relationship between ultimately bounded nets and convergence of sequences of operators on locally convex spaces was studied in [8, 9, 20]. The following proposition (which is, in fact, a version of [20, Corollary 2.14]) shows how $r_c(T)$ can be characterized in terms of the action of powers of T on ultimately bounded sequences. It also implies that $r_c(T)$ coincides with the number $\gamma(T)$ which was introduced in [8, 9, 20] for a continuous operator on locally convex spaces.

Proposition 4.7. *Let T be a linear operator on a topological vector space X, then*

$$r_c(T) = \inf\big\{\nu > 0\ :\ \lim_{n,\alpha} \tfrac{T^n}{\nu^n} x_\alpha = 0 \textit{ whenever } (x_\alpha) \textit{ is ultimately bounded}\big\}$$
$$= \inf\big\{\nu > 0\ :\ \big(\tfrac{T^n}{\nu^n} x_\alpha\big)_{n,\alpha} \textit{ is ultimately bounded whenever } (x_\alpha) \textit{ is ultimately bounded}\big\}.$$

Proof. To prove the first equality it suffices to show that $r_c(T) < 1$ if and only if $\lim_{n,\alpha} T^n x_\alpha = 0$ whenever (x_α) is an ultimately bounded net. Suppose that $r_c(T) < 1$, and let V be a zero neighborhood. One can find a zero neighborhood U such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \subseteq \varepsilon V$ for each $n > n_0$. Let (x_α) be an ultimately bounded net. There exists an index α_0 and a number $\delta > 0$ such that $x_\alpha \in \delta U$ whenever $\alpha > \alpha_0$. Then for $\varepsilon = \delta^{-1}$ one can find n_0 such that $T^n(U) \subseteq \delta^{-1}V$ for each $n > n_0$, so that $T^n x_\alpha \in \delta T^n(U) \subseteq V$ whenever $\alpha > \alpha_0$ and $n > n_0$. This means that $\lim_{n,\alpha} T^n x_\alpha = 0$.

Conversely, suppose that $\lim_{n,\alpha} T^n x_\alpha = 0$ for each ultimately bounded net (x_α), and assume that T^n does not converge equicontinuously to zero. Then there exists a zero neighborhood V such that for every zero neighborhood U one can find ε_U such that for every $m \in \mathbb{N}$ there exists $n_{U,m} > m$ with $T^{n_{U,m}}(U) \nsubseteq \varepsilon_U V$. Then there exists $x_{U,m} \in U$ such that

$$T^{n_{U,m}} x_{U,m} \notin \varepsilon_U V. \tag{1}$$

The collection of all zero neighborhoods ordered by inclusion is a directed set, so that $(x_{U,n})$ is an ultimately bounded net. Indeed, if W is a zero neighborhood then $x_{U,n} \in W$ for each zero neighborhood $U \subseteq W$ and every $n \in \mathbb{N}$. But it follows from (1) that the net $\bigl(T^n x_{U,m}\bigr)_{n,m,U}$ does not converge to zero.

To prove the second equality, let

$$\begin{aligned} \gamma_1 &= \inf\bigl\{\nu > 0 \,:\, \lim_{n,\alpha} \tfrac{T^n}{\nu^n} x_\alpha = 0 \text{ whenever } (x_\alpha) \text{ is ultimately bounded}\bigr\} \text{ and} \\ \gamma_2 &= \inf\bigl\{\nu > 0 \,:\, \bigl(\tfrac{T^n}{\nu^n} x_\alpha\bigr)_{n,\alpha} \text{ is ultimately bounded if } (x_\alpha) \text{ is ultimately bounded}\bigr\}. \end{aligned}$$

Since every net which converges to zero is necessarily ultimately bounded, it follows that $\gamma_1 \geqslant \gamma_2$. Now let $\nu > \gamma_2$, and let (x_α) be an ultimately bounded sequence. Suppose that $\gamma_2 < \mu < \nu$. Then $\bigl(\frac{T^n}{\mu^n} x_\alpha\bigr)_{n,\alpha}$ is ultimately bounded; that is, for each zero neighborhood V there exists an indices α_0 and n_0 and a positive ε such that $\frac{T^n}{\mu^n} x_\alpha \in \varepsilon V$ whenever $\alpha > \alpha_0$ and $n > n_0$. It follows that $\frac{T^n}{\nu^n} x_\alpha \in \frac{\mu^n \varepsilon}{\nu^n} V \subseteq V$ for $\alpha > \alpha_0$ and all sufficiently large n. This implies that $\lim\limits_{n,\alpha} \frac{T^n}{\nu^n} x_\alpha = 0$ so that $\nu \geqslant \gamma_1$. □

Question. *Are there similar ways for computing $r_l(T)$, $r_{bb}(T)$, $r_{nn}(T)$, and $r_{nb}(T)$ in terms of nets?*

Proposition 4.7 enables us to prove some important properties of r_c. The following lemma is analogous to Lemma 3.13 of [20].

Lemma 4.8. *If S and T are two commuting linear operators on a topological vector space X such that $r_c(S)$ and $r_c(T)$ are finite, then $r_c(ST) \leqslant r_c(S) r_c(T)$.*

Proof. Suppose $\mu > r_c(S)$ and $\nu > r_c(T)$ and let (x_α) be an ultimately bounded net in X. Then the net $(\frac{T^n x_\alpha}{\nu^n})_{n,\alpha}$ is ultimately bounded by Proposition 4.7. By applying Proposition 4.7 again we conclude that $(\frac{S^m T^n x_\alpha}{\mu^m \nu^n})_{m,n,\alpha}$ converges to zero. In particular, $\lim_{n,\alpha} \frac{(ST)^n x_\alpha}{(\mu\nu)^n} = \lim_{n,\alpha} \frac{S^n T^n x_\alpha}{\mu^n \nu^n} = 0$, and applying Proposition 4.7 one more time we get $\mu\nu > r_c(ST)$. □

Theorem 4.9. *If S and T are two commuting continuous operators on a locally convex space X then $r_c(S+T) \leqslant r_c(S) + r_c(T)$.*

Proof. Assume without loss of generality that both $r_c(S)$ and $r_c(T)$ are finite. Suppose that $\eta > r_c(S) + r_c(T)$ and take $\mu > r_c(S)$ and $\nu > r_c(T)$ such that $\eta > \mu + \nu$. Let (x_α) be an ultimately bounded net in X. By Proposition 4.7 it suffices to show that

$\lim_{n,\alpha} \frac{1}{\eta^n}(S+T)^n x_\alpha = 0$. Notice that the net $\left(\frac{T^n}{\nu^n}x_\alpha\right)_{n,\alpha}$ is ultimately bounded. This implies that the net $\left(\frac{S^m}{\mu^m}\frac{T^n}{\nu^n}x_\alpha\right)_{m,n,\alpha}$ converges to zero. Fix a seminorm p, then there exist indices n_0 and α_0 such that $p(S^m T^n x_\alpha) < \mu^m \nu^n$ whenever $m,n \geqslant n_0$ and $\alpha \geqslant \alpha_0$. Also, notice that we can split η into a product of two terms $\eta = \eta_1\eta_2$ such that $\eta_1 > 1$ while still $\eta_2 > \mu + \nu$. Further, if $n > 2n_0$ and $\alpha \geqslant \alpha_0$ then we have

$$p\left(\tfrac{1}{\eta^n}(S+T)^n x_\alpha\right) \leqslant$$
$$\tfrac{1}{\eta^n}\sum_{k=0}^{n_0}\tbinom{n}{k}p\big(S^kT^{n-k}x_\alpha\big) + \tfrac{1}{\eta^n}\sum_{k=n_0+1}^{n-n_0}\tbinom{n}{k}p\big(S^kT^{n-k}x_\alpha\big) + \tfrac{1}{\eta^n}\sum_{k=n-n_0+1}^{n}\tbinom{n}{k}p\big(S^kT^{n-k}x_\alpha\big).$$

Since $\binom{n}{k} = \frac{(n-k+1)\cdots(n-1)\cdot n}{1\cdot 2\cdots(k-1)\cdot k} \leqslant n^k$ and $\sum_{k=0}^{n}\binom{n}{k}\mu^k\nu^{n-k} = (\mu+\nu)^n$, we have

$$p\left(\tfrac{1}{\eta^n}(S+T)^n x_\alpha\right) \leqslant$$
$$\frac{n^{n_0}}{\eta^n}\sum_{k=0}^{n_0}p\big(S^kT^{n-k}x_\alpha\big) + \frac{1}{\eta^n}\sum_{k=n_0+1}^{n-n_0}\tbinom{n}{k}\mu^k\nu^{n-k} + \frac{n^{n_0}}{\eta^n}\sum_{k=n-n_0+1}^{n}p\big(S^kT^{n-k}x_\alpha\big)$$
$$\leqslant \frac{n^{n_0}}{\eta_1^n}\cdot\frac{1}{\eta_2^n}\sum_{k=0}^{n_0}\Big(p\big(T^{n-k}S^kx_\alpha\big) + p\big(S^{n-k}T^kx_\alpha\big)\Big) + \frac{(\mu+\nu)^n}{\eta^n}.$$

Notice that $\lim_n \frac{(\mu+\nu)^n}{\eta^n} = 0$ and that $\lim_n \frac{n^{n_0}}{\eta_1^n} = 0$. Since T is continuous, the net $(T^kx_\alpha)_\alpha$ is ultimately bounded for every fixed k, so that $\lim_{n,\alpha}\frac{1}{\eta_2^{n-k}}S^{n-k}T^kx_\alpha = 0$. It follows that for every k between 0 and n_0 the expression $\frac{1}{\eta_2^n}p\big(S^{n-k}T^kx_\alpha\big)$ is uniformly bounded for all sufficiently large n and α. Similarly, for every k between 0 and n_0 the expression $\frac{1}{\eta_2^n}p\big(T^{n-k}S^kx_\alpha\big)$ is uniformly bounded for all sufficiently large n and α. Therefore there exist indices n_1 and α_1 such that the finite sum

$$\frac{1}{\eta_2^n}\sum_{k=0}^{n_0}\Big(p\big(T^{n-k}S^kx_\alpha\big) + p\big(S^{n-k}T^kx_\alpha\big)\Big)$$

is uniformly bounded for all $n \geqslant n_1$ and $\alpha \geqslant \alpha_1$. It follows that $\lim\limits_{n,\alpha} p\Big(\frac{1}{\eta^n}(S+T)^nx_\alpha\Big) = 0$, so that $\eta > r_c(S+T)$. □

Corollary 4.10. *If T is a continuous operator on a locally convex space with $r_c(T) < \infty$ then $r_c\big(P(T)\big) < \infty$ for every polynomial $P(z)$.*

Definition 4.11. *We say that a sequence (x_n) in a topological vector space is* ***fast null*** *if $\lim\limits_{n\to\infty}\alpha^nx_n = 0$ for every positive real α.*

Lemma 4.12. *If T is a linear operator on a topological vector space with $r_c(T) < \infty$ then (T^nx_n) is fast null whenever (x_n) is fast null.*

Proof. Suppose (x_n) is a fast null sequence in a topological vector space and $r_c(T) < \infty$. Let $\nu > r_c(T)$. The sequence $\nu^n \alpha^n x_n$ converges to zero, hence is ultimately bounded. Thus by Proposition 4.7 we have

$$\lim_{n\to\infty} \alpha^n T^n x_n = \lim_{n\to\infty} \tfrac{T^n}{\nu^n} \nu^n \alpha^n x_n = 0.$$

□

5. Spectra and spectral radii

It is well known that for a continuous operator T on a Banach space its spectral radius $r(T)$ equals the geometric radius of the spectrum $\lvert\sigma(T)\rvert = \sup\{|\lambda| \;:\; \lambda \in \sigma(T)\}$. Furthermore, whenever $|\lambda| > r(T)$, the resolvent operator $R_\lambda = (\lambda I - T)^{-1}$ is given by the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$. We are going to show in the following five theorems that the spectral radii that we introduced are upper bounds for the actual radii of the correspondent spectra, and that when $|\lambda|$ is greater than or equal to any of these spectral radii, then the Neumann series converges in the corresponding operator topology to the resolvent operator.

In the following Theorems 5.1–5.5 we assume that T is a linear operator on a sequentially complete locally convex space, λ is a complex number and R_λ is the resolvent of T at λ in the sense of Definition 3.1.

Theorem 5.1. *If $|\lambda| > r_l(T)$ then the Neumann series converges pointwise to a linear operator R_λ^0, and $R_\lambda^0(\lambda I - T) = I$. Moreover, if T is continuous, then $R_\lambda^0 = R_\lambda$ and $\lvert\sigma^l(T)\rvert \leqslant r_l(T)$.*

Proof. For any $\lambda \in \mathbb{C}$ with $|\lambda| > r_l(T)$ one can find $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_l(T)$. Consider a point $x \in X$ and a basic neighborhood of zero U. Since by the definition of $r_l(T)$ the sequence $\left(\frac{T^n x}{(\lambda z)^n}\right)$ converges to zero, there exist a positive integer n_0 such that $\frac{T^n x}{(\lambda z)^n} \in U$ whenever $n \geqslant n_0$. Therefore, $\frac{T^n x}{\lambda^n} \in z^n U \subseteq |z|^n U$ because U is balanced. Thus, if $n \geqslant m \geqslant n_0$, then $\sum_{i=n}^{m} \frac{T^i x}{\lambda^i} \in \sum_{i=n}^{m} |z|^i U \subseteq \left(\sum_{i=n}^{m} |z|^i\right) U$ because U is convex. Since $|z| < 1$, we have $\sum_{i=n}^{m} |z|^i < 1$ for sufficiently large m and n, and so $\sum_{i=n}^{m} \frac{T^i x}{\lambda^i} \in U$ because U is balanced. Therefore $R_{\lambda,n} x = \frac{1}{\lambda} \sum_{i=0}^{n} \frac{T^i x}{\lambda^i}$ is a Cauchy sequence and hence it converges to some $R_\lambda^0 x$ because X is sequentially complete.

Clearly, R_λ^0 is a linear operator. Notice that $R_{\lambda,n}(\lambda x - Tx) = x - \frac{T^{n+1} x}{\lambda^{n+1}}$ for every x. As n goes to infinity, the left hand side of this identity converges to $R_\lambda^0(\lambda x - Tx)$, while the right hand side converges to x. Thus it follows that $R_\lambda^0(\lambda I - T) = I$.

Finally, notice that $R_{\lambda,n}$ commutes with T for every n. Therefore, if T is continuous, then

$$R_\lambda^0 T x = \lim_{n\to\infty} R_{\lambda,n} T x = \lim_{n\to\infty} T R_{\lambda,n} x = T\big(\lim_{n\to\infty} R_{\lambda,n} x\big) = T R_\lambda^0 x$$

for every x. This implies that $(\lambda I - T) R_\lambda^0 = R_\lambda^0(\lambda I - T) = I$, so that R_λ^0 is the (left and right) inverse of $\lambda I - T$. This means that $R_\lambda^0 = R_\lambda$ and $\lambda \in \rho^l(T)$. Thus, $\lvert\sigma^l(T)\rvert \leqslant r_l(T)$. □

Theorem 5.2. *If T is bb-bounded and $|\lambda| > r_{bb}(T)$, then the Neumann series converges uniformly on bounded sets, and its sum R_λ^0 is bb-bounded. Moreover, if T is continuous, then $R_\lambda^0 = R_\lambda$ and $\left|\sigma^{bb}(T)\right| \leqslant r_{bb}(T)$.*

Proof. Suppose that $|\lambda| > r_{bb}(T)$. The sum R_λ^0 of the Neumann series exists by Theorem 5.1. As in the proof of Theorem 5.1 we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Fix $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_{bb}(T)$, and consider a bounded set A. Let U be a basic neighborhood of zero, without loss of generality U we assume that U is closed. Since $\frac{T^n}{(\lambda z)^n}$ converges to zero uniformly on A, there exits $n_0 \in \mathbb{N}$ such that $\frac{T^n}{\lambda^n z^n}(A) \subseteq U$ for all $n > n_0$. Also, since $|z| < 1$, we can assume without loss of generality that $\sum_{i=n_0}^{\infty} |z|^i < |\lambda|$. Then

$$\frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in \frac{1}{\lambda}\Bigl(\sum_{i=n+1}^{m} |z|^i\Bigr) U \subseteq U$$

whenever $x \in A$ and $m > n > n_0$. Since U is closed, we have

$$R_\lambda^0 x - R_{\lambda,n} x = \lim_{m\to\infty} \frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in U$$

for each $x \in A$ and $n > n_0$, so that $(R_\lambda^0 - R_{\lambda,n})(A) \subseteq U$ whenever $n > n_0$. This shows that $R_{\lambda,n}$ converges to R_λ^0 uniformly on bounded sets. By Lemma 2.17 this implies that R_λ^0 is bb-bounded.

Furthermore, if T is continuous, then by Theorem 5.1 we have $R_\lambda = R_\lambda^0$, so that $\lambda \in \rho^{bb}(T)$, whence it follows that $\left|\sigma^{bb}(T)\right| \leqslant r_{bb}(T)$. □

The next theorem is similar to Theorem 2.18 of [20].

Theorem 5.3. *If T is a continuous and $|\lambda| > r_c(T)$, then the Neumann series converges equicontinuously to R_λ and R_λ is continuous. In particular, $\left|\sigma^c(T)\right| \leqslant r_c(T)$ holds.*

Proof. Let $|\lambda| > r_c(T)$. It follows from Theorem 5.1 that the Neumann series converges to R_λ. Again, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Let $z \in \mathbb{C}$ be such that $0 < |z| < 1$ and $\lambda z > r_c(T)$. Let U be a zero neighborhood, without loss of generality we can assume that U is closed. There exists a zero neighborhood V such that $\frac{T^n}{\lambda^n z^n}(V) \subseteq U$ for every $n \geqslant 0$. Let $\varepsilon > 0$. Then $\sum_{i=n_0}^{\infty} |z|^i < \varepsilon|\lambda|$ for some n_0. Thus

$$\frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in \frac{1}{\lambda}\Bigl(\sum_{i=n+1}^{m} |z|^i\Bigr) \varepsilon U \subseteq U$$

whenever $x \in V$ and $m > n > n_0$. Since U is closed, we have

$$R_\lambda x - R_{\lambda,n} x = \lim_{m\to\infty} \frac{1}{\lambda} \sum_{i=n+1}^{m} \frac{T^i x}{\lambda^i} \in \varepsilon U$$

for each $x \in V$ and $n > n_0$, so that $(R_\lambda - R_{\lambda,n})(V) \subseteq \varepsilon U$ whenever $n > n_0$. This shows that $R_{\lambda,n}$ converges to R_λ equicontinuously, and Lemma 2.19 yields that R_λ is continuous. □

Theorem 5.4. *If T is nn-bounded and $|\lambda| > r_{nn}(T)$, then the Neumann series nn-converges to R_λ and R_λ is nn-bounded. In particular, $\left|\sigma^{nn}(T)\right| \leqslant r_{nn}(T)$.*

Proof. Let $|\lambda| > r_{nn}(T)$. By Theorem 5.1 the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$ converges to R_λ. Again, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Fix some z such that $0 < |z| < 1$ and $\lambda z > r_{nn}(T)$. There exists a base $\mathcal{N}_0$ of closed convex zero neighborhoods such that for every $U \in \mathcal{N}_0$ there is a scalar $\beta > 0$ such that $\frac{T^n}{(\lambda z)^n}(U) \subseteq \beta U$ for all $n \geqslant 0$. Fix $U \in \mathcal{N}_0$. Then for each $n \geqslant 0$ we have $\frac{T^n}{\lambda^n z^n}(U) \subseteq \beta U$ for some $\beta > 0$, so that $\frac{T^n x}{\lambda^n} \in |z|^n \beta U$ whenever $x \in U$. It follows that

$$R_{\lambda,n}x = \tfrac{1}{\lambda}\sum_{i=0}^{n} \tfrac{T^i x}{\lambda^i} \in \tfrac{\beta}{\lambda}\Bigl(\sum_{i=0}^{n} |z|^i\Bigr)U.$$

Then $R_\lambda x \in \frac{\beta}{\lambda(1-|z|)}U$, so that $R_\lambda(U) \subseteq \frac{\beta}{\lambda(1-|z|)}U$, which implies that R_λ is nn-bounded, and, therefore, $\left|\sigma^{nn}(T)\right| \leqslant r_{nn}(T)$.

Fix $\varepsilon > 0$. Then $\sum_{i=N}^{\infty}|z|^i < |\lambda|$ for some N. Then for every $U \in \mathcal{N}_0$ we have

$$\tfrac{1}{\lambda}\sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \tfrac{1}{\lambda}\Bigl(\sum_{i=n+1}^{m} |z|^i\Bigr)\varepsilon U \subseteq U$$

whenever $x \in U$ and $N < n < m$. Since U is closed, we have

$$R_\lambda x - R_{\lambda,n}x = \lim_{m\to\infty} \tfrac{1}{\lambda}\sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \varepsilon U$$

for each $x \in U$ and $n > N$, so that $(R_\lambda - R_{\lambda,n})(U) \subseteq \varepsilon U$ whenever $N < n$. This shows that $R_{\lambda,n}$ nn-converges to R_λ. □

Theorem 5.5. *If T is nb-bounded and $|\lambda| > r_{nb}(T)$, then the Neumann series converges to R_λ uniformly on a zero neighborhood. Further, $\left|\sigma^{nb}(T)\right| \leqslant r_{nb}(T)$.*

Proof. Let $|\lambda| > r_{nb}(T)$. By Theorem 5.1 the Neumann series $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}}$ converges to R_λ. Since $r_{bb}(T) \leqslant r_{nb}(T)$ then R_λ is bb-bounded by Theorem 5.2. But then $\sum_{i=0}^{\infty} \frac{T^i}{\lambda^{i+1}} = \frac{1}{\lambda}I + \frac{1}{\lambda}R_\lambda T$. Notice that $R_\lambda T$ is nb-bounded as a product of a bb-bounded and an nb-bounded operator (see 2.6).

Suppose that $|\lambda| > r_{nb}(T)$. Fix $z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_{nb}(T)$. Then the sequence $\bigl(\frac{T^n}{\lambda^n z^n}\bigr)$ converges to zero uniformly on U, where U is a basic neighborhood of zero. We will show that the Neumann series converges uniformly on U. As in the proof of Theorem 5.1, we denote the partial sums of the Neumann series by $R_{\lambda,n}$. Let V be a basic neighborhood of zero, without loss of generalityt we assume that V is closed. Since $\bigl(\frac{T^n}{\lambda^n z^n}\bigr)$ converges to zero uniformly on U, there exits $n_0 \in \mathbb{N}$ such that $\frac{T^n}{\lambda^n z^n}(U) \subseteq V$ for all

$n > n_0$. Also, since $|z| < 1$, we can assume without loss of generality that $\sum_{i=n_0}^{\infty} |z|^i < |\lambda|$. Then

$$\tfrac{1}{\lambda} \sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in \tfrac{1}{\lambda}\Big(\sum_{i=n+1}^{m} |z|^i\Big) V \subseteq V$$

whenever $x \in A$ and $m > n > n_0$. Since V is closed, we have

$$R_\lambda x - R_{\lambda,n} x = \lim_{m\to\infty} \tfrac{1}{\lambda} \sum_{i=n+1}^{m} \tfrac{T^i x}{\lambda^i} \in V$$

for each $x \in U$ and $n > n_0$, so that $(R_\lambda - R_{\lambda,n})(U) \subseteq V$ whenever $n > n_0$. □

In rest of this section we present some remarks on Theorems 5.1–5.5. In particular, we discuss the necessity of sequential completeness and local convexity, and consider several examples and special cases.

5.6. It is easy to see that each spectral radius is exactly the radius of convergence of the Neumann series in the corresponding operator convergence. Indeed, in each of Theorems 5.1–5.5 the convergence of the Neumann series implies that the terms of the series tend to zero. It follows that $|\lambda|$ is greater than or equal to the corresponding spectral radius.

5.7. *Sequential completeness.* Recall that if X is a Banach space, then the norm topology on X and the weak* topology on X^* are sequentially complete. The weak topology of X is sequentially complete when X is reflexive. Also, it is known that the weak topologies of ℓ_1 and of $L_1[0,1]$ are sequentially complete. Since all these topologies are locally convex, Theorems 5.1–5.5 are applicable to each of them.

It can be easily verified that $C([0,1])$ with the topology of pointwise convergence is not sequentially complete (consider $x_n(t) = t^n$). Similarly, the sequence spaces ℓ_p for $0 < p \leqslant \infty$, c, c_0, and c_{00} (the space of eventually vanishing sequences) are not sequentially complete in the topology of coordinate-wise convergence: the sequence $x_n = \sum_{i=1}^{n} i e_i$ serves as a counterexample.

Example 5.8. *Theorems 5.1–5.5 fail without sequential completeness.* Consider the space c_0 with the topology of coordinate-wise convergence. Let T be the forward shift operator on c_0, that is, $Te_k = e_{k+1}$, where e_k is the k-th unit vector of c_0. Let V be a zero neighborhood, we can assume without loss of generality that $V = \{x \in c_0 : |x_{i_1}| < 1, \dots, |x_{i_k}| < 1\}$ where $i_1 < i_2 < \dots < i_k$ are positive integers. If $x \in c_0$ then $T^n x$ has zero components 1 through n. In particular, for every positive ν we have $\frac{T^n x}{\nu^n} \in V$ whenever $n > i_k$. Therefore $\big(\frac{T^n}{\nu^n}\big)$ converges uniformly on c_0 for every $\nu > 0$, so that $r_{nb}(T) = 0$. It follows from Proposition 4.3 that $r_l(T) = r_{bb}(T) = r_c(T) = r_{nn}(T) = 0$. On the other hand, $\sum_{n=1}^{\infty} T^n e_1$ diverges in c_0. Since T is obviously continuous, this shows that Theorems 5.1–5.5, do not hold in c_0. Thus, sequential completeness condition is essential in the theorems.

5.9. *Monotone convergence property.* Notice that if T is a positive operator on a locally convex-solid vector lattice (i.e., a locally convex space which is also a vector lattice such that $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$ for every generating seminorm p) then we can substitute the sequential completeness condition in Theorems 5.1–5.5 by a weaker condition called the ***sequential monotone completeness property***: a locally convex-solid vector lattice is said to satisfy the sequential monotone completeness property if every monotone Cauchy sequence converges in the topology of X. For details, see [3]. Indeed, we used the sequential completeness at just one point — we used it in the proof of Theorem 5.1 to claim that since $R_{\lambda,n}x = \frac{1}{\lambda}\sum_{i=0}^{n}\frac{T^i x}{\lambda^i}$ is a Cauchy sequence, then it converges to some $R_\lambda x$. But if T is positive, then $R_{\lambda,n}x^+$ and $R_{\lambda,n}x^-$ are increasing sequences, and the sequential monotone completeness property ensures the convergence.

5.10. *Banach spaces.* If T is a (norm) continuous operator on a Banach space, then it follows from 3.2 and 4.5 that $\sigma^l(T) = \sigma^{bb}(T) = \sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T) = \sigma(T)$ and $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T) = r(T)$, where $\sigma(T)$ and $r(T)$ are the usual spectrum and the spectral radius of T. Further, it follows from Lemma 4.3 that $r_l(T) \leqslant r(T)$. On the other hand, since $r(T) = \bigl|\sigma(T)\bigr|$, then $r(T) \leqslant r_l(T)$ by Theorem 5.1, so that $r_l(T) = r(T)$.

5.11. The following argument is a counterpart to 3.5. Let T be a (norm) continuous operator on a Banach space X, $r(T)$ the usual spectral radius of T. Further, let $r_l(T)$ and $r_{bb}(T)$ be computed with respect to the weak topology of X. We claim that if the weak topology of X is sequentially complete, then $r_l(T) = r_{bb}(T) = r(T)$. Indeed, $r(T) \leqslant r_l(T)$ by 3.5 and Theorem 5.1 because $\sigma(T) = \sigma^l(T)$. In view of Proposition 4.3 it suffices to show that $r_{bb}(T) \leqslant r(T)$. Let $\nu > r(T)$, and let A be a weakly bounded subset of X. Then A is norm bounded, so that the sequence $\frac{T^n}{\nu^n}$ converges to zero uniformly on A in the norm topology. In particular, the set $\bigcup_{n=0}^{\infty}\frac{T^n}{\nu^n}(A)$ is norm bounded, hence weakly bounded, so that $\nu > r_{bb}(T)$.

Quasinilpotence. Recall that a norm continuous operator T on a Banach space X is said to be ***quasinilpotent*** if $r(T) = 0$ or, equivalently, if $\sigma(T) = \{0\}$. Quasinilpotent operators on Banach spaces have some nice properties. Therefore in the framework of topological vector spaces it is interesting to study operators having some of their spectra trivial or some of their spectral radii being zero. Notice, for example, that it follows from Proposition 4.6 that if T is an operator on a locally convex topological vector space, then $r_l(T) = 0$ if and only if $\lim_n \sqrt[n]{p(T^n x)} = 0$ for every seminorm p in a generating family of seminorms and for every $x \in X$. Further, if the space is in addition sequentially complete, then for such an operator we would have $\sigma^l(T) = \{0\}$ by Theorem 5.1.

Recall also that a norm continuous operator T on a Banach space X is said to be ***locally quasinilpotent at a point*** $x \in X$ if $\lim_n \sqrt[n]{\|T^n x\|} = 0$. Using Lemma 4.1, the concept of local quasinilpotence can be naturally generalized to topological vector spaces: an operator T on a topological vector space X is said to be ***locally quasinilpotent at a point*** $x \in X$ if $\lim_n \frac{T^n x}{\nu^n} = 0$ for every $\nu > 0$. It follows immediately from the definition of $r_l(T)$ that $r_l(T) = 0$ if and only if T is locally quasinilpotent at every $x \in X$. It is known that a continuous operator on a Banach space is quasinilpotent if and only if it is locally quasinilpotent at every point. We see now that this is just a corollary of 5.10. The following example shows that a similar result for general topological vector spaces is not valid, that is, $r_l(T)$ may be equal to zero without the other radii be equal to zero.

Example 5.12. *A continuous operator with $r_l(T) = 0$ but $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T) = \infty$.* Consider the space of all bounded real sequences $\ell_\infty = \big\{x = (x_1, x_2, \dots) : \sup |x_k| < \infty\big\}$ with the topology of coordinate-wise convergence. This topology can be generated by the family of coordinate seminorms $\{p_m\}_{m=1}^\infty$ where $p_m(x) = |x_m|$. Let e_k denote the k-th unit vector in ℓ_∞.

Define an operator $T\colon \ell_\infty \to \ell_\infty$ via $T\colon \sum_{k=1}^\infty a_k e_k \mapsto \sum_{k=2}^\infty \frac{(k-1)^{k-1}}{k^k} a_k e_{k-1}$. Then $T^n e_k = \frac{(k-n)^{k-n}}{k^k} e_{k-n}$ if $n < k$ and zero otherwise. Clearly T is continuous. In order to show that $r_l(T) = 0$ fix a positive real number ν and $x \in \ell_\infty$. Then

$$\Big|\Big(\tfrac{T^n x}{\nu^n}\Big)_m\Big| = \Big|\tfrac{m^m}{(m+n)^{m+n}\nu^n} x_{n+m}\Big| \leqslant \sup_n \tfrac{m^m}{(m+n)^{m+n}\nu^n} \cdot \sup_n |x_n| < \infty$$

It follows from Lemma 4.4(i) that $r_l(T) = 0$.

Now we show that $r_{bb}(T) = \infty$ by presenting a bounded set A in ℓ_∞ such that the sequence $\big(\frac{T^n}{\nu^n}\big)$ in not uniformly bounded on A for every positive ν. Let

$$A = \big\{\, x \in \ell_\infty \;:\; x_n \leqslant (2n)^{2n} \text{ for all } n \geqslant 0 \,\big\}.$$

Then $(2n)^{2n} e_n \in A$ for each $n > 0$ and $\big(\frac{T^{n-1}}{\nu^{n-1}}(2n)^{2n} e_n\big)_1 = \frac{(2n)^{2n}}{n^n \nu^n}$ is unbounded. Then by Lemma 4.4(ii) we have $r_{bb}(T) = \infty$, and it follows from Proposition 4.3 that $r_c(T) = r_{nn}(T) = r_{nb}(T) = \infty$.

It is not difficult to show that $\sigma^l(T) = \{0\}$, while $\sigma^c(T) = \sigma^{nn}(T) = \sigma^{nb}(T) = \mathbb{C}$.

Non-locally convex spaces. We proved the key Theorems 5.1–5.5 for locally convex spaces, but they are still valid for locally pseudo-convex spaces. The local convexity of X was used only once in the proof of Theorem 5.1, while Theorems 5.2–5.5 used Theorem 5.1. Hence it would suffice to modify the proof of Theorem 5.1 in such a way that it would work for locally pseudo-convex spaces instead of locally convex. Local convexity was used in the proof of Theorem 5.1 to show that if $\frac{T^n x}{(\lambda z)^n} \in U$ for all $n > n_0$ and some $n_0 \in \mathbb{N}$, then there exists $m_0 \in \mathbb{N}$ such that $\sum_{i=n}^m \frac{T^i x}{\lambda^i} \in U$ for all $m, n > m_0$. (Recall that T is a linear operator, $\lambda, z \in \mathbb{C}$ such that $0 < |z| < 1$ and $\lambda z > r_l(T)$, $x \in X$, and U is a basic neighborhood of zero.) If X is locally pseudo-convex, then we can assume that $U + U \subseteq \alpha U$ for some $\alpha > 0$, so that (X, U) is a locally bounded space. Let $\|\cdot\|$ be the Minkowski functional of U, then (see [14, pages 3 and 6]) for any $x_1, x_2, \dots, x_n$ in X we have

$$\|x_1 + ... + x_k\| \leqslant 4^{\frac{1}{p}} \big(\|x_1\|^p + \dots + \|x_k\|^p\big)^{\frac{1}{p}},$$

where $2^{\frac{1}{p}} = \alpha$. Notice that $\|\frac{T^n x}{\lambda^n}\| \leqslant |z|^n$ for all $n > n_0$. Since $|z| < 1$, then there exists m_0 such that $\sum_{i=n}^{m} |z|^{ip} < \frac{1}{4}$ whenever $n, m > m_0$. But then

$$\Bigl\|\sum_{i=n}^{m} \tfrac{T^i x}{\lambda^i}\Bigr\| \leqslant 4^{\frac{1}{p}} \Bigl(\sum_{i=n}^{m} \|\tfrac{T^i x}{\lambda^i}\|^p\Bigr)^{\frac{1}{p}} \leqslant 4^{\frac{1}{p}} \Bigl(\sum_{i=n}^{m} |z|^{ip}\Bigr)^{\frac{1}{p}} < 1,$$

so that $\sum\limits_{i=n}^{m} \frac{T^i x}{\lambda^i} \in U$.

The following example shows that Theorems 5.1-5.5 fail if we assume no convexity conditions at all.

Example 5.13. *An operator on a complete non locally pseudo-convex space, whose spectral radii are 1, and whose Neumann series nevertheless diverges at $\lambda = 2$.* Let X be the space of all measurable functions on $[0,1]$ with the topology of convergence in measure (which is not pseudo-convex). We identify the endpoints 0 and 1 and consider the interval as a circle. Fix an irrational α and define a linear operator T on X as the translation by α, i.e., $(Tf)(t) = f(t - \alpha)$. It is easy to see that $\frac{T^n f}{\nu^n}$ converges in measure to zero for every $f \in X$ if and only if $\nu > 1$. We conclude, therefore, that $r_l(T) = 1$. Moreover, since the sets of the form $W_{\varepsilon,\delta} = \bigl\{f \in X : \mu(f > \varepsilon) < \delta\bigr\}$ form a base of zero neighborhoods for the topology of convergence in measure, and $T(W_{\varepsilon,\delta}) \subseteq W_{\varepsilon,\delta}$, it follows that $r_{nn}(T) \leqslant 1$. Then by Proposition 4.3 we have $r_l(T) = r_{bb}(T) = r_c(T) = r_{nn}(T) = 1$. Nevertheless, we are going to construct a function $h \in X$ such that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n h}{2^n}$ does not converge in measure, which means that the conclusions of Theorems 5.1–5.5 do not hold for this space

For each $n = 1, 2, 3, \dots$ one can find a positive integer M_n such that the intervals $[k\alpha, k\alpha + \frac{1}{n}]$ (mod 1) for $k = 1, \dots, M_n$ cover the circle. Let $s_n = \sum_{i=1}^{n} M_i$, and let h be the step function taking value 2^{s_n} on the interval $(\frac{1}{n+1}, \frac{1}{n}]$. If $s_{n-1} < k \leqslant s_n$ for some positive integers n and k, then on $[0, \frac{1}{n}]$ we have $h \geqslant 2^{s_n} \geqslant 2^k$, so that $\frac{h}{2^k} \geqslant 1$ on $[0, \frac{1}{n}]$, and it follows that $\frac{T^k h}{2^k} \geqslant 1$ on $[k\alpha, k\alpha + \frac{1}{n}]$.

Now, given any positive integer N, we have $N \leqslant s_{n-1}$ for some n. Then for each $k = s_{n-1} + 1, \dots, s_n$ we have $\frac{T^k h}{2^k} \geqslant 1$ on the interval $[k\alpha, k\alpha + \frac{1}{n}]$. It follows that

$$\sum_{k=s_{n-1}+1}^{s_n} \tfrac{T^k h}{2^k} \geqslant 1 \quad \text{on the set} \quad \bigcup_{k=s_{n-1}+1}^{s_n} [k\alpha, k\alpha + \tfrac{1}{n}] = s_{n-1}\alpha + \bigcup_{k=1}^{M_n} [k\alpha, k\alpha + \tfrac{1}{n}] = [0,1],$$

so that the series $\sum_{n=0}^{\infty} \frac{T^n h}{2^n}$ does not converge in measure.

Other approaches to spectral theory. There are different approaches to a spectral theory of operators on topological vector spaces, e.g., [21] and [4]. For example, Allan [4] defines the spectrum of an element x in a locally-convex algebra B as the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible or the inverse is not bounded, where $y \in B$ is said to be bounded if $\{c^n y^n\}_{n=1}^{\infty}$ is a bounded set for some real $c > 0$. In our terms, this means that R_λ exists and has finite spectral radius. Allan's spectrum is, therefore, bigger than ours. Allan defines the radius of boundedness of $\beta(x)$, which in our terms is exactly the spectral radius, and he shows that $\beta(x)$ is less than or equal to the geometric radius of his spectrum. This result nicely complements our Theorems 5.1–5.5 where we showed that a spectral radius is greater than or equal to the geometric radius of the corresponding spectrum.

For example, if X is a locally-convex space then it can be easily verified that the collection of all continuous operators on X equipped with the topology of uniform convergence on bounded sets is a locally convex algebra. For a base of convex neighborhoods of zero in this algebra one can take the sets $\mathcal{V}_{A,U}$ of all continuous operators T such that $T(A) \subseteq U$, where $A \subseteq X$ is bounded and U is a convex basic neighborhood of zero neighborhood in X. Therefore, the result of Allan is applicable in the setup of Theorem 5.2.

6. NB-BOUNDED OPERATORS

Since nb-boundedness is the strongest of the boundedness conditions we have introduced, it is natural to expect that stronger results can be obtained for nb-bounded operators.

6.1. The following argument is often useful when dealing with nb-bounded operators. Suppose that X and Y are topological vector spaces and $T\colon X \to Y$ is nb-bounded. Then $T(U)$ is a bounded set in Y for some basic neighborhood of zero U. We claim that if Y is Hausdorff, then $\bigcap_{n=1}^{\infty} \frac{1}{n}U \subseteq \operatorname{Null} T$. Indeed, it suffices to show that if $x \in \frac{1}{n}U$ for every $n \geqslant 1$ then Tx belongs to every zero neighborhood V of Y. But $T(U) \subseteq \alpha V$ for some positive α (depending on V), and hence $Tx \in \frac{1}{n}T(U) \subseteq \frac{\alpha}{n}V \subseteq V$ whenever $n \geqslant \alpha$.

It follows that if T is one-to-one, then U cannot contain any nontrivial linear subspaces. In particular, if U is convex then the locally bounded space (X,U) is Hausdorff, hence quasinormable. In this case T is a continuous operator from (X,U) to Y, and, moreover, if $X = Y$, then T is continuous as an operator from (X,U) to (X,U).

In fact, many "classical" topological vector spaces have the property that every zero neighborhood contains a nontrivial linear subspace, e.g., topologies of pointwise or coordinate-wise convergence, weak topologies. On the other hand, there exist spaces with a base of zero neighborhoods which consists of neighborhoods not containing any nontrivial linear subspaces. For example, let X be the space of all analytic functions on $\mathbb{C}$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. The sets $U_{n,\varepsilon} = \bigl\{f \in X \;:\; \lvert f(z)\rvert < \varepsilon \text{ whenever } \lvert z\rvert \leqslant n\bigr\}$, ($n \geqslant 0$ and $\varepsilon > 0$) form a base of zero neighborhoods of this topology. It is easy to see that no $U_{n,\varepsilon}$ contains a non-trivial linear subspace. Note that this topology is generated by the countable sequence of norms $\lVert f\rVert_n = \sup_{\lvert z\rvert \leqslant n}\bigl\lvert f(z)\bigr\rvert$; clearly $\lVert\cdot\rVert_n$ is the Minkowski functional of $U_{n,1}$.

Theorem 6.2. *If X is a complete locally convex space then X is locally bounded if and only if X admits an nb-bounded bijection.*

Proof. If X is locally bounded then the identity map is an nb-bounded bijection. Suppose that T is an nb-bounded bijection on X. Then there exists a basic neighborhood of zero

U such that $T(U)$ is bounded. Without loss of generality we may assume that U is closed. Let $A = \overline{T(U)}$. Then A is convex, bounded, balanced, and absorbing. It follows that the space (X, A) is locally convex and locally bounded, — denote it by X_A. Notice also that the topology of X_A is finer than the original topology on X because A is bounded. In particular, X_A is Hausdorff.

We claim that X_A is complete. Indeed, if (x_α) is a Cauchy net in X_A, then it is also Cauchy in the original topology of X, which is complete, so that x_α converges to some x. Fix $\varepsilon > 0$. Then there exists α_0 such that $x_\alpha - x_\beta \in \varepsilon A$ whenever $\alpha, \beta \geqslant \alpha_0$. Letting $\beta \to \infty$, since A is closed we have $x_\alpha - x \in \varepsilon A$, i.e., $x_\alpha \to x$ in X_A. Thus, X_A is complete, hence Banach.

Since A is bounded, we can find m such that $A \subseteq mU$. Then $T(A) \subseteq T(mU) \subseteq mA$, so that T is bounded in X_A. Then T^{-1} is also bounded in X_A by the Open Mapping Theorem, so that $U = T^{-1}\big(T(U)\big) \subseteq T^{-1}(A) \subseteq nA$ for some $n > 0$. Hence U is bounded. □

Proposition 6.3. *Let $T\colon X \to Y$ be an nb-bounded operator between Hausdorff topological vector spaces such that X is not locally bounded. If*

(i) *every zero neighborhood in X contains a non-trivial linear subspace, or*
(ii) *both X and Y are Fréchet spaces,*

then T is not a bijection.

Proof. If every zero neighborhood of X contains a non-trivial linear subspace, then T cannot be one-to-one by 6.1. Suppose now that X and Y are Fréchet and assume that T is a bijection. Let $S\colon Y \to X$ be the linear inverse of T. The Open Mapping Theorem implies that S is continuous and hence bb-bounded. It follows that the identity operator of X is nb-bounded being the composition of the nb-bounded operator T and the bb-bounded operator S. But the identity operator is nb-bounded if and only if the space is locally bounded, a contradiction. □

Weak topologies. We are going to show that an operator which is nb-bounded relative to a weak topology has to be of finite rank. In order to prove this we need the following well-known lemma. For completeness we provide a simple proof of it.

Lemma 6.4. *Let T be a linear operator on a vector space L, and let $f_1, \dots, f_n$ be linear functionals on L such that $Tx = 0$ whenever $f_i(x) = 0$ for every $i = 1, \dots, n$. Then T is a finite rank operator of rank at most n.*

Proof. Define a linear map π from L to $\mathbb{R}^n$ via $\pi(x) = (f_1(x), \dots, f_n(x))$. Then the dimension of the range $\pi(L)$ is at most n. Define also a linear map φ from $\pi(L)$ to L via $\varphi(\pi(x)) = Tx$. It can be easily verified that φ is well-defined. Then the range of T coincides with the range $\varphi(\pi(L))$, which is of dimension at most n. □

Proposition 6.5. *Let X be a locally convex space, and T an operator on X such that T is nb-bounded with respect to the weak topology of X. Then T is of finite rank.*

Proof. Suppose T maps some weak zero neighborhood $U = \big\{x \in X \ : \ |f_i(x)| < 1, i = 1, \dots, n\big\}$, $(f_1, \dots, f_n \in X')$, to a weakly bounded set. Since the weak topology is Hausdorff, it follows from 6.1 that $\bigcap_{n=1}^{\infty} \frac{1}{n} U \subseteq \ker T$. In particular, $Tx = 0$ whenever $f_i(x) = 0$ for every $i = 1, \dots, n$. Then Lemma 6.4 implies that T is a finite rank operator. $\square$

Spectra and spectral radii of nb-bounded operators.

Proposition 6.6. *If T is an nb-bounded operator on a topological vector space then $\sigma^{bb}(T) = \sigma^{c}(T) = \sigma^{nn}(T) = \sigma^{nb}(T)$.*

Proof. If X is locally bounded then the result is trivial by 3.3. Suppose that X is not locally bounded. Then, in view of 3.2, it suffices to show that $\rho^{bb}(T) \subseteq \rho^{nb}(T)$. Let $\lambda \in \rho^{bb}(T)$, so that R_λ is bb-bounded. If $\lambda \neq 0$, then it follows from $R_\lambda(\lambda I - T) = I$ that $R_\lambda = \frac{1}{\lambda} R_\lambda T + \frac{1}{\lambda} I$. Thus, R_λ is a sum of an nb-bounded operator and a multiple of the identity operator, which yields $\lambda \in \rho^{nb}(T)$. To finish the proof, we will show that if X is not locally bounded then $\lambda = 0$ always belongs to $\sigma^{bb}(T)$ and, therefore, to $\sigma^{c}(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$. Indeed, if the resolvent $R_\lambda = T^{-1}$ were bb-bounded, then $I = T^{-1}T$ would be nb-bounded, which is impossible in a non-locally bounded space, a contradiction. $\square$

Proposition 6.7. *If T is an nb-bounded operator on a topological vector space, then $r_{bb}(T) = r_c(T) = r_{nn}(T) = r_{nb}(T)$.*

Proof. By Proposition 4.3 it suffices to show that $r_{bb}(T) \geqslant r_{nb}(T)$. Since T is nb-bounded, then $T(U)$ is a bounded set for some zero neighborhood U. Let $\nu > r_{bb}(T)$ and fix a zero neighborhood V. Then νV is again a zero neighborhood. In particular, since the sequence $\frac{T^n}{\nu^n}$ converges to zero uniformly on bounded sets, we have $\frac{T^n}{\nu^n}\big(T(U)\big) \subseteq \nu V$ for all sufficiently large n. Then $\frac{T^{n+1}}{\nu^{n+1}}(U) \subseteq V$, so that $\frac{T^n}{\nu^n}$ converges to zero uniformly on U. Therefore $\nu \geqslant r_{nb}(T)$, so that $r_{bb}(T) \geqslant r_{nb}(T)$. $\square$

6.8. In view of Propositions 6.6 and 6.7 we can write $\sigma(T)$ instead of $\sigma^{bb}(T)$, $\sigma^{c}(T)$, $\sigma^{nn}(T)$, and $\sigma^{nb}(T)$ and $r(T)$ instead of $r_{bb}(T)$, $r_c(T)$, $r_{nn}(T)$, and $r_{nb}(T)$.

We have established in Theorems 5.1–5.5 that under certain conditions the spectral radii of a linear operator are greater than or equal to the geometric radii of the corresponding spectra. Of course we would like to know when the equalities hold. It is well known that the equality $\big|\sigma(T)\big| = r(T)$ holds for every continuous operator on a Banach space. Moreover, it was shown in [10] that this equality also holds for every continuous operator on a quasi-Banach space (a complete quasinormed space). Further, by means of Proposition 4.7 the main result of [9] is equivalent to the following statement: $r(T) = \big|\sigma(T)\big|$ *for every nb-bounded operator T on a complete locally convex space*. Here we present a direct proof of this. Our proof is a simplified version of the proof of [9].

Theorem 6.9. *If T is an nb-bounded linear operator on a sequentially complete locally convex space, then $\big|\sigma(T)\big| = r(T)$.*

Proof. Suppose $T(U)$ is bounded for some basic neighborhood of zero U. It follows from Propositions 6.6, 6.7, and 4.3, and Theorem 5.5 that it suffices to show that $\left|\sigma^{nn}(T)\right| \geqslant r_{nb}(T)$. We are going to show that T induces a continuous operator $\widetilde{T}$ on some Banach space such that $\sigma\bigl(\widetilde{T}\bigr) \subseteq \sigma^{nn}(T) \cup \{0\}$ while $r\bigl(\widetilde{T}\bigr) \geqslant r_{nb}(T)$, and then appeal to the fact that the spectral radius of a continuous operator on a Banach space equals the geometric radius of the spectrum.

Consider T as a continuous operator on the locally bounded space $X_U = (X, U)$. Then $\sigma_U(T)$ is defined by 3.3 and $r_U(T)$ is defined by 4.5. We claim that $r_U(T) \geqslant r_{nb}(T)$. To see this, suppose $r_U(T) < \nu$. Then $\frac{T^n}{\nu^n}(U) \subseteq U$ for all sufficiently large n. Let V be another zero neighborhood, then $T(U) \subseteq \alpha V$ for some $\alpha > 0$, so that $\frac{T^n}{\nu^n}(U) = \frac{T}{\nu}\frac{T^{n-1}}{\nu^{n-1}}(U) \subseteq \frac{1}{\nu}T(U) \subseteq \frac{\alpha}{\nu}V$ for sufficiently large n. This implies that $\nu \geqslant r_{nb}(T)$, and it follows that $r_U(T) \geqslant r_{nb}(T)$.

On the other hand, we claim that $\sigma_U(T) \subseteq \sigma^{nn}(T)$. Suppose $\lambda \in \rho^{nn}(T)$. Then R_λ is nn-bounded with respect to some base $\mathcal{N}_0$ of zero neighborhoods. We can assume without loss of generality that $U \in \mathcal{N}_0$, so that $R_\lambda(U) \subseteq \beta U$ for some $\beta > 0$. It follows that $\lambda \in \rho_U(T)$.

Since U is convex, the space X_U is, in fact, a seminormed space. We can assume without loss of generality that it is a normed space, because otherwise we can consider the quotient space $X_U/(\operatorname{Null} T)$ and the quotient operator $\widehat{T}$ on this quotient space instead of T. Indeed, since $\bigcap_{n=1}^{\infty}\frac{1}{n}U \subseteq \operatorname{Null} T$ by 6.1, we conclude that the quotient space $X_U/(\operatorname{Null} T)$ is Hausdorff. It follows then that $X_U/(\operatorname{Null} T)$ is a normed space, and $\widehat{T}$ is norm bounded. The spectrum $\sigma_U(T)$ becomes even smaller when we substitute T with $\widehat{T}$. Indeed, suppose $\lambda \in \rho_U(T)$. Then the resolvent R_λ exists in X_U and is continuous. If $x \in \ker T$, then $x = R_\lambda(\lambda I - T)x = \lambda R_\lambda x$, so that R_λ leaves $\ker T$ invariant, and, therefore, induces a quotient operator $\widehat{R_\lambda}$ on $X_U/\ker T$ via $\widehat{R_\lambda}([x]) = [R_\lambda x]$. Clearly, $\widehat{R_\lambda}$ is continuous: if $[x_n] \to [x]$ in $X_U/\ker T$ then $x_n - z_n \to x$ in X_U for some $(z_n)_{n=1}^{\infty}$ in $\ker T$, so that $[R_\lambda x_n] = [R_\lambda(x_n - z_n)] \to [R_\lambda x]$. On the other hand, $r_U(\widehat{T}) \geqslant r_U(T)$, because if $\nu > r_U(\widehat{T})$ then $\frac{\widehat{T}^n}{\nu^n}([U]) \subseteq [U]$ for all sufficiently large n, $\frac{T^n}{\nu^n}(U) \subseteq U + \ker T$, and so $\frac{T^{n+1}}{\nu^{n+1}}(U) \subseteq \frac{1}{\nu}T(U) \subseteq \frac{\alpha}{\nu}U$ for some $\alpha > 0$. It follows that $\nu \geqslant r_U(T)$ and, therefore, $r_U(\widehat{T}) \geqslant r_U(T)$.

Finally, we consider the completion $\widetilde{X_U}$ of X_U, and extend T to a continuous linear operator $\widetilde{T}$ on the completion. The spectrum of $\widetilde{T}$ is smaller that the spectrum of T, because if $\lambda \in \rho_U(T)$ then the resolvent R_λ can be extended by continuity to $\widetilde{R_\lambda}$ on $\widetilde{X}$, and $\widetilde{R_\lambda}$ is a continuous inverse to $\lambda I - \widetilde{T}$, so that $\lambda \in \rho(\widetilde{T})$. On the other hand, $r(\widetilde{T}) \geqslant r_U(T)$ because if $\nu > r(\widetilde{T})$ then $\frac{\widetilde{T}^n}{\nu^n}(\widetilde{U}) \subseteq \widetilde{U}$ for all sufficiently large n, which implies $\frac{T^n}{\nu^n}(U) \subseteq U$ since T is a restriction of $\widetilde{T}$ on X. $\square$

7. Compact operators

As with bounded operators, there is more than one way to define a compact operator on an arbitrary topological vector space. A subset of a topological vector space is called ***precompact*** if its closure is compact. Given a linear operator T on a topological vector space, T is called ***Montel*** if it maps every bounded set into a precompact set and ***compact*** if it maps some neighborhood into a precompact set. To be consistent, we should have probably called these operators "b-compact" and "n-compact" respectively, but the names "Montel" and "compact" are commonly accepted. Obviously, every compact operator is Montel and nb-bounded (hence continuous); every Montel operator is bb-bounded.

7.1. If T is compact or Montel, then sequential completeness is not needed in Theorems 5.1–5.5. Indeed, we used sequential completeness just once, namely, in the proof of Theorem 5.1 to justify the convergence of the sequence $R_{\lambda,n}x = \frac{1}{\lambda}\sum_{i=0}^{n}\frac{T^i x}{\lambda^i}$. But since the sequence $(R_{\lambda,n}x)_n$ is Cauchy and, therefore, bounded, the sequence $(TR_{\lambda,n}x)_n$ has a convergent subsequence whenever T is compact or Montel. Furthermore, it follows from $R_{\lambda,n+1}x = \frac{1}{\lambda}(I + TR_{\lambda,n})x$ that $(R_{\lambda,n}x)_n$ has a convergent subsequence hence converges.

Let K be a compact operator on an arbitrary topological vector space, and let $\sigma(K)$ and $r(K)$ be as in 6.8. It was proved in [15] that $\sigma(K) = \{0\}$ implies $r_l(K) = 0$. In the following theorem we use the technique of [15] to improve this result by showing that in general $r(K) \leqslant \lvert\sigma(K)\rvert$.

Theorem 7.2. *If K is a compact operator on a Hausdorff topological vector space X, then $r(K) \leqslant \lvert\sigma(K)\rvert$.*

Proof. Assume that $\lvert\sigma(K)\rvert < r(K)$. Without loss of generality (by scaling K) we can assume that $\lvert\sigma(K)\rvert < 1 < r(K)$. Since K is compact, there is a basic neighborhood of zero U such that $\overline{K(U)}$ is compact. Without loss of generality we can assume that U is closed. In particular $\overline{K(U)}$ is bounded, so that $\overline{K(U)} \subseteq \eta U$ for some $\eta > 0$. We can assume without loss of generality that $\eta > 1$. We define the following subsets of U:

$$U_1 = \overline{K(U)} \cap U, \quad U_{n+1} = K(U_n) \cap U \quad (n = 1, 2, \dots), \quad \text{and} \quad U_0 = \bigcap_{n=1}^{\infty} U_n.$$

Notice that U_1 is compact because $\overline{K(U)}$ is compact and U is closed. Also, if U_n is compact, then $K(U_n)$ is compact being the image of a compact set under a continuous operator. Therefore, every U_n for $n \geqslant 1$ is compact. Using induction, we can show that the sequence (U_n) is decreasing. Indeed, $U_1 \subseteq U$ by definition, $U_2 = K(U_1) \cap U \subseteq K(U) \cap U \subseteq U_1$, and if $U_n \subseteq U_{n-1}$, then $U_{n+1} = K(U_n) \cap U \subseteq K(U_{n-1}) \cap U = U_n$. It follows also that U_0 is compact and contains zero.

Notice that K maps every balanced set to a balanced set. Since U is balanced, U_n is balanced for each $n \geqslant 0$. If A is a balanced subset of U, then obviously $A \subseteq (\eta A) \cap U$, and when we apply the same reasoning to $\frac{1}{\eta}K(A)$ instead of A (which is also a balanced subset of U), we get $\frac{1}{\eta}K(A) \subseteq K(A) \cap U$. We use this to show by induction that $\frac{1}{\eta^n}K^n(U) \subseteq U_n$ for every $n \geqslant 1$. Indeed, for $n = 1$ we have $\frac{1}{\eta}K(U) \subset K(U) \cap U \subseteq U_1$. Suppose $\frac{1}{\eta^n}K^n(U) \subseteq U_n$ for some $n \geqslant 1$. Then

$$\tfrac{1}{\eta^{n+1}}K^{n+1}(U) \subseteq \tfrac{1}{\eta}K(U_n) \subseteq K(U_n) \cap U = U_{n+1},$$

which proves the induction step.

Next, we claim that there exists an open zero neighborhood V and an increasing sequence of positive integers (n_j) such that $U_{n_j} \setminus V$ is nonempty for every $j \geqslant 1$. Assume for the sake of contradiction that for every open zero neighborhood V we have $U_n \subseteq V$ for all sufficiently large n. Since $\frac{1}{2}U$ contains an open zero neighborhood, then there exists a positive integer N such that $U_n \subseteq \frac{1}{2}U$ whenever $n \geqslant N$. This implies that $U_{N+m} = K^m(U_N)$ for all $m \geqslant 0$. Indeed, this holds trivially for $m = 0$. Suppose that $U_{N+m} = K^m(U_N)$ for some $m \geqslant 0$. Then $U_{N+m+1} = K(U_{N+m}) \cap U = K^{m+1}(U_N) \cap U$, and this implies that $U_{N+m+1} = K^{m+1}(U_N)$ because $U_{N+m+1} \subseteq \frac{1}{2}U$. Now take any open zero neighborhood V, then $\frac{1}{\eta^N}V$ is again a zero neighborhood, and by assumption there exists a positive integer M such that $U_n \subseteq \frac{1}{\eta^N}V$ whenever $n \geqslant M$. Let $n \geqslant \max\{M, N\}$, then

$$V \supseteq \eta^N U_n = \eta^N K^{n-N}(U_N) \supseteq \eta^N K^{n-N}\left(\tfrac{1}{\eta^N}K^N(U)\right) = K^n(U),$$

which contradicts the hypothesis $r_{nb}(K) = r(K) > 1$.

It follows from $U_{n_j} \setminus V \neq \emptyset$ for every $j \geqslant 1$ that $U_n \setminus V \neq \emptyset$ for all sufficiently large n because U_n is a decreasing sequence. Since $U_n \setminus V$ is a decreasing sequence of nonempty compact sets, then $U_0 \setminus V = \bigcap_{n=1}^{\infty}(U_n \setminus V) \neq \emptyset$, so that $U_0 \neq \{0\}$.

For every $n \geqslant 1$ we have $U_0 \subseteq U_n$. It follows that $K(U_0) \subseteq K(U_n)$ and, therefore, $K(U_0) \subseteq \bigcap_{n=1}^{\infty} K(U_n)$. Actually, the reverse inclusion also holds. To see this, let $y \in \bigcap_{n=1}^{\infty} K(U_n)$. Then $y = Kx_n$, where $x_n \in U_n \subseteq U_1$. Since U_1 is compact, the sequence (x_n) has a cluster point, i.e., $x_{n_j} \to x$ for some subsequence (x_{n_j}) and some x. Since K is continuous we have $y = Kx$. On the other hand, since every U_{n_j} is closed we have $x \in U_{n_j}$, so that $x \in \bigcap_{n=1}^{\infty} U_{n_j} = U_0$. Thus $K(U_0) = \bigcap_{n=1}^{\infty} K(U_n)$.

Next, we claim that $U_0 \subseteq K(U_0) \subseteq \eta U_0$. Indeed,

$$U_0 = \bigcap_{n=2}^{\infty} U_n = \bigcap_{n=2}^{\infty}\bigl[K(U_{n-1}) \cap U\bigr] \subseteq \bigcap_{n=2}^{\infty} K(U_{n-1}) = K(U_0).$$

On the other hand, since U_n are decreasing and $\eta > 1$, we have $K(U_n) \subseteq K(U_{n-1}) \subseteq \eta K(U_{n-1})$ and $K(U_n) \subseteq K(U) \subseteq \eta U$, so that $K(U_n) \subseteq \eta K(U_{n-1}) \cap \eta U = \eta U_n$. This implies $K(U_0) \subseteq K(U_n) \subseteq \eta U_n$ for every n. Thus $K(U_0) \subseteq \eta U_0$.

Since $\overline{K(U)}$ is compact, hence bounded, then $\overline{K(U)} + \overline{K(U)}$ is also bounded. Thus there is a positive constant γ such that $\overline{K(U)} + \overline{K(U)} \subseteq \gamma U$. Without loss of generality we can assume $\gamma \geqslant 2$. It follows that

$$U_1 + U_1 = \overline{K(U)} \cap U + \overline{K(U)} \cap U \subseteq \overline{K(U)} + \overline{K(U)} \subseteq \gamma U.$$

We use induction to show that $U_n + U_n \subseteq \gamma U_{n-1}$. Indeed, since $A \cap B + C \cap D \subseteq (A + C) \cap (B + D)$ for any four sets A, B, C, and D, then

$$\begin{aligned} U_{n+1} + U_{n+1} &= K(U_n) \cap U_n + K(U_n) \cap U_n \\ &\subseteq \big[K(U_n) + K(U_n)\big] \cap (U_n + U_n) \subseteq K(U_n + U_n) \cap (U_n + U_n) \\ &\subseteq K(\gamma U_{n-1}) \cap \gamma U_{n-1} = \gamma\big[K(U_{n-1}) \cap U_{n-1}\big] = \gamma U_n. \end{aligned}$$

Finally, $U_0 + U_0 \subseteq \bigcap_{n=1}^{\infty}(U_n + U_n) \subseteq \bigcap_{n=1}^{\infty} \gamma U_n = \gamma U_0$.

Next, consider the set $F = \bigcup_{n=1}^{\infty} nU_0$. This set is closed under multiplication by a scalar, and $U_0 + U_0 \subseteq \gamma U_0$ implies that F is a linear subspace of X. We consider the locally bounded topological vector space (F, U_0) with multiples of U_0 as the base of zero neighborhoods. Since U_0 is balanced by definition, this topology is linear, and it is Hausdorff because U_0 is compact. Also, it is finer than the topology on F inherited from X because U_0 is compact and, therefore, bounded in X.

We claim that (F, U_0) is complete. Indeed, if (x_n) is a Cauchy sequence in (F, U_0) then there exists $k > 0$ such that $x_n \in kU_0$ for each $n > 0$. Since U_0 is compact, the sequence (x_n) has a subsequence which converges to some $x \in kU_0$ in the topology of X. Moreover, $\lim\limits_{n\to\infty} x_n = x$ because the sequence (x_n) is Cauchy in X. Fix $\varepsilon > 0$, then there exists n_0 such that $x_n - x_m \in \varepsilon U_0$ whenever $n, m \geqslant n_0$. Let $m \to \infty$, since U_0 is is closed we have $x_n - x \in \varepsilon U_0$, i.e., $x_n \to x$ in (F, U_0). Thus, (F, U_0) is complete and, therefore, quasi-Banach.

It follows from $U_0 \subseteq K(U_0) \subseteq \eta U_0$ that F is invariant under K and the restriction $\widetilde{K} = K|_F$ is continuous. We claim that $\sigma(\widetilde{K}) \subseteq \sigma(K) \cup \{0\}$. If $\lambda \in \rho(K)$ and $\lambda \neq 0$, then $(\lambda I - K)$ is a homeomorphism, so that $(\lambda I - K)(U)$ is a closed zero neighborhood, and $\alpha U_1 \subseteq (\lambda I - K)(U)$ for some positive real α because U_1 is bounded. Further, $\alpha K(U_1) \subseteq K(\lambda I - K)(U) \subseteq (\lambda I - K)K(U)$. Therefore

$$\alpha U_2 \subseteq \alpha K(U_1) \cap \alpha U_1 \subseteq (\lambda I - K)K(U) \cap (\lambda I - K)(U),$$

and since $\lambda I - K$ is one-to-one we get $\alpha U_2 \subseteq (\lambda I - K)(K(U) \cap U) \subseteq (\lambda I - K)(U_1)$. Similarly, we obtain $\alpha U_{n+1} \subseteq (\lambda I - K)(U_n)$ for all $n \geqslant 1$, and then $\alpha U_0 \subseteq (\lambda I - K)(U_0)$. This implies that the restriction of $\lambda I - K$ to F is onto, invertible, and the inverse is continuous. Thus, $\lambda \in \rho(\widetilde{K})$.

In particular this implies that $\big|\sigma(\widetilde{K})\big| \leqslant \big|\sigma(K)\big| < 1$. On the other hand, it follows from $U_0 \subseteq K(U_0)$ that $U_0 \subseteq \widetilde{K}^n(U_0)$ for all $n \geqslant 0$, so that $\widetilde{K}^n$ does not converge to zero uniformly on U_0, whence $r(\widetilde{K}) = r_{bb}(\widetilde{K}) \geqslant 1$. This produces a contradiction because it was proved in [10] that the spectral radius of a continuous operator on a quasi-Banach space equals the radius of the spectrum. □

Corollary 7.3. *If K is a compact operator on a locally convex (or pseudo-convex) space, then $r(K) = \big|\sigma(K)\big|$.*

8. Closed operators

In certain situations one has to deal with unbounded linear operators on Banach spaces. For example, the generator of a strongly continuous operator semigroup is generally a closed operator with dense domain (see e.g. [11, 6]). Throughout this section T will be a closed operator on a Banach space X with domain $\mathcal{D}(T)$. As usually, we define $\mathcal{D}(T^{n+1}) = \{x \in \mathcal{D}(T^n) \;:\; T^n x \in \mathcal{D}(T)\}$ and $D = \bigcap_{n=0}^{\infty} \mathcal{D}(T^n)$. In case when T is the infinitesimal generator of an operator semigroup, D is dense in the range of the semigroup, which is usually assumed to be all of X. The set D with the locally-convex topology τ given by the sequence of norms $\|x\|_n = \sum_{k=0}^{n} \|T^k x\|$ is a Fréchet space. Clearly, D is invariant under T, and the restriction operator $T_{|D}$ is continuous because $x_\alpha \xrightarrow{\tau} 0$ in D implies $\|Tx_\alpha\|_n \leqslant \|x_\alpha\|_{n+1} \to 0$ for each n.

We investigate the relation between the spectral properties of the original operator T on X and of the restriction $T_{|D}$ on D.[3] Recall that $\lambda \in \rho(T)$ if $R(\lambda;T) = (\lambda I - T)^{-1} \colon X \to \mathcal{D}(T)$ exists (it is automatically bounded by [11, Theorem 2.16.3]), and $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Lemma 8.1. *If $\lambda \in \rho(T)$ then $R(\lambda;T)_{|D}$ is a bijection from D to D commuting with T on $\mathcal{D}(T)$. Furthermore, for each $n \geqslant 0$ there is a constant C_n such that $\|R(\lambda;T)x\|_n \leqslant C_n \|x\|_{n-1}$ for each $x \in D$.*

Proof. It can be easily verified that $R(\lambda;T)$ is a bijection from $\mathcal{D}(T^n)$ onto $\mathcal{D}(T^{n+1})$ and, therefore, the restriction $R(\lambda;T)_{|D}$ is a bijection. Notice that since $R(\lambda;T)(\lambda I - T)x = x$ for each $x \in \mathcal{D}(T)$ and $(\lambda I - T)R(\lambda;T)x = x$ for each $x \in X$, then

$$\begin{aligned} TR(\lambda;T)x &= \lambda R(\lambda;T)x - x \text{ for each } x \in \mathcal{D}(T) \text{ and} \\ R(\lambda;T)Tx &= \lambda R(\lambda;T)x - x \text{ for each } x \in X, \end{aligned}$$

so that T and $R(\lambda;T)$ commute on $\mathcal{D}(T)$. It also follows that for each $x \in D$ we have

$$\begin{aligned} T^2R(\lambda;T)x &= \lambda TR(\lambda;T)x - Tx = \lambda^2 R(\lambda;T)x - \lambda x - Tx, \\ T^3R(\lambda;T)x &= \lambda^2 TR(\lambda;T)x - \lambda Tx - T^2x = \lambda^3 R(\lambda;T)x - \lambda^2 x - \lambda Tx - T^2 x, \\ &\vdots \\ T^kR(\lambda;T)x &= \lambda^k R(\lambda;T)x - \lambda^{k-1}x - \lambda^{k-2}Tx - \cdots - \lambda T^{k-2}x - T^{k-1}x. \end{aligned}$$

It follows that

$$\|T^kR(\lambda;T)x\| \leqslant |\lambda|^k \|R(\lambda;T)x\| + |\lambda|^{k-1}\|x\| + |\lambda|^{k-2}\|Tx\| + \cdots + \|T^{k-1}x\|$$

for each $x \in D$, so that

$$\|R(\lambda;T)x\|_n = \sum_{k=0}^{n} \|T^kR(\lambda;T)x\| \leqslant \\ \mu_n\|R(\lambda;T)x\| + \mu_{n-1}\|x\| + \mu_{n-2}\|Tx\| + \cdots + \mu_0\|T^{n-1}x\|,$$

[3] Another approach to this question can be found in [22].

where $\mu_k = 1 + |\lambda| + \cdots + |\lambda|^k$. Since $\|R(\lambda;T)x\| \leqslant \|R(\lambda;T)\|\|x\|$ it follows that $\|R(\lambda;T)x\|_n \leqslant C_n\big(\|x\| + \|Tx\| + \cdots + \|T^{n-1}x\|\big)$ for some C_n, so that $\|R(\lambda;T)x\|_n \leqslant C_n\|x\|_{n-1}$. ☐

Proposition 8.2. *The inclusion $\rho(T) \subseteq \rho^{nn}(T_{|D})$ holds. Moreover, if D is dense in X and T is the smallest closed extension of $T_{|D}$, then $\rho(T) = \rho^{nn}(T_{|D})$.*

Proof. Suppose that $\lambda \in \rho(T)$ and consider the resolvent operator $R(\lambda;T)$ on X. Then $\|R(\lambda;T)x\|_n \leqslant C_n\|x\|_{n-1} \leqslant C_n\|x\|_n$ hence $R(\lambda;T)_{|D}$ is nn-bounded and $\lambda \in \rho^{nn}(T_{|D})$.

Suppose now that D is dense in X, T is the smallest closed extension of $T_{|D}$, and $\lambda \in \rho^{nn}(T_{|D})$. Then there exists an nn-bounded operator $R(\lambda;T_{|D})\colon D \to D$ such that $R(\lambda;T_{|D})(\lambda I - T)x = (\lambda I - T)R(\lambda;T_{|D})x = x$ for each $x \in D$. Then there is a constant $C > 0$ such that $\|R(\lambda;T_{|D})x\| = \|R(\lambda;T_{|D})x\|_0 \leqslant C\|x\|_0 = C\|x\|$ for each $x \in D$. It follows that $R(\lambda;T_{|D})$ can be extended to a bounded operator R on X. Fix $x \in X$ and pick (x_n) in D such that $x_n \to x$. Then $R(\lambda;T_{|D})x_n \to Rx$ and $(\lambda I - T)R(\lambda;T_{|D})x_n = x_n \to x$. Since $\lambda I - T$ is closed we have $(\lambda I - T)Rx = x$. It follows, in particular, that $\lambda I - T$ is onto.

Since $\|(\lambda I - T)x\| \geqslant \frac{1}{C}\|x\|$ for each $x \in D$, it follows that for every nonzero $y \in X$ the pair $(y,0)$ doesn't belong to the closure of the graph of $\lambda I - T_{|D}$. But the closure of the graph of $\lambda I - T_{|D}$ is the graph of $\lambda I - T$ because $\lambda I - T$ is the smallest closed extension of $\lambda I - T_{|D}$. It follows that $\lambda I - T$ is one-to-one, hence $\lambda \in \rho(T)$. ☐

Suppose now that S is a bounded operator on X such that D is invariant under S and $STx = TSx$ for each $x \in D$. Then

$$\|Sx\|_m = \sum_{k=0}^{m}\|T^kSx\| \leqslant \|S\|\sum_{k=0}^{m}\|T^kx\| = \|S\| \cdot \|x\|_m,$$

so that $\|S_{|D}\|_n \leqslant \|S\|$. Moreover, if $m \leqslant k$ then $\|x\|_m \leqslant \|x\|_k$, so that the mixed seminorm $\mathfrak{m}_{km}(S_{|D}) \leqslant \|S\|$. It also follows from Proposition 4.6 that $r_{nn}(S_{|D}) \leqslant r(S)$.

Further, we claim that if $R = R(\lambda;T)$ for some $\lambda \in \rho(T)$, then $r_{nb}(R_{|D}) \leqslant r(R)$. Indeed, recursive application of Lemma 8.1 yields $\|R^nx\|_k \leqslant M_k\|R^{n-k}x\|$ for each $x \in D$ and $k \geqslant n$, where $M_k = \Pi_{i=1}^{k}C_i$. It follows that the mixed seminorm

$$\begin{aligned}\mathfrak{m}_{mk}(R^n_{|D}) &= \sup\bigl\{\|R^nx\|_k \;:\; x \in D,\ \|x\|_m \leqslant 1\bigr\} \\ &\leqslant \sup\bigl\{M_k\|R^{n-k}x\| \;:\; \|x\| \leqslant 1\bigr\} = M_k\|R^{n-k}\|.\end{aligned}$$

Therefore $\lim_n \sqrt[n]{\mathfrak{m}_{mk}(R^n_{|D})} \leqslant \lim_n \sqrt[n]{\|R^n\|} = r(T)$ for any $m,k \geqslant 0$. Now Proposition 4.6 yields $r_{nb}(R_{|D}) \leqslant r(R)$.

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