MEASURES OF NON-COMPACTNESS OF OPERATORS ON BANACH LATTICES

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Abstract. [2, 11] used representation spaces to study measures of non-compactness and spectral radii of operators on Banach lattices. In this paper, we develop representation spaces based on the nonstandard hull construction (which is equivalent to the ultrapower construction). As a particular application, we present a simple proof and some extensions of the main result of [6] on the monotonicity of the measure of non-compactness and the spectral radius of AM-compact operators. We also use the representation spaces to characterize d-convergence and discuss the relationship between d-convergence and the measures of non-compactness.

1. Introduction

Recall that an operator T between Banach lattices is said to be ***positive*** if it maps positive vectors to positive vectors. In this case, we write $T \geqslant 0$. We write $S \leqslant T$ if $T - S \geqslant 0$. We say that S is ***dominated*** by T if $|Sx| \leqslant T|x|$ for each x. An operator T between Banach lattices is said to be ***order bounded*** if it maps order intervals into order intervals. It can be easily verified that if T dominates S then both S and T are order bounded. An order bounded operator is ***AM-compact*** if it maps order intervals (or almost order bounded sets) into precompact sets. A set $A \subseteq E$ is ***almost order bounded*** if for every $\varepsilon > 0$ there exists $u \in E_+$ such that $A \subseteq [-u, u] + \varepsilon B_E$, where B_E stands for the unit ball of E. An operator T between Banach lattices is said to be ***semicompact*** if it maps bounded sets to almost order bounded sets. We refer the reader to [1, 8, 9, 17] for a detailed study of Banach lattices and positive operators. All Banach lattices in this paper are assumed to be complex unless specified otherwise, all operators are assumed to be linear and bounded.

A lot of work has been done on the problem of the relationship between compact operators and the order structure of a Banach lattice, see, e.g. [2, 12, 14, 15]. Still there are many open questions. In particular, the problem can be considered from the point of view of spectra of the operators. It is well known that if T is a positive operator on a Banach lattice then the spectral radius $r(T)$ belongs to the spectrum $\sigma(T)$. If S is dominated by T then $\|S\| \leqslant \|T\|$ and $r(S) \leqslant r(T)$. The central question of this paper is whether similar statements hold for the essential spectrum, essential spectral radius, and the measure of non-compactness.

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Recall that the Calkin algebra of a Banach space X is the quotient of the algebra of $\mathcal{L}(X)$ over the closed algebraic ideal of all compact operators. The ***essential spectrum*** $\sigma_{\mathrm{ess}}(T)$ and the ***essential spectral radius*** $r_{\mathrm{ess}}(T)$ of an operator T on X are defined as the spectrum and, respectively, the spectral radius of the canonical image of T in the Calkin algebra.

If A is a bounded subset of a Banach space X then the ***measure of noncompactness*** $\chi(A)$ (it is sometimes referred as the Hausdorff or ball measure of non-compactness) is defined via:

$$\chi(A) = \inf\{\, \delta > 0 \;:\; A \text{ can be covered with a finite number of balls of radius } \delta \,\}.$$

Clearly $\chi(A) = 0$ if and only if A is relatively compact. The measure of noncompactness of an operator $T\colon X \to Y$ between Banach spaces is defined via $\chi(T) = \chi(TB_X)$. Then χ is a seminorm on $\mathcal{L}(X,Y)$. It was shown in [10] that:

$$(*) \qquad r_{\mathrm{ess}}(T) = \lim_{n\to\infty} \sqrt[n]{\chi(T^n)}.$$

for every $T \in \mathcal{L}(X)$. We refer the reader to [3] for more details on measures of noncompactness.

Questions.

(i) Does $r_{\mathrm{ess}}(S) \in \sigma_{\mathrm{ess}}(S)$ for any positive operator S on a Banach lattice?
(ii) Is $r_{\mathrm{ess}}(S) \leqslant r_{\mathrm{ess}}(T)$ for any operators S and T provided T dominates S?
(iii) Is $\chi(S) \leqslant \chi(T)$ for any operators T and S provided T dominates S?

These questions were first addressed in [6], and the following results were obtained (see also [9, Section 4.3]):

Theorem 1 ([6])**.**

(i) *If S is a positive AM-compact operator on a Banach lattice then $r_{\mathrm{ess}}(S) \in \sigma_{\mathrm{ess}}(S)$.*
(ii) *If S and T are two operators on a Banach lattice such that $0 \leqslant S \leqslant T$ and S is AM-compact, then $r_{\mathrm{ess}}(S) \leqslant r_{\mathrm{ess}}(T)$.*
(iii) *If $S,T\colon E \to F$ are two operators between Banach lattices such that $0 \leqslant S \leqslant T$, S is AM-compact, and both E' and F have order continuous norms, then $\chi(S) \leqslant \chi(T)$.*

In the same paper, an example is given of two non-AM-compact operators $0 \leqslant S \leqslant T$ with $r_{\mathrm{ess}}(S) > r_{\mathrm{ess}}(T)$ and $r_{\mathrm{ess}}(S) \notin \sigma_{\mathrm{ess}}(S)$.

It is easy to see that if an operator $S\colon E \to E$ is dominated by a compact operator T then $r_{\mathrm{ess}}(S) = 0$. Indeed, $0 \leqslant S + T \leqslant 2T$, so that $(S+T)^3$ is compact by the Cube Theorem [1, Theorem 16.14], and it follows that $r_{\mathrm{ess}}(S) = 0$. But, as far as we know, it is still not known whether every operator dominated by an essentially quasinilpotent operator is itself essentially quasinilpotent.

An important technical tool used in [6] is a ***measure of non-semicompactness***, introduced analogously to the Hausdorff measure of noncompactness: if A is a norm bounded set in a Banach lattice, then

$$\rho(A) = \inf\{\, \delta > 0 \;:\; A \subseteq [-u,u] + \delta B_E \text{ for some } u \in E_+ \,\},$$

and $\rho(T) = \rho(TB_E)$ whenever T is an operator between Banach lattices. Clearly, $\rho(A) = 0$ iff A is almost order bounded, and $\rho(T) = 0$ iff T is semi-compact. Furthermore, $\rho(A) \leqslant \chi(A)$ for each bounded set A and $\rho(T) \leqslant \chi(T)$ for each bounded operator T. It was proved in [6, Theorem 2.5] that $\rho(T) = \chi(T)$ for every order bounded AM-compact operator $T\colon E \to F$ when E' and F have order continuous norms.

In this paper, we develop a certain representation space technique and, using this technique, we present a simple proof of Theorem 1 as well as some improvements of it. Our technique is based on the nonstandard hull construction of Nonstandard Analysis (which is equivalent to the ultrapower construction). Recall the construction briefly. If X is a (standard) Banach space, denote by ${}^*X$ the ***nonstandard extension*** of X. The symbols x, y, z, etc., will usually stand for elements of ${}^*X$. If T is a standard operator on X, we use the same symbol T instead of ${}^*T$ to denote the extension of T to ${}^*X$. The symbol $\mathrm{fin}({}^*X)$ stands for the subspace of all elements of ${}^*X$ of finite norm, while the ***monad of zero***, $\mu(0)$, consists of those elements of ${}^*X$ whose norm is infinitesimal. The Banach space $\widehat{X} = \mathrm{fin}({}^*X)/\mu(0)$ is called the ***nonstandard hull*** of X. If $x \in \mathrm{fin}({}^*X)$ then $\hat{x}$ will stand for the corresponding element in $\widehat{X}$. Every bounded operator between Banach spaces $T\colon X \to Y$ induces a bounded operators $\widehat{T}\colon \widehat{X} \to \widehat{Y}$ via $\widehat{T}\hat{x} = \widehat{Tx}$. Clearly, X is isometrically isomorphic to $\mathrm{ns}({}^*X)/\mu(0)$, where $\mathrm{ns}({}^*X)$ stands for the set of all near-standard elements of ${}^*X$. Thus, one can view X as a closed subspace of $\widehat{X}$. For $A \subseteq X$ we write $\widehat{A} = \{\hat{x} \;:\; x \in {}^*A\}$. It is known that a standard set $A \subseteq X$ is relatively compact iff ${}^*A \subseteq \mathrm{ns}({}^*X)$ iff $\widehat{A} \subseteq X$. If E is a Banach lattice then $\widehat{E}$ also is a Banach lattice. Further details on nonstandard analysis and nonstandard hulls can be found in [5, 7, 16]. A reader familiar with the technique of ultraproducts can view $\widehat{X}$ as an ultrapower of X. Clearly, all the proofs in this paper can be redone in terms of ultrapowers, but we believe that the language of Nonstandard Analysis is more appropriate for this problem.

Define $\widetilde{X} = \widehat{X}/X$. We will see in Section 2 that this space is a representation space for χ and σ_{ess}, that is, $\chi(T) = \|\widetilde{T}\|$ and $\sigma_{\mathrm{ess}}(T) = \sigma(\widetilde{T})$. For a Banach lattice E, let $I(E)$ be the order ideal generated by E in $\widehat{E}$, and let $i_E = \overline{I(E)}$. Now we define $\check{E} = \widehat{E}/i_E$. It will be shown in Section 2 that $\check{E}$ is a representation space for $\rho(T)$, that is, $\rho(T) = \|\check{T}\|$. We will also show that $T\colon E \to F$ is AM-compact if and only if $\widehat{T}$ maps i_E into F. In Section 3 we use these representation spaces to prove Theorem 1 and similar results. It should be mentioned that various representation spaces have been used to study the essential spectrum of an operator, see e.g., [2, 11].

Finally, in Section 4 we discuss the d-topology on a Banach lattice. We say that a net (x_α) ***d-converges*** to x in E if $|x_\alpha - x| \wedge y$ converges to zero in norm for every $y \in E_+$. We investigate the relation between the d-topology, the space $\widehat{E}$, and the questions stated in the beginning of the paper. We also discuss examples of d-topologies.

2. Representation Spaces

Given a Banach space X, we define $\widetilde{X} = \widehat{X}/X$. If $x \in {}^*X$, then $\tilde{x}$ will stand for the corresponding element in $\widetilde{X}$. Every operator $T\colon X \to Y$ induces an operators

$\widetilde{T}\colon \widetilde{X} \to \widetilde{Y}$ given by $\widetilde{T}\tilde{x} = \widetilde{Tx}$. An operator $T\colon X \to Y$ is compact iff the range of $\widehat{T}$ is contained in Y iff $\widetilde{T} = 0$.

Lemma 2. *If A is a bounded subset of X then* $\chi(A) = \max\limits_{y\in {}^*A}\|\tilde{y}\|$.

Proof. Fix $y \in {}^*A$. For every standard $\varepsilon > 0$ there is a finite set $F \subset X$ such that A (and hence ${}^*A$) is within $\chi(A) + \varepsilon$ of F. Then $\|\tilde{y}\| \leqslant \operatorname{dist}(\hat{y}, F) \leqslant \chi(A) + \varepsilon$, so that $\|\tilde{y}\| \leqslant \chi(A)$. Conversely, for every finite family of balls of radius less than $\chi(A)$ there exists a point in A which is not covered by the balls. By the idealization (saturation) principle, there exists $y \in {}^*A$ which does not belong to any standard ball of radius less than $\chi(A)$. Therefore $\|\tilde{y}\| \geqslant \chi(A)$. $\square$

From this point of view, $\chi(A)$ measures how far the set ${}^*A$ is from $\operatorname{ns}({}^*X)$. Then $\chi(T)$ measures how much closer sets become to $\operatorname{ns}({}^*X)$ after we apply T. Actually, the following lemma describes the relation. Similar results were proved in [16, 11].

Corollary 3. *If T is a bounded operator on X then* $\chi(T) = \|\widetilde{T}\|$.

Corollary 3 and formula (*) imply that $r_{\mathrm{ess}}(T) = r(\widetilde{T})$ for every bounded operator on a Banach space. Moreover, it was shown in [4] and in [16, Theorem 3.11] that $\widetilde{T}$ is invertible iff T is Fredholm, so that $\sigma(\widetilde{T}) = \sigma_{\mathrm{ess}}(T)$.

Now we turn to Banach lattices. It is well known that if E is a Banach lattice then $\widehat{E}$ is also a Banach lattice.

Remark 4. The following simple observation turns out to be quite handy in the context of our problem. Suppose that I is an (order) ideal in a vector lattice E and consider the quotient vector lattice E/I. It is known that the canonical epimorphism from E onto E/I is a lattice homomorphism, hence it maps order intervals onto order intervals. It follows that if $a, b, x \in E$ are such that $a \leqslant b$ and $[a] \leqslant [x] \leqslant [b]$, where $[a]$, $[x]$, and $[b]$ are the equivalence classes of a, x, and b respectively in the quotient vector lattice E/I, then $a \leqslant x' \leqslant b$ for some $x' \in [x]$. In particular, if E is a Banach lattice and $\hat{a} \leqslant \hat{x} \leqslant \hat{b}$ for some $a, x, b \in {}^*E$ such that $a \leqslant b$, then $a \leqslant x' \leqslant b$ for some $x' \in {}^*E$ such that $x' \approx x$.

The following important characterization was obtained in [5].

Theorem 5. *The following statements are equivalent:*

(i) $\widehat{E}$ *is Dedekind complete;*
(ii) $\widehat{E}$ *has the projection property;*
(iii) $\widehat{E}$ *has order continuous norm;*
(iv) c_0 *is not lattice finitely representable*[1] *in* E.

Notice that $\widetilde{E}$ need not be a Banach lattice because E might not be an (order) ideal in $\widehat{E}$. In fact, this happens only when E is atomic with order continuous norm. It

[1]Recall that a Banach lattice F is lattice finitely representable in a Banach lattice E if for each finite dimensional vector sublattice H in F and for each $\delta > 0$ there exists a lattice embedding $T\colon H \to E$ such that $\|T\|, \|T^{-1}\| \leqslant 1 + \delta$.

was first noticed in [5] that E is an ideal in $\widehat{E}$ if and only if the order intervals in E are compact. Indeed, $[-u,u]$ is a compact set in E for each $u \in E_+$ if and only if $[-u,u]_{^*E} \subset \mathrm{ns}(^*E)$. In view of Remark 4 this is equivalent to $[-\hat{u},\hat{u}]_{\widehat{E}} \subset E$ for each $u \in E_+$. It was shown in [14, 13] that the intervals in a Banach lattice E are compact if and only if E is atomic with order continuous norm. Thus, the following result holds.

Proposition 6. *The space $\widetilde{E}$ is a Banach lattice if and only if E is atomic with order continuous norm.*

Denote by $I(E)$ the ideal generated by E in $\widehat{E}$. This ideal was extensively studied in [5].

Theorem 7 ([5])**.** *The following statements are equivalent:*

(i) *$I(E)$ is Dedekind complete;*
(ii) *$I(E)$ has the projection property;*
(iii) *$I(E)$ has order continuous norm;*
(iv) *E has order continuous norm;*

It also follows from Proposition 6 that $I(E) = E$ if and only if E is atomic with order continuous norm.

Example 8. *$I(E)$ need not be norm closed.* Let $E = L_1[0,1]$, fix an infinite positive integer N, and consider the partition of $^*[0,1]$ into $2N$ equal subintervals. Let A be the union of all odd-numbered intervals, i.e., $A = \bigcup_{i=1}^{N}[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N}]$, and $f = \chi_A$, the characteristic function of A. We claim that zero is the greatest standard function in $\widehat{E}$ dominated by $\hat{f}$. Indeed, suppose that $\hat{g} \leqslant \hat{f}$ for some $g \in L_1[0,1]$ such that g is positive on a set of positive measure. Then there exists $\varepsilon > 0$ such that $m(C) > 0$, where $C = \{g > \varepsilon\}$ and m stands for the Lebesgue measure. Clearly $(\varepsilon\chi_C - f)^+ \leqslant (g - f)^+ \approx 0$. Let $\delta = \frac{\varepsilon \cdot m(C)}{4+\varepsilon}$. Then one can find a set $D \subseteq [0,1]$ such that D is a finite union of intervals and $m(C \bigtriangleup D) < \delta$. It follows that $\|(\varepsilon\chi_D - f)^+\| \leqslant \|(\varepsilon\chi_C - f)^+\| + \delta \lesssim \delta$. On the other hand, since $m(A \cap I) = \frac{1}{2}m(I)$ for every standard interval $I \subseteq [0,1]$, we have $\|(\varepsilon\chi_D - f)^+\| = \frac{\varepsilon}{2}m(D) \geqslant \frac{\varepsilon}{2}\big(m(C) - \delta\big) = 2\delta$, a contradiction. It can be shown in a similar fashion that $\chi_{[0,1]}$ is the smallest standard function that dominates f.

Now let $E = L_1(\mathbb{R})$. Again, fix an infinite positive integer N, and let A_1 be the set A from the previous paragraph. Cut the interval $^*[1,2]$ into $4N$ equal subintervals and let A_2 be the union of every fourth interval, i.e., $A_2 = 1 + \bigcup_{i=1}^{N}[\frac{i}{N} - \frac{1}{4N}, \frac{i}{N}]$. Similarly, for every $n \in {^*\mathbb{N}}$ let $A_n = n - 1 + \bigcup_{i=1}^{N}[\frac{i}{N} - \frac{1}{2^n N}, \frac{i}{N}]$. Next, let $B_n = \bigcup_{k=1}^{n} A_k$ for each $n \in {^*\mathbb{N}}$ and $B = \bigcup_{k \in {^*\mathbb{N}}} A_k$. Then $m(B) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots < \infty$. For each $n \in {^*\mathbb{N}}$ let h_n be the characteristic function of B_n, and let h be the characteristic function of B. Notice that $\|h - h_n\| = m(B \setminus B_n) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots$, so that $\|\hat{h} - \hat{h}_n\| \to 0$ as $n \to 0$ in $\mathbb{N}$. On the other hand, $h_n \leqslant \chi_{[0,n]}$ for every n, so that $\hat{h}_n \in I(E)$ for every standard n. But it follows from the previous paragraph, that h is not dominated by a standard function in $L_1(\mathbb{R})$, because this function would have to be greater or equal than 1 a.e. on $\mathbb{R}$. Therefore, $\hat{h}$ is in the closure of $I(E)$ but not in $I(E)$, so that $I(E)$ is not closed.

Denote by i_E the closure of $I(E)$ in $\widehat{E}$.

Remark 9. Since every closed ideal in a Banach lattice with order continuous norm is a band, it follows from Theorem 5 that if c_0 is not lattice finitely representable in E then i_E is a projection band.

Define a representation space $\check{E}$ for E via $\check{E} = \widehat{E}/i_E$. Apparently $\check{E}$ is a Banach lattice. It follows from Lemma 6 that $\check{E} = \widetilde{E}$ if and only if E is atomic with order continuous norm. On the other hand, if E has a strong order unit, then $I(E) = \widehat{E}$ and $\check{E}$ is trivial. It follows from $E \subseteq i_E \subseteq \widehat{E}$ that $\|\check{x}\| \leqslant \|\tilde{x}\| \leqslant \|\hat{x}\|$ for each $x \in \mathrm{fin}({}^*E)$.

Let $T\colon E \to F$ be an order bounded operator between Banach lattices. Suppose that $\hat{x} \in I(E)$. Then $|\hat{x}| \leqslant \hat{u}$ for some $u \in E_+$. By Remark 4 there exists $y \in {}^*E$ such that $x \approx y \in {}^*[-u,u]$. Then $Tx \approx Ty \in T^*[-u,u] \subseteq {}^*[-v,v]$ for some $v \in F_+$, so that $\widehat{T}\hat{x} \in I(F)$. Thus, $\widehat{T}$ maps i_E into i_F and, therefore, $\check{x} \mapsto (Tx)\check{}$ induces a bounded operator $\check{T}$ from $\check{E}$ to $\check{F}$.

We claim that $\check{T}$ is order bounded. Indeed, suppose that $\check{u} > 0$ and $\check{x} \in [-\check{u},\check{u}]$ for some $x, u \in \mathrm{fin}({}^*E)$. By Remark 4 we can assume that $u \in {}^*E_+$ and $x \in [-u,u]$. Then $Tx \in T[-u,u] \subseteq [-v,v]$ for some $v \in {}^*E_+$. Therefore, $\check{T}[-\check{u},\check{u}] \subseteq [-\check{v},\check{v}]$, hence $\check{T}$ is order bounded. Notice that v can be chosen in $\mathrm{fin}({}^*E)$ because of the following fact, which is due to A. Wickstead: *if $T\colon E \to F$ is order bounded then*

$$\sup_{\|z\|\leqslant 1, z\geqslant 0} \inf\bigl\{\|y\| \,:\, y \in E_+ \text{ and } T[-z,z] \subseteq [-y,y]\bigr\} \quad < \quad \infty.$$

Indeed, otherwise for each $n > 0$ one could find a positive z_n in B_E with

$$\inf\bigl\{\|y\| \,:\, y \in E_+ \text{ and } T[-z_n,z_n] \subseteq [-y,y]\bigr\} > n^3.$$

Let $z = \sum_{n=1}^{\infty} \frac{z_n}{n^2}$, it is easy to see that $T[-z,z]$ is not contained in any order interval in F.

Clearly, if T is positive then $\check{T}$ is positive and $\check{T} = 0$ if and only if T is semi-compact.

The following two results are analogous to Lemma 2 and Corollary 3.

Lemma 10. *If A is a bounded subset of E, then $\rho(A) = \max\limits_{y\in {}^*A} \|\check{y}\|$.*

Proof. Fix a standard $\gamma > \rho(A)$. Then $A \subseteq [-u,u] + \gamma B_E$ for some $u \in E_+$. For each $y \in {}^*A$ we have $y = v + h$ such that $v \in {}^*[-u,u]$ and $\|h\| \leqslant \gamma$. It follows from $\hat{v} \in i_E$ that $\|\check{y}\| = \|\hat{h}\| \leqslant \gamma$. Thus $\|\check{y}\| \leqslant \rho(A)$.

On the other hand, for every standard positive $\gamma < \rho(A)$ and for every $u \in E_+$ there is a point y in A which doesn't belong to $[-u,u] + \gamma B_E$. By saturation there exists $y \in {}^*A$ such that $y \notin {}^*[-u,u] + \gamma {}^*B_E$ for every standard positive $\gamma < \rho(A)$ and for every $u \in E_+$. Then $\|\check{y}\| \geqslant \rho(A)$. □

It follows, in particular (c.f. [12, Corollary 1.4]), that a bounded subset $A \subset E$ is almost order bounded if and only if $\widehat{A} \subseteq i_E$.

Corollary 11. *If $T\colon E \to F$ is operator between Banach lattices then $\rho(T) = \|\check{T}\|$.*

3. Applications

The following theorem follows immediately from Proposition 6 and Corollary 3.

Theorem 12. *If E and F are atomic Banach lattices with order continuous norms then:*

(i) *if T is a positive operator on E then $r_{\text{ess}}(T) \in \sigma_{\text{ess}}(T)$;*
(ii) *if $S, T \colon E \to E$ and T dominates S then $r_{\text{ess}}(S) \leqslant r_{\text{ess}}(T)$;*
(iii) *if $S, T \colon E \to F$ and T dominates S then $\chi(S) \leqslant \chi(T)$.*

Since every operator on an atomic Banach lattice is AM-compact, this result can be viewed as a special case of Theorem 1 except that we do not require E' to have order continuous norm in (iii).

Next, we are going to characterize AM-compact operators. Denote by φ_E the canonical epimorphism from $\widetilde{E}$ to $\check{E}$ given by $\varphi_E(\tilde{x}) = \check{x}$. By the definition of $\check{T}$ we have $\check{T}\varphi_E = \varphi_F \widetilde{T}$.

Theorem 13. *Let $T \colon E \to F$ be an order bounded operator between Banach lattices. The following statements are equivalent:*

(i) *T is AM-compact;*
(ii) *$\widehat{T}$ maps i_E into F;*
(iii) *There exists a map $\bar{T} \colon \check{E} \to \widetilde{F}$ such that $\widetilde{T} = \bar{T}\varphi_E$, i.e., $\widetilde{T}\tilde{x} = \bar{T}\check{x}$:*

$$\begin{tikzcd}
\widetilde{E} \arrow[r, "\widetilde{T}"] \arrow[d, "\varphi_E"'] & \widetilde{F} \\
\check{E} \arrow[ur, "\bar{T}"'] &
\end{tikzcd}$$

Proof. If $u \in E_+$ then $T[-u, u]$ is compact if and only if $\widehat{T[-u, u]} \subseteq F$. In view of Remark 4 we have $\widehat{T[-u, u]} = \widehat{T}[-\hat{u}, \hat{u}]$, so that (i)$\Leftrightarrow$(ii). To show (ii)$\Leftrightarrow$(iii) notice that $\ker \varphi_E = i_E/E$. If $\widehat{T}$ maps i_E into F then $i_E/E \subseteq \ker \widetilde{T}$, so that $\bar{T}\big(\varphi_E(\tilde{x})\big) = \widetilde{T}\tilde{x}$ defines an operator from $\check{E}$ to $\widetilde{F}$. Conversely, if such a $\bar{T}$ exists, then for every $\hat{x} \in i_E$ we have $\varphi_E(\tilde{x}) = 0$ so that $\widetilde{T}\tilde{x} = \bar{T}\varphi_E(\tilde{x}) = 0$, hence $\widehat{T}\hat{x} \in F$. $\square$

Remark 14. Notice that if T is AM-compact then $\|\widetilde{T}\| = \|\bar{T}\|$ because φ_E maps the unit ball of $\widetilde{E}$ onto the unit ball of $\check{E}$.

If E, F, and G are Banach lattices and $S \colon E \to F$ and $T \colon F \to G$ are order bounded operators such that T is AM-compact, then it follows from the diagram

$$\begin{tikzcd}
\widetilde{E} \arrow[r, "\widetilde{S}"] \arrow[d, "\varphi_E"'] & \widetilde{F} \arrow[r, "\widetilde{T}"] \arrow[d, "\varphi_F"'] & \widetilde{G} \\
\check{E} \arrow[r, "\check{S}"] & \check{F} \arrow[ur, "\bar{T}"] &
\end{tikzcd}$$

that $\|\widetilde{T}\widetilde{S}\| \leqslant \|\varphi_E\|\|\check{S}\|\|\bar{T}\| = \|\check{S}\|\|\widetilde{T}\|$ (c.f. [6, Lemma 3.1].

Now we are ready to present a simple proof of Theorem 1. We replace the condition $0 \leqslant S \leqslant T$ with the slightly weaker condition of S being dominated by T.

Proof of Theorem 1. If T is positive and AM-compact, then by Remark 14 we have $\|\widetilde{T}^n\| \leqslant \|\check{T}^{n-1}\|\|\widetilde{T}\|$, so that

$$r(\widetilde{T}) = \lim_{n\to\infty} \sqrt[n]{\|\widetilde{T}^n\|} \leqslant \lim_{n\to\infty} \sqrt[n]{\|\check{T}^{n-1}\|\|\widetilde{T}\|} = \lim_{n\to\infty} \sqrt[n]{\|\check{T}^{n-1}\|} = r(\check{T}).$$

On the other hand, we always have $r(\check{T}) \leqslant r(\widetilde{T})$, so that $r_{\mathrm{ess}}(T) = r(\widetilde{T}) = r(\check{T})$. Since $\check{T}$ is a positive operator on the Banach lattice $\check{E}$ we have $r(\check{T}) \in \sigma(\check{T})$. Finally, $\check{T}$ is a quotient of $\widetilde{T}$ so that $\sigma(\check{T}) \subseteq \sigma(\widetilde{T}) = \sigma_{\mathrm{ess}}(T)$. Thus, $r_{\mathrm{ess}}(T) \in \sigma_{\mathrm{ess}}(T)$.

Next, if T dominates S and S is AM-compact then by Remark 14 we have $\|\widetilde{S}^n\| \leqslant \|\check{S}^{n-1}\|\|\widetilde{S}\| \leqslant \|\check{T}^{n-1}\|\|\widetilde{S}\|$. This yields

$$r_{\mathrm{ess}}(S) = r(\widetilde{S}) = \lim_{n\to\infty} \sqrt[n]{\|\widetilde{S}^n\|} \leqslant \lim_{n\to\infty} \sqrt[n]{\|\check{T}^{n-1}\|} = r(\check{T}) \leqslant r(\widetilde{T}) = r_{\mathrm{ess}}(T).$$

If in addition E' and F have order continuous norms, then $\chi(S) = \rho(S) = \|\check{S}\|$ and $\|\check{T}\| = \rho(T) \leqslant \chi(T)$, but since $\check{E}$ is a Banach lattice and $\check{T}$ dominates $\check{S}$ we conclude that $\|\check{S}\| \leqslant \|\check{T}\|$ so that $\chi(S) \leqslant \chi(T)$. □

The following theorem is an analog of [2, Theorem 1.5] for $\widetilde{T}$ and $\check{T}$.

Theorem 15. *Let E be a Banach lattice and $T\colon E \to E$ an AM-compact operator. Then*

(i) $\sigma_p(\check{T}) \setminus \{0\} = \sigma_p(\widetilde{T}) \setminus \{0\}$;
(ii) *If $\lambda \neq 0$ then $\lambda I - \check{T}$ is onto if and only if $\lambda I - \widetilde{T}$ is onto;*
(iii) $\sigma(\check{T}) \setminus \{0\} = \sigma(\widetilde{T}) \setminus \{0\} = \sigma_{\mathrm{ess}}(T) \setminus \{0\}$.

Proof. (i) Suppose that $\lambda \neq 0$. If $\check{T}\check{x} = \lambda\check{x}$, $\check{x} \neq 0$ then $(\lambda I - \widehat{T})\hat{x} \in i_E$ so that $\widehat{T}(\lambda I - \widehat{T})\hat{x} \in E$ hence $(\lambda I - \widetilde{T})(\widetilde{T}\tilde{x}) = 0$. Notice that $\widetilde{T}\tilde{x} \neq 0$ because $\check{T}\check{x} = \lambda\check{x} \neq 0$. This yields $\lambda \in \sigma_p(\widetilde{T})$. Conversely, if $\widetilde{T}\tilde{x} = \lambda\tilde{x}$, $\tilde{x} \neq 0$ then $\check{T}\check{x} = \lambda\check{x}$. Notice that $\check{x} \neq 0$ because otherwise we would have $\hat{x} \in i_E$ which would imply $\widehat{T}\hat{x} \in E$ and $\widetilde{T}\tilde{x} = 0$.

(ii) If $\lambda I - \widetilde{T}$ is onto, then $\lambda I - \check{T}$ is also onto as a quotient of $\lambda I - \widetilde{T}$. To show the converse, take $\hat{y} \in \widehat{E}$, then there exists $\hat{x} \in \widehat{E}$, such that $(\lambda I - \check{T})\check{x} = \check{y}$. Then $(\lambda I - \widehat{T})\hat{x} - \hat{y} \in i_E$ so that $\widehat{T}\big((\lambda I - \widehat{T})\hat{x} - \hat{y}\big) \in E$. This yields $(\lambda I - \widetilde{T})(\widetilde{T}\tilde{x} + \tilde{y}) - \lambda\tilde{y} = 0$. Hence $(\lambda I - \widetilde{T})(\frac{1}{\lambda})(\widetilde{T}\tilde{x} + \tilde{y}) = \tilde{y}$ and, therefore, $\lambda I - \widetilde{T}$ is onto.

Finally, (iii) follows immediately from (i) and (ii). □

4. d-topologies

Definition 16. We say that a net (x_α) in a Banach lattice E ***d-converges*** to $x \in E$ if $|x_\alpha - x| \wedge y$ converges to zero in norm for every $y \in E_+$. The topology generated by this convergence will be referred to as the ***d-topology*** of E.

Let $\mu_d(0)$ denotes the monad of zero for the d-topology, while E^d stands for the disjoint complement of E in $\widehat{E}$.

Lemma 17. *For a point $x \in {}^*E$ the following are equivalent:*

(i) $x \in \mu_d(0)$;
(ii) $\hat{x} \in E^d$;

(iii) x *is* ***nearly disjoint*** *with every* $y \in E_+$*, i.e.,* $|x| \wedge y \approx 0$.

Proof. To see that (ii)⇔(iii) observe that $\hat{x} \in E^d$ if and only if for every $y \in E_+$ we have $|\hat{x}| \wedge \hat{y} = 0$ or, equivalently, $|x| \wedge y \approx 0$. On the other hand, (x_α) d-converges to zero in E if and only if $|x_\alpha| \wedge y$ converges to zero in norm for every $y \in E_+$, which is equivalent to $|x_\alpha| \wedge y \approx 0$ for every infinite α, so that (i)⇔(iii). □

Notice that the embedding of E^d into $\check{E}$ given by $\hat{x} \mapsto \check{x}$ is an isometry (we will see in Example 21 that it need not be onto). Indeed, if $\hat{x} \in E^d$ then clearly $\|\check{x}\| \leqslant \|\hat{x}\|$. On the other hand, $\hat{x} \perp \hat{y}$ for each $\hat{y} \in i_E$, so that $\|\hat{x} + \hat{y}\| \geqslant \|\hat{x}\|$, whence $\|\check{x}\| = \|\tilde{x}\|$. It is easy to see that $\|\check{x}\| = \|\tilde{x}\|$ for every $\hat{x} \in E \oplus E^d$. It follows then from Lemmas 2 and 10 that if A is a bounded set in A then $\chi(A) = \rho(A)$ whenever $\widehat{A} \subseteq E \oplus E^d$. By Lemma 17 $\hat{x} \in E \oplus E^d$ means that x is nearstandard relative to the d-topology, so that $\widehat{A} \subseteq E \oplus E^d$ *if and only if* A *is relatively d-compact* (i.e., relatively compact with respect to the d-topology). Thus, we arrive to the following result.

Proposition 18. *If* $A \subseteq E$ *is d-compact, then* $\chi(A) = \rho(A)$.

We say that an operator between Banach lattices is ***d-compact*** if it maps bounded sets into relatively d-compact sets. Observe that if T is order bounded and d-compact then it is AM-compact. Indeed, if $y \in E_+$ then $T[-y, y]$ is relatively d-compact and order bounded, but the d-topology agrees with the norm topology on order bounded sets, so that $T[-y, y]$ is relatively compact. It follows, in particular, that Theorem 1 applies to d-compact operators. We claim that in the case of d-compact operators the order continuity condition in Theorem 1(iii) can be removed. Indeed, the following result follows immediately from Proposition 18.

Proposition 19. *If* $S, T \colon E \to F$ *are two operators between Banach lattices such that* T *dominates* S *and* S *is d-compact then* $\chi(S) \leqslant \chi(T)$.

Next, we are going to present several examples of d-topologies. First, notice that if E has a strong order unit then the d-topology coincides with the norm topology of E.

Example 20. $E = C_0(\Omega)$ *where* Ω *is a normal topological space.* The d-convergence in E is exactly the ***ucc*** topology, i.e., the topology of uniform convergence on compacta. Indeed, let (x_α) be a net in E such that $x_\alpha \xrightarrow{\text{ucc}} 0$, and $y \in E_+$. Fix $\varepsilon > 0$. Then one can find a compact set $K \subseteq \Omega$ such that $y(t) \leqslant \varepsilon$ whenever $t \notin K$. There exists an index α_0 such that $|x_\alpha(t)| \leqslant \varepsilon$ whenever $t \in K$ and $\alpha \geqslant \alpha_0$. Then $|x_\alpha(t)| \wedge y(t) \leqslant \varepsilon$ for every $t \in \Omega$, so that (x_α) is d-null.

Conversely, suppose that (x_α) is d-null in E and K is a compact subset of Ω. Let $y \in E$ such that $0 \leqslant y \leqslant \mathbf{1}$ and $y(t) = 1$ whenever $t \in K$. Since $|x_\alpha| \wedge y \to 0$, it follows that for every $\varepsilon > 0$ there exists an index α_0 such that $|x_\alpha(t)| \wedge y(t) \leqslant \varepsilon$ whenever $\alpha \geqslant \alpha_0$. This implies $|x_\alpha(t)| \leqslant \varepsilon$ whenever $t \in K$ and $\alpha \geqslant \alpha_0$.

Notice that if $x \in {}^*E$ then $\hat{x} \in E^d$ if and only if $x(t) \approx 0$ whenever $t \in {}^*K$ for some compact $K \subseteq \Omega$, or, equivalently, if $x(t) \approx 0$ for every nearstandard $t \in {}^*\Omega$.

Example 21. $E = c_0$. We will show that $\widehat{E} \neq i_E \oplus E^d$. It follows from the previous example that the d-convergence on c_0 is exactly coordinate-wise convergence. Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $x = \sum_{k=1}^{N} e_k$. Assume that $\hat{x} = \hat{y} + \hat{z}$ for some $y, z \in {}^*E$ such that

$\hat{y} \in i_E$ and $\hat{z} \in E^d$. Then $\|\hat{y} - \hat{v}\| < \frac{1}{4}$ for some $\hat{v} \in I(E)$. Set $D = \{i \in {}^*\mathbb{N} : |z_i| < \frac{1}{2}\}$. Then $\mathbb{N} \subseteq D$ because $z_i \approx 0$ for each standard i. By the Overspill Principle there exists $n \in D$ such that $n \notin \mathbb{N}$ and $n \leqslant N$. Then $y_n \approx x_n - z_n \geqslant \frac{1}{2}$ and, therefore, $v_n \geqslant \frac{1}{4}$. But v_n must be infinitesimal because v is dominated by a standard sequence, a contradiction. Thus, $\hat{x} \notin i_E \oplus E^d$ so that $i_E \oplus E^d$ is a proper subset of $\widehat{E}$. It also follows that the embedding of E^d into $\check{E}$ given be $\hat{x} \mapsto \check{x}$ is not onto, because otherwise we would have $\check{x} = \check{z}$ for some $\hat{z} \in E^d$, and this would imply $\hat{x} = \hat{y} + \hat{z}$ for $\hat{y} = \hat{x} - \hat{z} \in i_E$.

Furthermore, if $A = \{e_k\}_{k \in \mathbb{N}}$, it can be easily verified that A is d-compact in c_0. On the other hand, if we consider its convex hull $\operatorname{co} A$ then clearly $x \in {}^*\operatorname{co} A$, so that $\operatorname{co} A$ is not relatively d-compact. Thus the convex hull of a d-compact set need not be d-compact, and the d-topology need not be locally convex.

Remark 22. It follows from Remark 9 that if E is a Banach lattice such that c_0 is not lattice finitely representable in E, then $\widehat{E} = i_E \oplus E^d$. In this case the map $\hat{x} \mapsto \check{x}$ is an isometry between E^d and $\check{E}$. If, in addition, E is atomic with order continuous norm (e.g., $E = \ell_p$, $1 \leqslant p < \infty$) then $i_E = E$, so that $\widehat{E} = E \oplus E^d$. It follows that every bounded set in E is relatively d-compact, and every E-valued bounded operator is d-compact.

Example 23. *$E = L_p(\mu)$ where $1 \leqslant p < \infty$ and μ is a finite measure.* The d-convergence in E is exactly the convergence in measure. To show this, let $x_\alpha \xrightarrow{\mu} 0$ in E. Clearly $|x_\alpha| \wedge \mathbf{1}$ converges to zero in L_p-norm. Similarly, $|x_\alpha| \wedge s \to 0$ for every simple function $s \in E$. Let $y \in E_+$. For each positive ε there exists a simple function $s \in E$ such that $\|s - y\| \leqslant \varepsilon$. Then:

$$\big\||x_\alpha| \wedge y\big\| \leqslant \big\||x_\alpha| \wedge s\big\| + \big\||x_\alpha| \wedge s - |x_\alpha| \wedge y\big\| \leqslant \big\||x_\alpha| \wedge s\big\| + \|s - y\| \leqslant \big\||x_\alpha| \wedge s\big\| + \varepsilon.$$

This yields $|x_\alpha| \wedge y \to 0$, so that x_α d-converges to zero.

Conversely, suppose (x_α) d-converges to zero in E. Fix $\varepsilon > 0$. Then $|x_\alpha| \wedge \varepsilon\mathbf{1} \to 0$, so that we can find α_0 such that $\big\||x_\alpha| \wedge \varepsilon\mathbf{1}\big\| \leqslant \varepsilon^{\frac{p+1}{p}}$ whenever $\alpha \geqslant \alpha_0$. This yields:

$$\mu\Big(\big\{|x_\alpha| > \varepsilon\big\}\Big) = \frac{1}{\varepsilon^p} \int\limits_{|x_\alpha| > \varepsilon} \varepsilon^p \, d\mu = \frac{1}{\varepsilon^p} \int\limits_{|x_\alpha| > \varepsilon} \big(|x_\alpha| \wedge \varepsilon\big)^p \, d\mu \leqslant \frac{1}{\varepsilon^p} \int \big(|x_\alpha| \wedge \varepsilon\big)^p \, d\mu \leqslant \varepsilon$$

whenever $\alpha \geqslant \alpha_0$.

For $x \in {}^*E$ the following lemma guarantees that $\hat{x} \in E^d$ if and only if $|x| \wedge \mathbf{1} \approx 0$.

Lemma 24. *If e is a quasi-interior point in a Banach lattice E, then for $x \in {}^*E$ we have $\hat{x} \in E^d$ if and only if $|x| \wedge e \approx 0$.*

Proof. Without loss of generality we assume $x \geqslant 0$. It follows from Lemma 17 that $\hat{x} \in E^d$ implies $x \wedge e \approx 0$. Conversely, assume that x is nearly disjoint with e and let $y \in E_+$. By [1, Theorem 15.13] we have $\|y - y \wedge ne\| \to 0$. Fix $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|y - y \wedge ne\| \leqslant \varepsilon$. Therefore:

$$\begin{aligned}\|x \wedge y\| &\leqslant \|x \wedge y - x \wedge y \wedge ne\| + \|x \wedge y \wedge ne\| \\ &\leqslant \|y - y \wedge ne\| + \|x \wedge ne\| \leqslant \varepsilon + n\|x \wedge e\| \leqslant 2\varepsilon.\end{aligned}$$

It follows that x is nearly disjoint with y, so that by Lemma 17 we conclude that $\hat{x} \in E^d$. □

The following interesting observation was communicated to the author by W.B. Johnson: *if $0 < q < p < \infty$ then a bounded sequence in $L_p(\mu)$ converges to zero in measure if and only if it converges to zero in $\|\cdot\|_q$.* Indeed, it is well known that convergence in $\|\cdot\|_q$ implies convergence in measure. On the other hand, suppose that (x_n) is a norm-bounded d-null sequence in $L_p(\mu)$ and $0 < q < p$, then:

$$\begin{aligned}\int_{|x_n|<1} |x_n|^q &\leqslant \Bigl(\int_{|x_n|<1} |x_n|^p\Bigr)^{\frac{q}{p}} \Bigl(\int_{|x_n|<1} \mathbf{1}\Bigr)^{\frac{p-q}{p}} \\ &\leqslant \Bigl(\int_{|x_n|<1} |x_n|^p \wedge \mathbf{1}\Bigr)^{\frac{q}{p}} \|\mu\|^{\frac{p-q}{p}} = \bigl\||x_n| \wedge \mathbf{1}\bigr\|_p^q \cdot \|\mu\|^{\frac{p-q}{p}} \to 0\end{aligned}$$

and

$$\int_{|x_n|\geqslant 1} |x_n|^q \leqslant \Bigl(\int_{|x_n|\geqslant 1} |x_n|^p\Bigr)^{\frac{q}{p}} \Bigl(\int_{|x_n|\geqslant 1} \mathbf{1}\Bigr)^{\frac{p-q}{p}} \leqslant \|x_n\|_p^q \cdot \mu\bigl(|x_n| \geqslant 1\bigr)^{\frac{p-q}{p}} \to 0,$$

so that $\|x_n\|_q \to 0$. It follows that a bounded subset A in $L_p(\mu)$ is (relatively) d-compact if and only if it is (relatively) norm compact in $L_q(\mu)$.

Example 25. *A positive d-compact operator which is not compact.* Consider the sequence of intervals $A_n = \bigl(\frac{1}{2^n}, \frac{1}{2^{n-1}}\bigr]$. Let $f_n = 2^{\frac{n}{2}} \chi_{A_n}$. Then $\|f_n\|_2 = 1$ for each n. It is easy to see that the operator $T \colon \ell_2 \to L_2[0,1]$ given by $Te_n = f_n$ is an isometric embedding, hence not compact. However, T is d-compact because it is compact as an operator from ℓ_2 to $L_1[0,1]$. Indeed, $T = \sum_{n=1}^{\infty} e'_n \otimes f_n$, but $\sum_{n=1}^{\infty} \|e'_n\|_{\ell'_2} \|f_n\|_1 = \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} < +\infty$, so that T considered as an operator from ℓ_2 to $L_1[0,1]$ is nuclear, and hence compact.

We have mentioned that every order bounded d-compact operator is AM-compact. The following example was pointed out to the author by W.B. Johnson.

Example 26. *A d-compact operator which is not AM-compact.* Define $T \colon L_2[0,1] \to \ell_2$ via $Tx = \bigl(\int x r_n\bigr)_{n=0}^{\infty}$, where r_n is the n-th Rademacher function. It follows from Remark 22 that T is d-compact. On the other hand, $Tr_n = e_n$, so that T is not AM-compact.

Recall that a set A in a Banach lattice E is said to be ***PL-compact*** if it is relatively compact with respect to the seminorm $f\bigl(|\cdot|\bigr)$ for every $f \in E'_+$. It was shown in [6, Proposition 2.1] that $\rho(A) = \chi(A)$ for every PL-compact set A in a Banach lattice with order continuous norm. Furthermore, if E' and F have order continuous norm then [17, Theorem 125.3] guarantees that an order bounded operator $T \colon E \to F$ is

AM-compact if and only if TB_E is PL-compact. In particular, if T is d-compact then TB_E is PL-compact.

Example 27. *A set which is d-compact but not PL-compact.* Consider the set $A = \{e_k\}_{k=1}^{\infty}$ in ℓ_1. It can be easily seen that A is d-compact. Nevertheless A is not PL-compact because if we take $f(x) = \sum_{i=1}^{\infty} x_i$ then $f(|\cdot|)$ coincides with the norm on E, while A is not relatively norm compact.

Finally we would like to mention that it is crucial to describe the sets that satisfy $\chi(A) = \rho(A)$, because if $\chi(TB_E) = \rho(TB_E)$ then $\chi(T) = \rho(T)$ and, therefore we can answer the questions stated in the beginning of the paper in the affirmative. Indeed, it follows from (*) and Corollary 11 that $r_{\text{ess}}(S) = r(\check{S}) \in \sigma(\check{S}) \subseteq \sigma(\widetilde{S}) = \sigma_{\text{ess}}(S)$ because $\check{E}$ is a Banach lattice. Furthermore, if T dominates S then $\chi(S) = \rho(S) = \|\check{S}\| \leqslant \|\check{T}\| = \rho(T) \leqslant \chi(T)$. Along with (*) this yields $r_{\text{ess}}(S) \leqslant r_{\text{ess}}(T)$.

We have $\rho(A) \leqslant \chi(A)$ for every bounded set A. It was already mentioned that $\rho(A) = \chi(A)$ for every PL-compact set A in a Banach lattice with order continuous norm and for every d-compact set in any Banach lattice. Recall that

$$\rho(A) = \max_{\hat{y} \in \widehat{A}} \|\check{y}\| \quad \text{and} \quad \chi(A) = \max_{\hat{y} \in \widehat{A}} \|\tilde{y}\|,$$

and denote by A_χ the set of all the points of $\widehat{A}$ where the latter maximum is attained. Clearly, if $\|\check{x}\| = \|\tilde{x}\|$ for some $\hat{x} \in A_\chi$ then $\chi(A) = \rho(A)$. In particular, A is d-compact then the entire $\widehat{A}$ is contained in $E \oplus E^d$ and $\|\check{x}\| = \|\tilde{x}\|$ for every $\hat{x} \in E \oplus E^d$. But clearly d-compactness is a way too strong condition. It would suffice for just A_χ and $E \oplus E^d$ to have nonempty intersection. Expressed in standard terms, this idea gives rise to the following proposition.

Proposition 28. *Suppose that A is a bounded set in a Banach lattice E. If there exists a d-convergent sequence $(x_n)_{n=1}^{\infty}$ in A such that $\chi(A) = \chi\bigl(\{x_n\}_{n=1}^{\infty}\bigr)$ then $\chi(A) = \rho(A)$.*

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ d-converges to some $x \in E$. Since $\chi(A) = \max_n \|\tilde{x}_n\|$ there exists $n_0 \in {}^*\mathbb{N}$ such that $\chi(A) = \|\tilde{x}_{n_0}\|$. If $n_0 \in \mathbb{N}$ then $x_{n_0} \in E$. If $n_0 \in {}^*\mathbb{N} \setminus \mathbb{N}$ then Lemma 17 implies $\hat{x}_{n_0} - \hat{x} \in E^d$. In either case $\hat{x}_{n_0} \in E \oplus E^d$ so that $\|\tilde{x}_{n_0}\| = \|\check{x}_{n_0}\|$, and, therefore, $\chi(A) \leqslant \rho(A)$. ☐

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