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# Multi-norms and Banach lattices

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## Abstract

In 2012, Dales and Polyakov introduced the concepts of multi-norms and dual multi-norms based on a Banach space. Particular examples are the lattice multi-norm $(\|\cdot\|_n^L)$ and the dual lattice multi-norm $(\|\cdot\|_n^{DL})$ based on a Banach lattice. Here we extend these notions to cover '$p$–multi-norms' for $1 \leqslant p \leqslant \infty$, where $\infty$–multi-norms and 1–multi-norms correspond to multi-norms and dual multi-norms, respectively. We shall prove two representation theorems. First we modify a theorem of Pisier to show that an arbitrary multi-normed space can be represented as $((Y^n, \|\cdot\|_n^L) : n \in \mathbb{N})$, where $Y$ is a closed subspace of a Banach lattice; we then give a version for certain $p$–multi-norms. Second, we obtain a dual version of this result, showing that an arbitrary dual multi-normed space can be represented as $(((X/Y)^n, \|\cdot\|_n^{DL}) : n \in \mathbb{N})$, where $Y$ is a closed subspace of a Banach lattice $X$; again we give a version for certain $p$–multi-norms.

We shall discuss several examples of $p$–multi-norms, including the weak $p$–summing norm and its dual and the canonical lattice $p$–multi-norm based on a Banach lattice. We shall determine the Banach spaces $E$ such that the $p$–sum power-norm based on $E$ is a $p$–multi-norm. This relies on a famous theorem of Kwapień; we shall present a simplified proof of this result. We shall relate $p$–multi-normed spaces to certain tensor products.

Our representation theorems depend on the notion of 'strong' $p$–multi-norms, and we shall define these and discuss when $p$–multi-norms and strong $p$–multi-norms pass to subspaces, quotients, and duals; we shall also consider whether these multi-norms are preserved when we interpolate between couples of $p$–multi-normed spaces. We shall discuss multi-bounded operators between $p$–multi-normed spaces, and identify the classes of these spaces in some cases, in particular for spaces of operators between Banach lattices taken with their canonical lattice $p$–multi-norms.



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# 1. Introduction

**1.1. Multi-norms and dual multi-norms.** A theory of multi-norms based on a normed space was introduced by Dales and Polyakov in [20]. The study of multi-norms and dual multi-norms was continued in [8, 18, 19], and there is a survey in [16]; a recent contribution is [7]. We recall the basic definitions of this theory.

We write $\mathbb{N}$ for the set of natural numbers; for $n \in \mathbb{N}$, the collection of permutations of the set $\mathbb{N}_n = \{1, \dots, n\}$ is denoted by $\mathfrak{S}_n$. The underlying field $\mathbb{F}$ of a linear space is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. As in the earlier papers, $E^n$ denotes the $n$-fold Cartesian power of a linear space $E$, taken with the coordinatewise linear operations.

The first definition that we give brings in a new term, 'power-norm'; the word 'special-norm' was used in [20, §2.2.1] and [52]. Thus a 'power-norm' is a sequence of norms defined on the powers of $E$.

Definition 1.1. Let $E$ be a linear space over $\mathbb{F}$. A *power-norm based on* $E$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on $E^n$ for each $n \in \mathbb{N}$ and such that the following Axioms (A1)–(A3) are satisfied for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$:

(A1) $\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\|_n = \|\boldsymbol{x}\|_n \quad (\sigma \in \mathfrak{S}_n)$;

(A2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leqslant (\max_{i=1,\dots,n} |\alpha_i|) \|\boldsymbol{x}\|_n \quad (\alpha_1, \dots, \alpha_n \in \mathbb{F})$;

(A3) $\|(x_1, \dots, x_n, 0)\|_{n+1} = \|\boldsymbol{x}\|_n$.

In this case, $(E^n, \|\cdot\|_n) = ((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *power-normed space.*

The power-norm is a *multi-norm* and $(E^n, \|\cdot\|_n)$ is a *multi-normed space* if, in addition to (A1)–(A3), we have

(A4) $\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_n)\|_n$

for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$.

The power-norm is a *dual multi-norm* and $(E^n, \|\cdot\|_n)$ is a *dual multi-normed space* if, in addition to (A1)–(A3), we have

(B4) $\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, 2x_n)\|_n$

for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$.

Let $(E^n, \|\cdot\|_n)$ be a power-normed space. Then, in particular, $(E, \|\cdot\|_1)$ is a normed space; we shall usually write $\|x\|$ for $\|(x)\|_1$ for $x \in E$, so giving the *base norm* on $E$. The power-norm is *based on* $E$. In the case where $(E, \|\cdot\|)$ is a Banach space, each space $(E^n, \|\cdot\|_n)$ is also a Banach space, and $(E^n, \|\cdot\|_n)$ is termed a *power-Banach space*, etc.

Many properties of multi-norms and of dual multi-norms were described in [20]; these properties included some strong connections with the theory of absolutely summing operators and with the theory of tensor norms.

For example, as in [20] and [18], there are a *maximum multi-norm* and *minimum multi-norm* based on a normed space $E$; these are denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, respectively, and they are defined by the property that

$$\|\boldsymbol{x}\|_n^{\min} \leqslant \|\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_n^{\max} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for every multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on $E$. The formula for $\|\cdot\|_n^{\min}$ is

$$\|\boldsymbol{x}\|_n^{\min} = \max_{i=1,\dots,n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N}).$$

By [20, Theorem 3.33], the dual of $\|\cdot\|_n^{\max}$ is $\mu_{1,n}$, the weak 1-summing norm, to be defined in §1.5, and so

$$\|\boldsymbol{x}\|_n^{\max} = \sup\left\{\left|\sum_{i=1}^{n}\langle x_i,\, \lambda_i\rangle\right| : \lambda_1, \dots, \lambda_n \in E',\ \mu_{1,n}(\lambda_1, \dots, \lambda_n) \leqslant 1\right\}$$

for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

There are also maximum and minimum dual multi-norms based on a normed space $E$; the maximum dual multi-norm is the sequence $(\|\cdot\|_n)$ defined by

$$\|\boldsymbol{x}\|_n = \sum_{i=1}^{n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N}).$$

See [20, p. 59].

In fact, in this work, we shall refer to '$\infty$–multi-norms' and '1–multi-norms' for 'multi-norms' and 'dual multi-norms', respectively, as special cases of '$p$–multi-norms'; see the definitions in §2.2.

**1.2. Description of the main results.** Our aim in this memoir is to generalize the notions of multi-norms and dual multi-norms to that of a $p$–multi-norm for $1 \leqslant p \leqslant \infty$; in the cases where $p = \infty$ and $p = 1$, we shall recover the classes of multi-norms and dual multi-norms, respectively. A $p$–multi-norm is a power-norm with an additional property; the precise definition will be given in §2.2.

Again $p$–multi-norms have a strong connection with certain cross-norms defined on tensor products. The study of $p$–multi-norms involves consideration of the normed space on which the $p$–multi-norm is based, and we shall obtain new results in this direction, especially involving '$p$–spaces'.

A key example of a $p$–multi-norm is that of the canonical lattice $p$–multi-norm defined on a real or complex Banach lattice: this $p$–multi-norm will be defined in Definition 4.22. There is a sense in which this $p$–multi-norm is generic. Indeed, our main representation theorem is Theorem 5.7, which roughly says the following. Take $p$ with $1 < p < \infty$. Then a

$p$–multi-norm based on a Banach space and satisfying extra conditions is the same as the canonical lattice $p$–multi-norm defined on a closed subspace of a certain Banach lattice. The analogous result for multi-norms themselves is Theorem 5.5: a multi-norm based on a Banach space is the same as the canonical Banach-lattice multi-norm defined on a closed subspace of a certain Banach lattice. This latter theorem is a result of Pisier, stated as [45, Théorème 2.1]. The analogous result for certain dual multi-norms is Theorem 5.6.

Our generalization of Pisier's theorem to $p$–multi-norms requires, in fact, that the $p$–multi-norm be a 'strong' $p$–multi-norm that is '$p$–convex'. We shall explain these extra terms in §2.5 and §2.6, respectively. In §2.5, we shall show that each $p$–multi-norm based on a Banach space is a strong $p$–multi-norm whenever $p$ is equal to 2 or $\infty$ and that, for every other value of $p$ with $1 \leqslant p \leqslant \infty$, there is a Banach space $E$ and a $p$–multi-norm based on $E$ that is not a strong $p$–multi-norm; we shall give a number of examples of $p$–multi-norms that are and are not strong $p$–multi-norms.

There is a dual representation theorem, given as Theorem 5.10; it shows that certain $p$–multi-norms, including dual multi-norms, based on a Banach space are the same as the quotient $p$–multi-norm based on a space $X/Y$, where $Y$ is a closed subspace of a Banach lattice $X$ and we take the canonical lattice $p$–multi-norm based on $X$.

Throughout we shall consider when properties of $p$–multi-norms based on Banach spaces pass to the corresponding power-norms based on subspaces, on quotients, on dual spaces, and on spaces that are the intermediate space formed by complex interpolation between a compatible couple of Banach spaces. Most of these results are not needed for the main representation theorems of Chapter 5.

Chapter 1 gives background, mainly in the theory of Banach spaces; a reader may wish to skim the results of this chapter and return to consult it when the particular background is relevant.

For example, we shall recall in Chapter 1 some standard theory of tensor products of Banach spaces, concentrating on the projective and injective tensor products. In §1.3, we shall define the $p$–sum norm based on a normed space, and, in §1.5, we shall introduce weak $p$–summing norms and their duals; these are examples of power-norms. A source of examples for us will be spaces in the class $SQ(p)$, where $1 \leqslant p \leqslant \infty$; these are Banach spaces that are isometrically isomorphic to closed subspaces of quotients of $L^p$-spaces, and we shall introduce this class in §1.6. In §1.7, we shall use an example of Schechtman to exhibit a space $S_p$ for $1 \leqslant p \leqslant 2$ that is isomorphic to a member of the class $SQ(p)$, but not isomorphic to a closed subspace of $L^p(\Omega)$ for any measure space $\Omega$. Some results here may be new.

The $p$–spaces of Herz are introduced in §1.8. Spaces in the class $SQ(p)$ are, by a theorem of Kwapień, exactly these $p$–spaces; the theorem of Kwapień seems to be important, and we shall present a proof of this result in §1.9. Finally, in §1.10, we shall recall some theory of complex interpolation spaces between compatible couples of Banach spaces.

In Chapter 2, we shall begin our study of $p$–multi-norms, which are special types of power-norms, giving the definition and various examples. In Theorem 2.8, we shall relate $p$–multi-norms to the $p$–spaces of Herz; indeed, we shall show that, for $p$ with $1 \leqslant p < \infty$, the $p$–sum norm based on a Banach space $E$ is a $p$–multi-norm if and only if $E$ is a $p$–space if and only if $E$ belongs to the class $SQ(p)$. Suppose that there are $p_0$–multi-norms and $p_1$–multi-norms based on Banach spaces $E_0$ and $E_1$, respectively. In §2.3, we shall discuss when there is a $p$–multi-norm based on suitable intermediate spaces between $E_0$ and $E_1$. In §2.4, we shall characterize $p$–multi-norms in terms of certain tensor products of Banach spaces, thus showing that our theory can be regarded as belonging to the latter subject. We shall also introduce, in §2.5 and §2.6, two strengthenings of the notion of a $p$–multi-norm to give strong $p$–multi-norms and $p$–convex and $p$–concave multi-norms, respectively; we shall give a variety of examples that show that, in various settings, there are $p$–multi-norms that are not strong $p$–multi-norms. Throughout the chapter, we shall explain when $p$–multi-norms and their strengthened versions based on Banach spaces pass to closed subspaces, to quotient spaces, to dual spaces, and to interpolation spaces.

The natural morphisms in the category of multi-normed spaces are the multi-bounded maps, and these are introduced in Chapter 3; we shall give various examples, and define $p$–multi-norms on spaces of multi-bounded operators.

In Chapter 4, we shall turn to our main topic, that of $p$–multi-norms in the setting of Banach lattices, in particular introducing in §4.3 the canonical lattice $p$–multi-norm based on a Banach lattice. In §4.1 and §4.2, we shall recall and somewhat extend some background on Banach lattices and regular and order-bounded operators between Banach lattices, in particular discussing pre-regular operators. In §4.4, we shall show that complex interpolation between Banach lattices gives a Banach lattice and that two canonical lattice $p_0$– and $p_1$–multi-norms on a Banach lattice produce a canonical lattice $p$–multi-norm for an appropriate value of $p$. In §4.5, we shall show how spaces of $p$–multi-bounded operators between Banach lattices with their respective canonical lattice $p$–multi-norms are related to spaces of pre-regular operators.

Finally, in Chapter 5, we shall give our representation theorems, together with some examples that show their limits.

**1.3. Notation and terminology.** First, we recall some standard definitions and notations primarily involving normed and Banach spaces that we shall use.

The cardinality of a set $S$ is $|S|$. The closed unit interval $[0,1]$ is denoted by $\mathbb{I}$. The conjugate index of $p \in [1,\infty]$ is denoted by $p'$, so that $1/p + 1/p' = 1$; we shall often set $q = p'$. Throughout we shall interpret the expression

$$\left(\sum_{i=1}^{n} \alpha_i^p\right)^{1/p},$$

where $\alpha_1, \dots, \alpha_n \geqslant 0$ and $n \in \mathbb{N}$, as $\max\{\alpha_1, \dots, \alpha_n\}$ when $p = \infty$.

Let $E$ be a linear space over a field $\mathbb{F}$ (always $\mathbb{R}$ or $\mathbb{C}$). Then we write $I_E$ for the identity operator on $E$. However the identity on $\mathbb{F}^n$ is usually denoted by $I_n$ for each $n \in \mathbb{N}$. The linear span of a subset $S$ of $E$ is denoted by

$$\operatorname{lin} S\,.$$

Let $E$ and $F$ be linear spaces. Then $E \oplus F$ is the direct sum of $E$ and $F$, and $\mathcal{L}(E,F)$ is the linear space (over $\mathbb{F}$) of $\mathbb{F}$–linear maps from $E$ into $F$.

Definition 1.2. Let $E$ and $F$ be linear spaces, and take $n \in \mathbb{N}$. The $n^{th}$ *amplification* of $T \in \mathcal{L}(E, F)$ is given by

$$T^{(n)} : (x_1, \dots, x_n) \mapsto (Tx_1, \dots, Tx_n)\,, \quad E^n \to F^n\,. \tag{1.3.1}$$

Let $T \in \mathcal{L}(E, F)$, and take $n \in \mathbb{N}$. Then the mapping $T^{(n)}$ is clearly also linear, and it is injective or surjective if and only if $T$ has the corresponding property. We may write equation (1.3.1) as:

$$T^{(n)} : \boldsymbol{x} \mapsto T^{(n)}\boldsymbol{x}\,, \quad E^n \to F^n\,.$$

Let $E$ be a linear space, and take $S, T \in \mathcal{L}(E)$. Then clearly

$$(S \circ T)^{(n)} = S^{(n)} \circ T^{(n)} \in \mathcal{L}(E^n) \quad (n \in \mathbb{N})\,. \tag{1.3.2}$$

The action of a linear functional $\lambda$ on an element $x$ of a linear space $E$ is usually denoted by $\langle x, \lambda\rangle$, so that the $n^{\mathrm{th}}$ amplification of $\lambda$ is the linear map $\lambda^{(n)}$, which is defined on $E^n$ by

$$\lambda^{(n)} : (x_1, \dots, x_n) \mapsto (\langle x_1, \lambda\rangle, \dots, \langle x_n, \lambda\rangle) = \langle \boldsymbol{x}, \lambda\rangle\,, \quad E^n \to \mathbb{F}^n\,, \tag{1.3.3}$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$.

Take $m, n \in \mathbb{N}$. For elements $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$, we write

$$(\boldsymbol{x}, \boldsymbol{y}) = (x_1, \dots, x_m, y_1, \dots, y_n) \in E^{m+n}\,;$$

this is called the *concatenation* of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Suppose that $F$ is a linear subspace of a linear space $E$. Then we shall often write

$$J_F : F \to E \quad \text{and} \quad Q_F : E \to E/F \tag{1.3.4}$$

for the natural embedding and the quotient map, respectively. Take $n \in \mathbb{N}$. Then $F^n$ is a linear subspace of $E^n$, and we identify $(E/F)^n$ with the quotient space $E^n/F^n$ via

$$(x_1 + F, \dots, x_n + F) = \boldsymbol{x} + F^n \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)\,. \tag{1.3.5}$$

Consequently, the quotient map $Q_{F^n} : E^n \to E^n/F^n$ is identified with the $n^{\mathrm{th}}$ amplification $Q_F^{(n)}$ of the quotient map $Q_F : E \to E/F$.

Let $E$ and $F$ be linear spaces. A bijection in $\mathcal{L}(E, F)$ is a *linear isomorphism*. Take $T \in \mathcal{L}(E, F)$. Then $T$ induces a linear map

$$\overline{T} : x + \ker T \mapsto Tx\,, \quad E/\ker T \to F\,. \tag{1.3.6}$$

Of course, $\overline{T}$ is a linear isomorphism from $E/\ker T$ onto $T(E)$; this is the *fundamental isomorphism theorem*. For $n \in \mathbb{N}$, we have $\ker T^{(n)} = (\ker T)^n$, and the identification of $E^n/\ker T^{(n)}$ with $(E/\ker T)^n$ implies that the induced map $\overline{T^{(n)}}$ is identified with the

$n^{\text{th}}$ amplification $\overline{T}^{(n)}$ of $\overline{T}$, as the following diagram illustrates:

$$\begin{tikzcd}
E^n/\ker T^{(n)} \arrow[dd, equal] \arrow[drrr, "\overline{T^{(n)}}"] & & & \\
& E^n \arrow[ul] \arrow[dl] \arrow[rr, "T^{(n)}"] & & F^n. \\
(E/\ker T)^n \arrow[urrr, "\overline{T}^{(n)}"'] & & &
\end{tikzcd} \tag{1.3.7}$$

Let $E$ and $F$ be linear spaces, and take $n \in \mathbb{N}$. For $T_1, \ldots, T_n \in \mathcal{L}(E, F)$, define $\Delta_{(T_1,\ldots,T_n)} \in \mathcal{L}(E, F^n)$ and $\Sigma_{(T_1,\ldots,T_n)} \in \mathcal{L}(E^n, F)$ by

$$\Delta_{(T_1,\ldots,T_n)}(x) = (T_1x, \ldots, T_nx) \quad (x \in E) \tag{1.3.8}$$

and

$$\Sigma_{(T_1,\ldots,T_n)}(x_1, \ldots, x_n) = T_1x_1 + \cdots + T_nx_n \quad (x_1, \ldots, x_n \in E)\,, \tag{1.3.9}$$

respectively.

Take $m, n \in \mathbb{N}$. Then $\mathbb{M}_{m,n} = \mathbb{M}_{m,n}(\mathbb{F})$ denotes the space of all $m \times n$ matrices over $\mathbb{F}$, with $\mathbb{M}_n$ for $\mathbb{M}_{n,n}$; we shall write $T \in \mathbb{M}_{m,n}$ as $(T_{i,j})$. The *transpose* of $T = (T_{i,j}) \in \mathbb{M}_{m,n}$ is the matrix $T^t = (T_{j,i}) \in \mathbb{M}_{n,m}$. A matrix $T \in \mathbb{M}_{m,n}$ is *row-special* (respectively, *column-special*) if it has at most one non-zero entry in each row (respectively, column); $T$ is *special* if it has at most one non-zero entry in each row and in each column. Suppose that $E$ is a linear space over $\mathbb{F}$. Then we further regard a matrix in $\mathbb{M}_{m,n}(\mathbb{F})$ as defining a linear map from $E^n$ to $E^m$ in the obvious way.

Now let $(E, \|\cdot\|)$ be a normed space over a field $\mathbb{F}$. We write

$$B_E\,, \quad B_E^\circ\,, \quad \text{and} \quad S_E$$

for the closed unit ball, the open unit ball, and the unit sphere of $E$, respectively. The dual space of $E$ (consisting of all continuous linear functionals on $E$) is denoted by $E'$, and the duality is implemented by the bilinear map

$$(x,\, \lambda) \mapsto \langle x,\, \lambda\rangle\,, \quad E \times E' \to \mathbb{F}\,;$$

the dual norm to $\|\cdot\|$ on $E'$ is often denoted by $\|\cdot\|'$. The *weak topology* on $E$ is $\sigma(E, E')$ and the *weak* topology* on $E'$ is $\sigma(E', E)$. The *bidual* of $E$ is $E'' = (E')'$, and the canonical embedding of $E$ into $E''$ is $\kappa_E$; we shall usually identify $E$ with $\kappa_E(E)$ and sometimes write $\widehat{x}$ for $\kappa_E(x)$, where $x \in E$.

Let $E$ be a normed space, take $n \in \mathbb{N}$, and let $\|\cdot\|_n$ be a norm on $E^n$ defining the product topology. Suppose that $F$ is a closed linear subspace of $E$. Then $F^n$ is a closed linear subspace of $(E^n, \|\cdot\|_n)$, and using the identification (1.3.5) we obtain a norm on $(E/F)^n = E^n/F^n$ that is given by the following explicit formula:

$$\|\boldsymbol{x} + F^n\|_n = \big\|(x_1 + F, \ldots, x_n + F)\big\|_n = \inf_{y_1,\ldots,y_n \in F} \|(x_1 + y_1, \ldots, x_n + y_n)\|_n \tag{1.3.10}$$

for $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$.

Suppose that $E$ and $F$ are normed spaces, and take $p$ with $1 \leqslant p \leqslant \infty$. Then we write

$$E \oplus_p F$$

for the direct sum $E \oplus F$, taken with the norm given by $\|x + y\| = (\|x\|^p + \|y\|^p)^{1/p}$ for $x \in E$ and $y \in F$. The dual space of $E \oplus_p F$ is identified with $E' \oplus_{p'} F'$.

Suppose that $E$ and $F$ are normed spaces. Then we write $\mathcal{B}(E, F)$ for the normed space (with respect to the operator norm) of all bounded linear operators from $E$ to $F$, with $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$. The space $\mathcal{B}(E, F)$ is a Banach space whenever $F$ is a Banach space, and $\mathcal{B}(E)$ is a unital Banach algebra when $E$ is Banach. For details on Banach algebras, see [15]. An operator of norm at most 1 is a *contraction*. For $T \in \mathcal{B}(E, F)$, we write $T' \in \mathcal{B}(F', E')$ for the *dual* of $T$, so that $T'$ is defined by the formula

$$\langle x,\, T'\lambda \rangle = \langle Tx,\, \lambda \rangle \quad (x \in E,\, \lambda \in F')\,;$$

of course, $\|T'\| = \|T\|$. For $y \in F$ and $\lambda \in E'$, set

$$(y \otimes \lambda)(x) = \langle x,\, \lambda \rangle y \quad (x \in E)\,. \tag{1.3.11}$$

Then $y \otimes \lambda \in \mathcal{B}(E, F)$ with $\|y \otimes \lambda\| = \|y\|\,\|\lambda\|$, and

$$\mathcal{F}(E, F) = \mathrm{lin}\,\{y \otimes \lambda : y \in F,\, \lambda \in E'\}$$

is the subspace in $\mathcal{B}(E, F)$ consisting of the *finite-rank operators*. Let $T \in \mathcal{B}(E, F)$, take $n \in \mathbb{N}$, and suppose that $\|\cdot\|$ and $|||\cdot|||$ are norms on $E^n$ and $F^n$, respectively, defining the product topologies. Then $T^{(n)} : (E^n, \|\cdot\|) \to (F^n, |||\cdot|||)$ is a bounded linear operator. A bijection $T \in \mathcal{B}(E, F)$ such that $T^{-1} \in \mathcal{B}(F, E)$ is an *isomorphism*; the spaces $E$ and $F$ are *isomorphic*, written

$$E \sim F\,,$$

when there is such an isomorphism from $E$ onto $F$. Take $C \geqslant 1$. Then $E$ and $F$ are $C$*–isomorphic* when there is an isomorphism $T \in \mathcal{B}(E, F)$ with $\|T\|\,\|T^{-1}\| \leqslant C$; in this case, we write

$$E \underset{C}{\sim} F\,.$$

In the case where $E$ and $F$ are Banach spaces, it is of course immediate from *Banach's isomorphism theorem* that each bijection $T \in \mathcal{B}(E, F)$ is an isomorphism.

Suppose that $E$ and $F$ are isomorphic normed spaces. Then the *Banach–Mazur distance* from $E$ to $F$ is

$$d(E, F) = \inf\{\|T\|\,\|T^{-1}\| : T \in \mathcal{B}(E, F) \text{ is an isomorphism}\}\,;$$

the spaces $E$ and $F$ are *almost isometric* if $d(E, F) = 1$. The infimum in the definition of $d(E, F)$ is attained when $E$ and $F$ are both finite-dimensional spaces, but this is not true in general. We have $d(E, F) \leqslant C$ whenever $E \underset{C}{\sim} F$. Clearly

$$d(E, G) \leqslant d(E, F)\, d(F, G) \tag{1.3.12}$$

for three normed spaces $E$, $F$, and $G$ such that $E \sim F \sim G$.

The following definition is taken from [2, Definition 11.1.1].

DEFINITION 1.3. Let $E$ and $F$ be infinite-dimensional Banach spaces. Then $E$ is *finitely representable* in $F$ if, for each finite-dimensional subspace $X$ of $E$ and each $\varepsilon > 0$, there is a finite-dimensional subspace $Y$ of $F$ with $\dim Y = \dim X$ such that $d(X, Y) < 1 + \varepsilon$.

Let $E$, $F$, and $G$ be infinite-dimensional Banach spaces, and suppose that $E$ is finitely representable in $F$ and that $F$ is finitely representable in $G$. Then it is noted in [2, Proposition 11.1.4] that $E$ is finitely representable in $G$. Examples of spaces that are finitely representable in other spaces will be given in §1.6.

Let $E$ and $F$ be normed spaces. An operator $T \in \mathcal{B}(E, F)$ is an *embedding* if it is an isomorphism onto a subspace of $F$ (where the subspace has the relative norm from $F$), and $E$ *embeds* in $F$ if there is such an embedding. Thus $T \in \mathcal{B}(E, F)$ is an embedding if and only if there exists $c > 0$ such that $\|Tx\| \geqslant c\,\|x\| \quad (x \in E)$. We define the *embedding constant* of $T \in \mathcal{B}(E, F)$ by the formula:

$$\beta(T) = \beta(T : E \to F) = \inf\{\|Tx\| : x \in S_E\}\,,$$

so that $\beta(T) > 0$ when $T$ is an embedding. When we consider an embedding $T : E \to F$ as an isomorphism onto its range, we see that $T$ has an inverse $T^{-1} : T(E) \to E$ and that

$$\left\|T^{-1} : T(E) \to E\right\| = 1/\beta(T)\,.$$

Suppose that $E$, $F$, $G$, and $H$ are normed spaces, and take $R \in \mathcal{B}(E, F)$ to be a surjection, $S \in \mathcal{B}(F, G)$, and $T \in \mathcal{B}(G, H)$. Then $TSR \in \mathcal{B}(E, H)$ and

$$\beta(TSR) \leqslant \beta(S)\,\|R\|\,\|T\|\,. \tag{1.3.13}$$

Indeed, take $\varepsilon > 0$, and then take $y \in S_F$ with $\|Sy\| < \beta(S) + \varepsilon$. Since $R$ is a surjection, there exists $x \in E$ with $Rx = y$, and then $1 \leqslant \|R\|\,\|x\|$, so that

$$\beta(TSR) \leqslant \frac{\|TSRx\|}{\|x\|} \leqslant \|T\|\,\|Sy\|\,\|R\| < (\beta(S) + \varepsilon)\,\|R\|\,\|T\|\,.$$

Inequality (1.3.13) follows.

Let $E$ and $F$ be normed spaces, and suppose that $T \in \mathcal{B}(E, F)$ is an open map, and hence a surjection. Then we define the *modulus of surjectivity* of $T \in \mathcal{B}(E, F)$ by

$$r(T) = \inf\{c > 0 : B_F^\circ \subset cT(B_E^\circ)\} \quad (T \in \mathcal{B}(E, F))\,,$$

so that $r(T) > 0$. In this case, the induced map $\overline{T} : E/\ker T \to F$ is an isomorphism and

$$r(T) = \left\|\overline{T}^{-1}\right\|\,. \tag{1.3.14}$$

Let $E$ and $F$ be Banach spaces. Then the following are standard results: for each embedding $T \in \mathcal{B}(E, F)$, the map $T'$ is a surjection and $r(T') = 1/\beta(T)$; for each surjection $T \in \mathcal{B}(E, F)$, the map $T'$ is an embedding and $\beta(T') = 1/r(T)$.

Two normed spaces $E$ and $F$ are *isometrically isomorphic*, written

$$E \cong F\,,$$

when there is a linear isometry from $E$ onto $F$; an embedding of $E$ into $F$ is an *isometric embedding* if it is an isometry, and then $E$ *embeds isometrically in* $F$.

Let $E$ and $F$ be normed spaces, and take $T \in \mathcal{B}(E, F)$. Then $T$ is a *quotient operator* if $T(B_E^\circ) = B_F^\circ$ and an *exact quotient operator* if $T(B_E) = B_F$. Each exact quotient operator is a quotient operator; the converse is not necessarily true. We shall use the following standard result [27, p. 333].

PROPOSITION 1.4. *Let $E$ and $F$ be normed spaces, and take $T \in \mathcal{B}(E, F)$.*

(i) *The induced operator $\overline{T}\colon E/\ker T \to F$ is an isometric isomorphism if and only if $T$ is a quotient operator.*

(ii) *The operator $T$ is an isometric embedding if and only if $T'$ is an exact quotient operator if and only if $T'$ is a quotient operator.* ∎

Suppose that $F$ is a closed subspace of a normed space $E$. Then the *annihilator* of $F$ in $E'$ is the weak*-closed subspace of $E'$ defined by

$$F^{\perp} = \{\lambda \in E' : \langle x,\, \lambda\rangle = 0 \ \ (x \in F)\}\,,$$

so that $F'$ is identified with $E'/F^{\perp}$. Thus $J_F' = Q_{F^{\perp}}$ in the notation of equation (1.3.4).

DEFINITION 1.5. Let $E$ be a normed space. Then a normed space $F$ is a *subquotient* of $E$ whenever there is a closed subspace $G$ of $E$ such that $F$ is isometrically isomorphic to a subspace of the quotient space $E/G$.

Equivalently, the normed space $F$ is a subquotient of a normed space $E$ whenever $F$ is isometrically isomorphic to a quotient of a subspace of $E$.

We shall also use the following result.

PROPOSITION 1.6. *Let $E$, $F$, and $G$ be normed spaces. Suppose that there are a quotient operator $Q : E \to F$ and a contraction $J : E \to G$ such that $J(E)$ is dense in $G$ and $\|Qx\| \leqslant \|Jx\| \ \ (x \in E)$. Then $F$ is isometrically isomorphic to a quotient of $G$.*

*Proof.* Take $z \in J(E)$. Then there exists $x \in E$ with $Jx = z$; we set $Tz = Qx$. Since $\|Qx\| \leqslant \|Jx\| \ \ (x \in E)$, the element $Tz$ is well-defined in $F$ and $\|Tz\| \leqslant \|Jx\| = \|z\|$. Clearly the map $T : J(E) \to F$ is linear. Since $J(E)$ is dense in $G$, the map $T$ extends to a contraction $T : G \to F$. Take $y \in F$ with $\|y\| < 1$. Since $Q$ is a quotient operator, there exists $x \in E$ with $\|x\| < 1$ and $Qx = y$. Then $\|Jx\| < 1$ and $T(Jx) = y$. This shows that $T$ is a quotient operator, and so the map $\overline{T} : G/\ker T \to F$ is an isometric isomorphism by Proposition 1.4(i). ∎

Let $E$ be a normed space. A closed subspace $F$ of $E$ is *complemented* if there is a closed subspace $G$ of $E$ such that $E = F \oplus G$; an idempotent in the algebra $\mathcal{B}(E)$ is a *projection on* $E$.

Now suppose that $E$ is a Banach space and that $F$ is a complemented subspace of $E$. Then there is a projection $P$ on $E$ with $P(E) = F$ and $E = P(E) \oplus (I_E - P)(E)$; the space $F$ is $\lambda$-*complemented* (for $\lambda \geqslant 1$) if there is such a projection $P$ with $\|P\| \leqslant \lambda$, and $\lambda(F, E)$, the *projection constant of $F$ in $E$*, is the infimum of such numbers $\lambda$.

A Banach space $E$ is *injective* if, for every Banach space $G$, every closed subspace $F$ of $G$, and every $T \in \mathcal{B}(F, E)$, there is an extension $\widetilde{T} \in \mathcal{B}(G, E)$ of $T$; the space $E$ is $\lambda$-*injective* if, further, we can ensure that $\left\|\widetilde{T}\right\| \leqslant \lambda \|T\|$. For example, the space $\ell^{\infty}(S)$ of bounded, scalar-valued functions on a non-empty set $S$ is always 1–injective. See [17, Proposition 2.5.5], for example.

For $p$ with $1 \leqslant p \leqslant \infty$, we write $\ell^p$ for the usual Banach space of scalar-valued, $p$–summable sequences, with

$$\|(\alpha_j)\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |\alpha_j|^p\right)^{1/p} < \infty \quad ((\alpha_j) \in \ell^p)\,;$$

for $n \in \mathbb{N}$, the $n$–dimensional versions of these spaces are denoted by $\ell^p_n$. The Banach space of all scalar-valued null sequences is $c_0$; the linear subspace of sequences which are eventually 0 is $c_{00}$, so that $c_{00}$ is dense in $c_0$ and $\ell^p$ for $1 \leqslant p < \infty$.

We shall write $\delta_i$ for the sequence $(\delta_{i,j} : j \in \mathbb{N})$ for $i \in \mathbb{N}$, where $\delta_{i,j}$ is the Kronecker delta. Later, we shall identify finite sequences $(\alpha_1, \dots, \alpha_n)$ in $\mathbb{F}^n$ with the element $(\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in c_{00}$, and regard $c_{00}$ and $\ell^p_n$ as subspaces of $\ell^p$, so that

$$\ell^p_n = \mathrm{lin}\,\{\delta_1, \dots, \delta_n\} \quad (n \in \mathbb{N})\,.$$

For $n \in \mathbb{N}$, we write $P_n : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^n$ for the linear map which is the projection onto the first $n$ coordinates. Let $E = \ell^p$ (for $1 \leqslant p \leqslant \infty$) or $E = c_0$. Then $P_n \mid E \in \mathcal{B}(E)$ with $\|P_n \mid E\| = 1$ in each case; we note that $\lim_{n\to\infty} P_n T = T$ in $(\mathcal{B}(E), \|\cdot\|)$ for each compact operator $T$ on $E$. We also regard each $T = (T_{i,j}) \in \mathbb{M}_{m,n}$, where $m, n \in \mathbb{N}$, as an operator on $c_{00}$ via the formula

$$T\alpha = T\left(\sum_{j=1}^{\infty} \alpha_j \delta_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} T_{i,j}\alpha_j\right)\delta_i = TP_n\alpha \quad (\alpha = (\alpha_j) \in c_{00})\,.$$

More generally we have the following definition.

Definition 1.7. Let $E$ be a normed space, and take $n \in \mathbb{N}$ and $p$ with $1 \leqslant p \leqslant \infty$. Define

$$\|\boldsymbol{x}\|_{\ell^p_n(E)} = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \qquad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n\,, n \in \mathbb{N})\,. \tag{1.3.15}$$

Clearly $(E^n, \|\cdot\|_{\ell^p_n(E)})$ is a normed space that is a Banach space when $E$ is a Banach space. The norm $\|\cdot\|_{\ell^p_n(E)}$ is called the *$p$–sum norm* on $E$, and we write $\ell^p_n(E)$ for $E^n$ taken with this norm. Let $F$ be a closed subspace of $E$. Then clearly the restriction of the $p$–sum norm on $E^n$ to $F^n$ and the quotient of the $p$–sum norm on $(E/F)^n$ are the $p$–sum norms on $F^n$ and $(E/F)^n$, respectively. The dual space to $\ell^p_n(E)$ is $\ell^{p'}_n(E')$.

Let $E$ be a normed space. We define the following space:

$$\ell^p(E) = \left\{(x_n) \in E^{\mathbb{N}} : \|(x_n)\|_{\ell^p(E)} = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} < \infty\right\}\,,$$

so that $\ell^p(E)$ is a normed space; the specified norm on $\ell^p(E)$ is also called the *$p$–sum norm*. In the case where $1 \leqslant p < \infty$, the dual space to $\ell^p(E)$ is $\ell^{p'}(E')$, and so the dual of the $p$–sum norm based on $E$ is the $p'$–sum norm based on $E'$.

The following result is easily checked.

Proposition 1.8. *Let $p$ with $1 \leqslant p \leqslant \infty$, and take $m, n \in \mathbb{N}$.*

(i) *We have* $\|T : \ell^p_n \to \ell^p_m\| = \left\|T^t : \ell^{p'}_m \to \ell^{p'}_n\right\|$ $(T \in \mathbb{M}_{m,n})$.

(ii) *For each row-special matrix* $T \in \mathbb{M}_{m,n}$*, we have*

$$\|T : \ell^p_n \to \ell^p_m\| = \max\left\{\left(\sum_{i=1}^m |T_{i,j}|^p\right)^{1/p} : j \in \mathbb{N}_n\right\}.$$

(iii) *For each column-special matrix* $T \in \mathbb{M}_{m,n}$*, we have*

$$\|T : \ell^p_n \to \ell^p_m\| = \max\left\{\left(\sum_{j=1}^m |T_{i,j}|^{p'}\right)^{1/p'} : i \in \mathbb{N}_m\right\}.$$

■

Let $\Gamma$ be an index set. Then the space of functions on $\Gamma$ with finite support is denoted by $c_{00}(\Gamma)$. Now take $p$ with $1 \leqslant p \leqslant \infty$. Then we write $\ell^p(\Gamma)$ for the corresponding space, and define elements $\delta_\gamma$ in these spaces for $\gamma \in \Gamma$ by $\delta_\gamma(s) = 1$ if $s = \gamma$ and $\delta_\gamma(s) = 0$ if $s \in \Gamma \backslash \{\gamma\}$. Thus $c_{00}(\Gamma)$ is dense in $(\ell^p(\Gamma), \|\cdot\|_{\ell^p(\Gamma)})$ for $1 \leqslant p < \infty$. In particular, the *uniform norm* $\|\cdot\|_\infty$ on a set $\Gamma$ is defined by

$$\|f\|_\infty = \sup\{|f(s)| : s \in \Gamma\} \quad (f \in \ell^\infty(\Gamma)).$$

Let $K$ be a compact (Hausdorff) space. Then $(C(K), \|\cdot\|_\infty)$ is the uniform algebra (with the pointwise operations and the norm $\|\cdot\|_\infty$) of all scalar-valued, continuous functions on $K$; if it be necessary to specify the scalar field, we shall write $C(K,\mathbb{R})$ or $C(K,\mathbb{C})$, as appropriate. For a study of $C(K)$ as a Banach space, see [17], for example.

We shall use the fact that each Banach space $E$ is a quotient of a space $\ell^1(\Gamma)$ for some index set $\Gamma$. Indeed, we can take $\Gamma = B_E$ and define

$$Q : \sum \alpha_\gamma \delta_\gamma \mapsto \sum \alpha_\gamma \gamma, \quad \ell^1(\Gamma) \to E.$$

We recall two elementary and well-known facts that we shall use.

PROPOSITION 1.9. *Let $E$ be a finite-dimensional normed space, and take $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$ and an embedding $J : E \to \ell^\infty_n$ such that*

$$\|x\| \leqslant \|Jx\|_\infty \leqslant (1+\varepsilon)\|x\| \quad (x \in E), \tag{1.3.16}$$

*and so $d(E, J(E)) \leqslant 1 + \varepsilon$.*

*Proof.* We may suppose that $\varepsilon < 1$.

The set $S_{E'} := \{\lambda \in E' : \|\lambda\| = 1\}$ is compact, and so totally bounded, in the metric space $(E', \|\cdot\|)$, and hence there exist $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in S_{E'}$ such that, for each $\lambda \in S_{E'}$, there exists $i \in \mathbb{N}_n$ with $\|\lambda - \lambda_i\| < \varepsilon/2$. Set

$$Jx = (1+\varepsilon)(\langle x,\, \lambda_1\rangle, \dots, \langle x,\, \lambda_n\rangle) \quad (x \in E).$$

Then $Jx \in \ell^\infty_n$ $(x \in E)$, the map $J : E \to \ell^\infty_n$ is linear, and (1.3.16) follows easily, so that $J$ is an embedding. ■

PROPOSITION 1.10. *Let $E$ be a normed space, take $k \in \mathbb{N}$, and suppose that $\{x_1, \dots, x_k\}$ is a linearly independent set in $E$. Then there exists $\varepsilon > 0$ such that $\{y_1, \dots, y_k\}$ is a linearly independent set whenever $y_j \in E$ and $\|x_j - y_j\| < \varepsilon$ for $j \in \mathbb{N}_k$.*

*Proof.* Set $F = \mathrm{lin}\,\{x_1, \dots, x_k\}$, a finite-dimensional subspace of $E$, and consider the linear bijection

$$T : (\zeta_1, \dots, \zeta_k) \mapsto \sum_{j=1}^{k} \zeta_j x_j\,, \quad \ell_k^1 \to F\,.$$

Set $M = \left\|T^{-1}\right\| > 0$, fix $\varepsilon \in (0, 1/M)$, and consider elements $y_1, \dots, y_k \in E$ such that $\|x_j - y_j\| < \varepsilon \ (j \in \mathbb{N}_k)$. Suppose that $\zeta_1, \dots, \zeta_k \in \mathbb{F}$ with $\sum_{j=1}^{k} \zeta_j y_j = 0$. Then

$$\frac{1}{M} \sum_{j=1}^{k} |\zeta_j| \leqslant \left\| \sum_{j=1}^{k} \zeta_j x_j \right\| = \left\| \sum_{j=1}^{k} \zeta_j (x_j - y_j) \right\| \leqslant \varepsilon \sum_{j=1}^{k} |\zeta_j|\ .$$

Since $\varepsilon < 1/M$, this is a contradiction unless $\sum_{j=1}^{k} |\zeta_j| = 0$. Hence $\zeta_1 = \cdots = \zeta_k = 0$, and so $\{y_1, \dots, y_k\}$ is linearly independent. ■

Now let $E$ be a normed space, and take $n \in \mathbb{N}$. We shall consider norms $|||\cdot|||$ on $E^n$ that satisfy the following two conditions:

$$|||\boldsymbol{x}||| \geqslant \max_{i=1,\dots,n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n) \tag{1.3.17}$$

and

$$|||(0, \dots, 0, x_i, 0, \dots, 0)||| = \|x_i\| \quad (x_i \in E,\ i \in \mathbb{N}_n)\,. \tag{1.3.18}$$

Each norm that satisfies these conditions defines the product topology of $E^n$. Certainly $|||\cdot||| := \|\cdot\|_n$ satisfies these conditions whenever $(\|\cdot\|_m)$ is a power-norm based on $E$.

The maps $\Delta_{(T_1,\dots,T_n)} \in \mathcal{L}(E, F^n)$ and $\Sigma_{(T_1,\dots,T_n)} \in \mathcal{L}(E^n, F)$ were defined in equations (1.3.8) and (1.3.9), respectively. The results of the following proposition will be developed further in §3.2.

Proposition 1.11. *Let $E$ and $F$ be normed spaces, and take $n \in \mathbb{N}$.*

(i) *Suppose that the norm $|||\cdot|||$ on $F^n$ satisfies* (1.3.17) *and* (1.3.18)*. Then the map*

$$(T_1, \dots, T_n) \mapsto \Delta_{(T_1,\dots,T_n)}\,, \quad \mathcal{B}(E, F)^n \to \mathcal{B}(E, F^n)\,,$$

*is a linear isomorphism.*

(ii) *Suppose that the norm $|||\cdot|||$ on $E^n$ satisfies* (1.3.17) *and* (1.3.18)*. Then the map*

$$(T_1, \dots, T_n) \mapsto \Sigma_{(T_1,\dots,T_n)}\,, \quad \mathcal{B}(E, F)^n \to \mathcal{B}(E^n, F)\,,$$

*is a linear isomorphism.*

*Proof.* Take $T_1, \dots, T_n \in \mathcal{B}(E, F)$.

(i) For each $x \in E$, we have

$$\left|\left|\left|\Delta_{(T_1,\dots,T_n)}(x)\right|\right|\right| \leqslant \sum_{i=1}^{n} |||\,(0, \dots, 0, T_i x, 0, \dots, 0)\,||| \leqslant \left(\sum_{i=1}^{n} \|T_i\|\right) \|x\|$$

by (1.3.18), and so $\Delta_{(T_1,\dots,T_n)} \in \mathcal{B}(E, F^n)$ with $\left\|\Delta_{(T_1,\dots,T_n)}\right\| \leqslant \sum_{i=1}^{n} \|T_i\|$.

Clearly the specified map is a linear injection. For $i \in \mathbb{N}_n$, let $\pi_i : F^n \to F$ be the coordinate projection onto the $i^{\text{th}}$ coordinate, and take $T \in \mathcal{B}(E, F^n)$; by (1.3.17), $\pi_i$ is a contraction, and so $\pi_i \circ T \in \mathcal{B}(E, F)$. Further $T = \Delta_{(\pi_1 \circ T, \dots, \pi_n \circ T)}$, and so the specified map is a surjection.

(ii) For each $x_1, \dots, x_n \in E$, we have

$$\left\|\Sigma_{(T_1,\dots,T_n)}(x_1,\dots,x_n)\right\| \leqslant \left(\sum_{i=1}^n \|T_i\|\right) \max_{i=1,\dots,n} \|x_i\| \leqslant \left(\sum_{i=1}^n \|T_i\|\right) |||(x_1,\dots,x_n)|||$$

by (1.3.17), and so $\Sigma_{(T_1,\dots,T_n)} \in \mathcal{B}(E^n, F)$ with $\left\|\Sigma_{(T_1,\dots,T_n)}\right\| \leqslant \sum_{i=1}^n \|T_i\|$.

Clearly the specified map is a linear injection. For $i \in \mathbb{N}_n$, let $\iota_i : E \to E^n$ be the embedding into the $i^{\text{th}}$ coordinate, and take $T \in \mathcal{B}(E^n, F)$; by (1.3.18), $\iota_i$ is an isometry, and so $T \circ \iota_i \in \mathcal{B}(E, F)$. Since $T = \Sigma_{(T\circ\iota_1,\dots,T\circ\iota_n)}$, the specified map is a surjection. ∎

Let $E$ be a normed space, take $n \in \mathbb{N}$, and suppose that $E^n$ is endowed with a norm $|||\cdot|||$ which satisfies equations (1.3.17) and (1.3.18). As a special case of clause (ii), above, take $\lambda_1, \dots, \lambda_n \in E'$, and define $\boldsymbol{\lambda} = \Sigma_{(\lambda_1,\dots,\lambda_n)} \in (E^n, |||\cdot|||)'$, so that

$$\langle \boldsymbol{x},\, \boldsymbol{\lambda}\rangle = \sum_{i=1}^n \langle x_i,\, \lambda_i\rangle \quad (\boldsymbol{x} = (x_1,\dots,x_n) \in E^n)\,. \tag{1.3.19}$$

Then

$$\max_{i=1,\dots,n} \|\lambda_i\| \leqslant |||\boldsymbol{\lambda}|||' \leqslant \sum_{i=1}^n \|\lambda_i\|\;,$$

where $|||\cdot|||'$ is the dual norm to $|||\cdot|||$, and so, by identifying $\boldsymbol{\lambda} \in (E^n, |||\cdot|||)'$ with $(\lambda_1,\dots,\lambda_n) \in (E')^n$, we have defined a norm on $(E')^n$. We have identified $\kappa_{E^n}$ with $\kappa_E^{(n)}$, and so we regard $\kappa_{E^n}(E^n)$ as a subspace of $(E'')^n$.

Suppose in addition that $T$ is an operator from $E$ into a normed space $F$ and that $F^n$ is also endowed with a norm $|||\cdot|||$ which satisfies equations (1.3.17) and (1.3.18). Then the above identification of the dual spaces of $(E^n, |||\cdot|||)$ and $(F^n, |||\cdot|||)$ with $(E')^n$ and $(F')^n$, respectively, implies that the dual of the $n^{\text{th}}$ amplification of $T$ is identified with the $n^{\text{th}}$ amplification of the dual of $T$, so that

$$(T^{(n)})' = (T')^{(n)} \quad (n \in \mathbb{N})\,. \tag{1.3.20}$$

**1.4. Tensor products.** We recall some definitions concerning tensor products of normed spaces; for the theory of such tensor products, see [22, 23, 24, 25, 32, 55] and [15, Appendix 3].

Suppose that $E$ and $F$ are linear spaces over the same field $\mathbb{F}$, and denote their (algebraic) tensor product by $E \otimes F$. Each element $z \in E \otimes F$ has a representation as $z = \sum_{j=1}^n x_j \otimes y_j$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$, and $y_1, \dots, y_n \in F$; in the case where $z \neq 0$, we may suppose that the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are linearly independent.

Let $F$ be a subspace of a linear space $E$, and let $G$ be a linear space. Then $F \otimes G$ is a subspace of $E \otimes G$ and the quotient space $(E \otimes G)/(F \otimes G)$ can be identified with $(E/F) \otimes G$.

Let $E$, $F$, and $G$ be linear spaces, and take $S$ to be a bilinear map from $E \times F$ into $G$. Then there is a unique linear map $T_S : E \otimes F \to G$ such that

$$T_S(x \otimes y) = S(x, y) \quad (x \in E,\, y \in F)\,.$$

Let $E$, $F$, $X$, and $Y$ be linear spaces, and suppose that $S \in \mathcal{L}(E, X)$ and $T \in \mathcal{L}(F, Y)$. Then there is a unique linear map $S \otimes T : E \otimes F \to X \otimes Y$ such that

$$(S \otimes T)(x \otimes y) = Sx \otimes Ty \quad (x \in E,\, y \in F)\,.$$

Similarly, suppose that $\lambda$ and $\mu$ are linear functionals on $E$ and $F$, respectively. Then $\lambda \otimes \mu$ is the unique linear functional on $E \otimes F$ such that

$$(\lambda \otimes \mu)(x \otimes y) = \langle x,\, \lambda\rangle\langle y,\, \mu\rangle \quad (x \in E,\, y \in F)\,. \tag{1.4.1}$$

Suppose that $E$ is a finite-dimensional space with a basis $\{e_1, \dots, e_n\}$ and that $F$ is a linear space. Then each element $z \in E \otimes F$ has a unique representation in the form $z = \sum_{j=1}^{n} e_j \otimes y_j$, where $y_1, \dots, y_n \in F$. For example, the space $\mathbb{F}^n$ has the standard basis $\{\delta_1, \dots, \delta_n\}$, and so we can identify $(y_1, \dots, y_n) \in F^n$ with $\sum_{j=1}^{n} \delta_j \otimes y_j$ in $\mathbb{F}^n \otimes F$.

Let $F$ and $G$ be linear spaces, and take $T \in \mathcal{L}(F, G)$ and $n \in \mathbb{N}$. As above, we identify $F^n$ and $G^n$ with $\mathbb{F}^n \otimes F$ and $\mathbb{F}^n \otimes G$, respectively. Then the $n^{\text{th}}$ amplification $T^{(n)}$ of $T$ is identified with the operator $I_n \otimes T : \mathbb{F}^n \otimes F \to \mathbb{F}^n \otimes G$. More generally, for $A \in \mathbb{M}_{m,n}$, where $m, n \in \mathbb{N}$, the action

$$A \otimes T : \mathbb{F}^n \otimes F \to \mathbb{F}^m \otimes G \tag{1.4.2}$$

corresponds to the map

$$\boldsymbol{x} \mapsto A(T^{(n)}\boldsymbol{x}) = T^{(m)}(A\boldsymbol{x})\,, \quad F^n \to G^m\,. \tag{1.4.3}$$

In particular, the map $A \otimes I_F : \mathbb{F}^n \otimes F \to \mathbb{F}^m \otimes F$ corresponds to the map $A : F^n \to F^m$, with the above identification.

Let $E$ and $F$ be normed spaces. The *projective tensor norm* $\|\cdot\|_\pi$ on $E \otimes F$ is defined by

$$\|z\|_\pi = \inf\left\{\sum_{j=1}^{m} \|x_j\|\,\|y_j\| : z = \sum_{j=1}^{m} x_j \otimes y_j,\, m \in \mathbb{N}\right\} \quad (z \in E \otimes F)\,,$$

where the infimum is taken over all representations of $z$ as an element of $E \otimes F$. Then $(E \otimes F, \|\cdot\|_\pi)$ is a normed space; it is complete if either $E$ or $F$ is finite dimensional and the other is a Banach space, but it is not complete if both $E$ and $F$ are infinite-dimensional spaces; the Banach space which is its completion is denoted by

$$(E \mathbin{\widehat{\otimes}} F, \|\cdot\|_\pi)\,.$$

The *injective tensor norm* $\|\cdot\|_\varepsilon$ on $E \otimes F$ is defined by

$$\|z\|_\varepsilon = \sup\left\{\left|\sum_{j=1}^{m} \langle x_j,\, \lambda\rangle\langle y_j,\, \mu\rangle\right| : \lambda \in B_{E'},\, \mu \in B_{F'}\right\}\,,$$

where $z = \sum_{j=1}^{m} x_j \otimes y_j$ is any representation of $z$ in $E \otimes F$. Then $(E \otimes F, \|\cdot\|_\varepsilon)$ is a normed space; the Banach space which is its completion is denoted by

$$(E \mathbin{\check{\otimes}} F, \|\cdot\|_\varepsilon)\,.$$

We note that always $\|z\|_\varepsilon \leqslant \|z\|_\pi$ $(z \in E \otimes F)$; it is straightforward to see that, in the case where $\dim E = n$, we have

$$\|z\|_\pi \leqslant n\,\|z\|_\varepsilon \quad (z \in E \otimes F)\,, \tag{1.4.4}$$

and so the identity map from $(E \otimes F, \|\cdot\|_\varepsilon)$ onto $(E \otimes F, \|\cdot\|_\pi)$ is an isomorphism in this special case.

A norm $\|\cdot\|$ on $E \otimes F$ is a *cross-norm* if

$$\|x \otimes y\| = \|x\| \, \|y\| \quad (x \in E,\, y \in F)\,,$$

and a *sub-cross-norm* if

$$\|x \otimes y\| \leqslant \|x\| \, \|y\| \quad (x \in E,\, y \in F)\,.$$

A sub-cross-norm $\|\cdot\|$ on $E \otimes F$ is *reasonable* if the linear functional $\lambda \otimes \mu$ that was defined in equation (1.4.1) is bounded on $(E \otimes F, \|\cdot\|)$ with $\|\lambda \otimes \mu\| \leqslant \|\lambda\| \, \|\mu\|$ for each $\lambda \in E'$ and $\mu \in F'$. The projective and injective tensor norms on $E \otimes F$ are both cross-norms; indeed, the projective tensor norm is the maximum cross-norm on $E \otimes F$.

The following result is [55, Proposition 6.1].

PROPOSITION 1.12. *Let $E$ and $F$ be normed spaces.*

(i) *A norm $\|\cdot\|$ on $E \otimes F$ is a reasonable sub-cross-norm if and only if*

$$\|z\|_\varepsilon \leqslant \|z\| \leqslant \|z\|_\pi \quad (z \in E \otimes F)\,.$$

(ii) *Each reasonable sub-cross-norm $\|\cdot\|$ on $E \otimes F$ is a cross-norm, and the dual norm $\|\cdot\|'$ is a cross-norm on $E' \otimes F'$.* ■

Let $E$ and $F$ be normed spaces. The dual space $(E \widehat{\otimes} F)'$ is isometrically isomorphic to $\mathcal{B}(E, F')$ via the map $\iota$ defined by

$$\langle y,\, (\iota\lambda)(x)\rangle = \langle x \otimes y,\, \lambda\rangle \quad (x \in E,\, y \in F,\, \lambda \in (E \widehat{\otimes} F)')\,. \tag{1.4.5}$$

By [22, p. 47], there is a natural isometric embedding of $E' \check{\otimes} F'$ in $(E \widehat{\otimes} F)'$, but this embedding is not usually a surjection. However, in the case where either $E$ or $F$ is a finite-dimensional space, we have the two identifications

$$(E \otimes F, \|\cdot\|_\varepsilon)' \cong (E' \otimes F', \|\cdot\|_\pi) \tag{1.4.6}$$

and

$$(E \otimes F, \|\cdot\|_\pi)' \cong (E' \otimes F', \|\cdot\|_\varepsilon)\,. \tag{1.4.7}$$

See [22, Theorem 6.4], for example.

Now let $E$, $F$, and $G$ be Banach spaces, and take $S$ to be a bounded bilinear map from $E \times F$ into $G$. Then there is a unique bounded linear map $T_S : E \widehat{\otimes} F \to G$ such that $T_S(x \otimes y) = S(x, y)$ $(x \in E,\, y \in F)$; further, $\|T_S\| = \|S\|$. The bilinear map

$$R : (y,\, \lambda) \mapsto y \otimes \lambda\,, \quad F \times E' \to \mathcal{B}(E, F)\,,$$

where $y \otimes \lambda$ was defined in (1.3.11), is bounded, and so we obtain a bounded linear operator $T_R : F \widehat{\otimes} E' \to \mathcal{B}(E, F)$. The range of $T_R$ is the space of *nuclear operators*, denoted by $(\mathcal{N}(E, F), \|\cdot\|_\nu)$, where $\|\cdot\|_\nu$ is the nuclear norm; see [22, §3.6].

We shall use the following standard theorem; see [55, Propositions 2.3 and 3.2], for example. For the final statement, see [22, (4.3) and (5.8)].

Theorem 1.13. *Let $E$, $F$, $X$, and $Y$ be Banach spaces, and suppose that $S \in \mathcal{B}(E, X)$ and $T \in \mathcal{B}(F, Y)$. Then there are unique operators*

$$S \otimes_\pi T \in \mathcal{B}(E \widehat{\otimes} F, X \widehat{\otimes} Y) \quad \text{and} \quad S \otimes_\varepsilon T \in \mathcal{B}(E \check{\otimes} F, X \check{\otimes} Y)$$

*with*

$$(S \otimes_\pi T)(x \otimes y) = Sx \otimes Ty \quad (x \in E,\ y \in F)$$

*and*

$$(S \otimes_\varepsilon T)(x \otimes y) = Sx \otimes Ty \quad (x \in E,\ y \in F)\,,$$

*respectively. Further,*

$$\|S \otimes_\pi T\| = \|S \otimes_\varepsilon T\| = \|S\|\,\|T\|\,.$$

*Suppose that $S$ and $T$ are injective. Then $S \otimes_\varepsilon T$ is always injective, and $S \otimes_\pi T$ is injective whenever either $E$ or $F$ has the approximation property.* ∎

We shall usually write $S \otimes T$ for either $S \otimes_\pi T$ or $S \otimes_\varepsilon T$, as appropriate.

In particular, suppose that $F$ is a closed subspace of a Banach space $E$ and that $G$ is a Banach space. Then the linear map

$$I_G \otimes_\pi J_F : (G \widehat{\otimes} F, \|\cdot\|_\pi) \to (G \widehat{\otimes} E, \|\cdot\|_\pi)$$

is a contraction, but it is not always an embedding. More generally, the projective tensor product 'preserves quotients, but not necessarily subspaces' and the injective tensor product 'preserves subspaces, but not necessarily quotients'. This phenomenon is discussed in the literature; for example, see [23, Theorem 2.3.1] and [55, §§2.1, 3.1]. The following result is contained in [22, Chapters 3 and 4] and [55, §2.1, §3.1, and Exercise 3.3].

Proposition 1.14. *Let $E$ and $G$ be Banach spaces, and suppose that $F$ is a closed subspace of $E$.*

(i) *The linear map $I_G \otimes Q_F : G \widehat{\otimes} E \to G \widehat{\otimes} (E/F)$ is a quotient operator.*

(ii) *The linear map $I_G \otimes J_F : G \widehat{\otimes} F \to G \widehat{\otimes} E$ is an isometry if and only if each $T \in \mathcal{B}(F, G')$ extends to an operator $\widetilde{T} \in \mathcal{B}(E, G')$ with $\left\|\widetilde{T}\right\| = \|T\|$.*

(iii) *For each measure space $\Omega$, the linear map $I_{L^1(\Omega)} \otimes J_F : L^1(\Omega) \widehat{\otimes} F \to L^1(\Omega) \widehat{\otimes} E$ is an isometry.*

(iv) *The linear map $\kappa_G \otimes \kappa_E : G \widehat{\otimes} E \to G'' \widehat{\otimes} E''$ is an isometry.*

(v) *The linear map $I_G \otimes J_F : G \check{\otimes} F \to G \check{\otimes} E$ is an isometry.* ∎

The next proposition is closely related to clause (ii), above; it may be well-known (see, e.g., [47, Section 3]), but we prove it for the sake of completeness.

Proposition 1.15. *Let $F$ be a finite-dimensional subspace of a Banach space $E$, and let $G$ be a Banach space. Then the linear map*

$$I_G \otimes_\pi J_F : G \widehat{\otimes} F \to G \widehat{\otimes} E$$

*is an embedding, and $\beta(I_G \otimes_\pi J_F) \geqslant 1/\lambda(F, E)$. Moreover, in the case where $G = F'$, we have*

$$\beta(I_{F'} \otimes_\pi J_F) = 1/\lambda(F, E)\,.$$

*Proof.* The first part of this proposition is easy, and hence we need to show only that $\beta(I_{F'} \otimes_\pi J_F) \leqslant 1/\lambda(F,E)$. Set

$$T = I_{F'} \otimes_\pi J_F : F' \widehat{\otimes} F \to F' \widehat{\otimes} E$$

and $c = 1/\beta(T)$. By equation (1.4.5) and the fact that $F$ is reflexive, we may consider the surjection $T'$ as an operator from $\mathcal{B}(E,F)$ onto $\mathcal{B}(F)$. Since $r(T') = 1/\beta(T) = c$, it follows that, for each $U \in \mathcal{B}(F)$, there exists $V \in \mathcal{B}(E,F)$ with $T'(V) = U$ and $\|V\| \leqslant c\,\|U\|$. But $T'$ is the restriction map, and so, taking $U = I_F$, we conclude that $\lambda(F,E) \leqslant c$.

The result follows. ∎

**1.5. Weak $p$–summing norms.** Let $E$ be a normed space. In this section, we shall recall the definition of the weak $p$–summing norms on $E$ and give some of their basic properties. Throughout this section, $1 \leqslant p \leqslant \infty$ and $q = p'$.

The following standard definition was given in [20, Definition 4.1.1] and [19, §2.3]; for further discussion, see also [22, 24, 32].

Let $E$ be a normed space, and take $n \in \mathbb{N}$. Following the notation of [18, 20, 32], we define $\mu_{p,n}(\boldsymbol{x})$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ by

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{ \left(\sum_{i=1}^n |\langle x_i,\,\lambda\rangle|^p\right)^{1/p} : \lambda \in B_{E'} \right\} = \sup\left\{ \|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell_n^p} : \lambda \in B_{E'} \right\}.$$

Then $(E^n, \mu_{p,n})$ is a normed space; it is a Banach space when $E$ is a Banach space. We write $\mu_{p,n}^E$ when it is necessary to identify the space $E$. For example,

$$\mu_{\infty,n}(\boldsymbol{x}) = \max_{i=1,\dots,n} \|x_i\| = \|\boldsymbol{x}\|_{\ell_n^\infty(E)} \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)\,. \tag{1.5.1}$$

Definition 1.16. Let $E$ be a normed space, and take $p$ with $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$. Then $\mu_{p,n}$ is the *weak $p$–summing norm* on $E$ (at dimension $n$).

Let $E$ be a normed space. Clearly $(\mu_{p,n})$ is a power-norm based on $E$ and, for each $n \in \mathbb{N}$, we have

$$\max_{i=1,\dots,n} \|x_i\| \leqslant \mu_{p,n}(x_1, \dots, x_n) \leqslant \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \quad (x_1, \dots, x_n \in E)\,. \tag{1.5.2}$$

Also $\mu_{p_1,n}(\boldsymbol{x}) \geqslant \mu_{p_2,n}(\boldsymbol{x})$ $(\boldsymbol{x} \in E^n)$ whenever $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $n \in \mathbb{N}$. By [32, p. 26] or [55, (6.4)], it follows that, for each $p \in [1,\infty]$, $n \in \mathbb{N}$, and $x_1, \dots, x_n \in E$, we have

$$\mu_{p,n}(x_1, \dots, x_n) = \sup\left\{ \left\|\sum_{j=1}^n \zeta_j x_j\right\| : \zeta_1, \dots, \zeta_n \in \mathbb{F},\ \left(\sum_{j=1}^n |\zeta_j|^q\right)^{1/q} \leqslant 1 \right\}. \tag{1.5.3}$$

Now take $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in E'$, and set $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. Then

$$\mu_{p,n}(\boldsymbol{\lambda}) = \sup\left\{ \left(\sum_{i=1}^n |\langle x,\,\lambda_i\rangle|^p\right)^{1/p} : x \in B_E \right\}. \tag{1.5.4}$$

Let $E$ and $F$ be normed spaces, and take $n \in \mathbb{N}$ and $T \in \mathcal{B}(E,F)$. Then

$$\mu_{p,n}(T^{(n)}\boldsymbol{x}) \leqslant \mu_{p,n}(\boldsymbol{x})\,\|T\| \quad (\boldsymbol{x} \in E^n)\,. \tag{1.5.5}$$

Let $E$ be a normed space. Then the space of sequences $x = (x_i) \in E^{\mathbb{N}}$ such that

$$\|x\| := \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x_i,\, \lambda\rangle|^p\right)^{1/p} : \lambda \in B_{E'}\right\} < \infty$$

is denoted by $(\ell^{\,p}_{\text{weak}}(E), \|\cdot\|_{\ell^{\,p}_{\text{weak}}(E)})$ in [23, p. 16], by $(\ell_p^{weak}(E), \|\cdot\|_p^{weak})$ in [24, p. 32], and by $(\ell_p^w(E), \|\cdot\|_p^w)$ in [55, (6.4)].

Let $E$ be a normed space, and take $n \in \mathbb{N}$. For $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, consider the map

$$T_{\boldsymbol{x}} : (\zeta_1, \dots, \zeta_n) \mapsto \sum_{i=1}^{n} \zeta_i x_i\,, \quad \ell^{\,q}_n \to E\,.$$

Then $T_{\boldsymbol{x}} \in \mathcal{B}(\ell^{\,q}_n, E)$ and $\mu_{p,n}(\boldsymbol{x}) = \|T_{\boldsymbol{x}}\|$. The norm on $E^n$ corresponding to the injective tensor norm on $\ell^{\,p}_n \otimes E$ is denoted by $\|\cdot\|_{\varepsilon,n}$, and so, for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\begin{aligned}\|\boldsymbol{x}\|_{\varepsilon,n} = \left\|\sum_{i=1}^{n} \delta_i \otimes x_i\right\|_{\varepsilon} &= \sup\left\{\left|\sum_{i=1}^{n} \alpha_i \langle x_i,\, \lambda\rangle\right| : \left(\sum_{i=1}^{n} |\alpha_i|^q\right)^{1/q} \leqslant 1,\ \lambda \in B_{E'}\right\}\\ &= \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i,\, \lambda\rangle|^p\right)^{1/p} : \lambda \in B_{E'}\right\} = \mu_{p,n}(\boldsymbol{x})\,.\end{aligned}$$

Hence

$$\|\boldsymbol{x}\|_{\varepsilon,n} = \mu_{p,n}(\boldsymbol{x}) \quad (\boldsymbol{x} \in E^n)\,. \tag{1.5.6}$$

It follows that

$$(E^n, \mu_{p,n}) \cong \mathcal{B}(\ell^{\,q}_n, E) \cong (\ell^{\,p}_n \otimes E, \|\cdot\|_{\varepsilon,n})\,. \tag{1.5.7}$$

In the case where $E$ is a finite-dimensional normed space, we also have

$$(E^n, \mu_{p,n}) \cong \mathcal{B}(E', \ell^{\,p}_n)\,.$$

Indeed, the element $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ corresponds to the element $T'_{\boldsymbol{x}} \in \mathcal{B}(E', \ell^{\,p}_n)$, where

$$T'_{\boldsymbol{x}}(\lambda) = \lambda^{(n)}(\boldsymbol{x}) = \langle \boldsymbol{x},\, \lambda\rangle \quad (\lambda \in E')\,.$$

Let $E$ be a normed space with a closed subspace $F$, and take $p$ with $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$, and $\boldsymbol{x} \in F^n$. Then it follows immediately from the Hahn–Banach theorem that we obtain the same values for $\mu_{p,n}(\boldsymbol{x})$ whether it be evaluated with respect to $E$ or $F$. Thus 'a weak $p$–summing norm passes to subspaces', in the sense that

$$J_F^{(n)} : (F^n, \mu_{p,n}^F) \to (E^n, \mu_{p,n}^E) \tag{1.5.8}$$

is an isometry for each $n \in \mathbb{N}$; *cf.* Proposition 1.14(v). Now suppose that $1 < p < \infty$ with $p \neq 2$ and that $n \in \mathbb{N}$. Then it is not necessarily the case that the norm $\mu_{p,n}^{E/F}$ on the quotient space $(E/F)^n$ of $E^n$ is equal to the quotient of the norm $\mu_{p,n}^E$ on $E^n$; we shall show this in Example 1.30, below.

Let $E$ be a normed space, and again take $p$ with $1 \leqslant p \leqslant \infty$ and set $q = p'$. For $n \in \mathbb{N}$ and $\boldsymbol{x} \in E^n$, define

$$\nu_{p,n}(\boldsymbol{x}) = \sup\left\{|\langle \boldsymbol{x},\, \boldsymbol{\lambda}\rangle| : \boldsymbol{\lambda} \in (E')^n,\ \mu_{q,n}(\boldsymbol{\lambda}) \leqslant 1\right\}\,.$$

Then we see that $(E^n, \nu_{p,n})$ is a normed space; we write $\nu^E_{p,n}$ when it is necessary to identify the space $E$. For example,

$$\nu_{1,n}(\boldsymbol{x}) = \sum_{j=1}^{n} \|x_j\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\, n \in \mathbb{N})\,. \tag{1.5.9}$$

Clearly $(\nu_{p,n})$ is a power-norm based on $E$. The norm $\nu_{p,n}$ is the restriction to $E^n$ of the dual norm of $\mu_{q,n}$, where $\mu_{q,n}$ is defined on $(E')^n$. Since $(\ell^q_n \check{\otimes} E')' = \ell^p_n \widehat{\otimes} E''$ by (1.4.6), it follows that

$$\nu_{p,n}(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\pi,n} \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N})\,, \tag{1.5.10}$$

where $\|\cdot\|_{\pi,n}$ denotes the projective tensor norm on $\ell^p_n \otimes E$ and we are using Proposition 1.14(iv). Hence

$$(E^n, \nu_{p,n}) \cong (\ell^p_n \otimes E, \|\cdot\|_{\pi,n}) \quad (n \in \mathbb{N})\,. \tag{1.5.11}$$

Definition 1.17. Let $E$ be a normed space, and take $p$ with $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$. Then $\nu_{p,n}$ is the *dual weak $p$–summing norm* on $E$ (at dimension $n$).

Take $n \in \mathbb{N}$. Then it is clear that the dual space to $(E^n, \mu_{p,n})$ is $((E')^n, \nu_{q,n})$ and that the dual space to $(E^n, \nu_{p,n})$ is $((E')^n, \mu_{q,n})$.

It follows from equation (1.5.2) by duality that, for each $n \in \mathbb{N}$, we have

$$\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \leqslant \nu_{p,n}(x_1, \dots, x_n) \leqslant \sum_{i=1}^{n} \|x_i\| \quad (x_1, \dots, x_n \in E)\,. \tag{1.5.12}$$

Example 1.18. Let $E$ be a normed space, and take $n \in \mathbb{N}$. Then we have defined the $p$–sum norm $\|\cdot\|_{\ell^p_n(E)}$ on the space $E^n$ in Definition 1.7. As in §1.4, we identify $\ell^p_n \otimes E$ with $E^n$, and so we obtain a norm on $\ell^p_n \otimes E$ corresponding to the $p$–sum norm. It follows from (1.5.6), (1.5.2), (1.5.12), (1.5.10) that

$$\|\boldsymbol{x}\|_{\varepsilon,n} = \mu_{p,n}(\boldsymbol{x}) \leqslant \|\boldsymbol{x}\|_{\ell^p_n(E)} \leqslant \nu_{p,n}(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\pi,n} \quad (\boldsymbol{x} \in E^n)\,, \tag{1.5.13}$$

and so, by Proposition 1.12, $\|\cdot\|_{\ell^p_n(E)}$ defines a reasonable cross-norm on $\ell^p_n \otimes E$.

(In fact, it follows from [55, (6.9)] that

$$d_p(z) \leqslant \|z\|_{\ell^p_n(E)} \leqslant g_p(z) \quad (z \in \ell^p_n \otimes E)\,,$$

where $d_p$ and $g_p$ denote certain 'Chevet–Saphar tensor norms'.)

Now suppose that $E$ and $F$ are normed spaces and that $T \in \mathcal{B}(E, F)$. Also in §1.4, we identified the operator $I_n \otimes T : \ell^p_n \otimes E \to \ell^p_n \otimes F$ with the $n^{\text{th}}$ amplification $T^{(n)}$ of $T$. It is clear from the definitions that

$$\|I_n \otimes T : \ell^p_n \otimes E \to \ell^p_n \otimes F\| = \left\|T^{(n)} : \ell^p_n(E) \to \ell^p_n(F)\right\| = \|T\|\,. \tag{1.5.14}$$

In the language of §3.1, this will say that $T$ is a multi-bounded operator with respect to the $p$–sum norms based on $E$ and $F$, respectively, and that $\|T\|_{mb} = \|T\|$. ■

Proposition 1.19. *Let $E$ and $F$ be normed spaces, and take $T \in \mathcal{B}(E,F)$ and $n \in \mathbb{N}$. Then*

$$\left\| T^{(n)} : (E^n, \mu_{p,n}^E) \to (F^n, \mu_{p,n}^F) \right\| = \left\| T^{(n)} : (E^n, \nu_{p,n}^E) \to (F^n, \nu_{p,n}^F) \right\| = \|T\| .$$

*Proof.* Recall that we are identifying the $n^{\text{th}}$ amplification $T^{(n)}$ of $T$ with the operator $I_n \otimes T : \ell_n^p \otimes E \to \ell_n^p \otimes F$. By Theorem 1.13, $\|I_n \otimes_\pi T\| = \|I_n \otimes_\varepsilon T\| = \|T\|$. The result now follows from the identifications of the weak $p$–summing norm and the dual weak $p$–summing norm in (1.5.6) and (1.5.10), respectively. ∎

The next result follows from Proposition 1.14(i) and equation (1.5.11).

Proposition 1.20. *Let $F$ be a closed subspace of a Banach space $E$, let $Q_F : E \to E/F$ be the quotient map, and take $p$ with $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$. Then*

$$Q_F^{(n)} : (E^n, \nu_{p,n}^E) \to ((E/F)^n, \nu_{p,n}^{E/F})$$

*is a quotient operator.* ∎

Take $F$ to be a closed subspace of a Banach space $E$, and suppose that $1 < p < \infty$ with $p \neq 2$ and that $n \in \mathbb{N}$. Then it is not necessarily the case that the norm $\nu_{p,n}^F$ on the subspace $F^n$ of $E^n$ is equal to the restriction to $F^n$ of the norm $\nu_{p,n}^E$ on $E^n$. We shall also show this in Example 1.30, below.

**1.6. Subspaces and subquotients of $L^p$-spaces.** Let $(\Omega, \mu)$ be a measure space, and take $p$ with $1 \leqslant p \leqslant \infty$. We write $L^p(\Omega, \mu)$ or $L^p(\Omega)$ for the usual Banach space of scalar-valued, $p$–integrable (with respect to the measure $\mu$) functions. In particular, we write $L^p(\mathbb{I})$ for the usual space of $p$–integrable (with respect to Lebesgue measure) functions on $\mathbb{I}$. Again we write $L^p(\Omega, \mu, \mathbb{R})$ or $L^p(\Omega, \mu, \mathbb{C})$, etc., when necessary.

We shall need some results which determine the Banach spaces that are either subspaces or subquotients of Banach spaces of this form, and we summarize the story here. Following Pisier in [51], we make the following definition.

Definition 1.21. Take $p$ with $1 \leqslant p \leqslant \infty$. Then the class of Banach spaces that are subquotients of Banach spaces of the form $L^p(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space, is denoted by $SQ(p)$.

Each Banach space $E$ is a quotient of a space of the form $\ell^1(\Gamma)$, and so $SQ(1)$ is the class of all Banach spaces. Set $B = B_{E'}$. Then the map $x \mapsto \kappa_E(x) \mid B,\ E \to \ell^\infty(B)$, is an isometric embedding, and so $SQ(\infty)$ is the class of all Banach spaces. Also $SQ(2)$ is the class of all Hilbert spaces. Let $E$ be a Banach space. Then clearly $E' \in SQ(p')$ if and only if $E \in SQ(p)$.

The first result is standard; see [2, Theorem 6.4.19 and Proposition 11.1.9], for example. (The result is stated just for real Banach spaces in these sources; the result for complex Banach spaces follows easily.) An early source for the final clause is a paper of Dor [26, Theorem 2.1].

Proposition 1.22. (i) *Suppose that $1 \leqslant p \leqslant 2$ and $1 \leqslant r < \infty$. Then $\ell^r$ and $L^r(\mathbb{I})$ each embed in $L^p(\mathbb{I})$ if and only if $p \leqslant r \leqslant 2$.*

(ii) *Suppose that* $2 < p < \infty$ *and* $1 \leqslant r < \infty$*. Then* $\ell^r$ *and* $L^r(\mathbb{I})$ *each embed in* $L^p(\mathbb{I})$ *if and only if* $r = 2$ *or* $r = p$*.*

*Moreover, in both cases,* $\ell^r$ *and* $L^r(\mathbb{I})$ *embed isometrically in* $L^p(\mathbb{I})$ *whenever they embed in* $L^p(\mathbb{I})$*.* ■

COROLLARY 1.23. *Suppose that* $2 \leqslant p \leqslant \infty$ *and* $1 \leqslant r < \infty$*. Then* $\ell^r$ *and* $L^r(\mathbb{I})$ *are each isometrically isomorphic to a quotient of* $L^p(\mathbb{I})$ *if and only if* $2 \leqslant r \leqslant p$*.*

*Proof.* Set $q = p'$ and $s = r'$. Suppose that $2 \leqslant r \leqslant p$. Then $1 \leqslant q \leqslant s \leqslant 2$, and so, by Proposition 1.22(i), $\ell^s$ and $L^s(\mathbb{I})$ embed isometrically in $L^q(\mathbb{I})$. Hence $\ell^r$ and $L^r(\mathbb{I})$ are isometrically isomorphic to a quotient of $L^p(\mathbb{I})$. The converse is similar. ■

It follows from Proposition 1.22(i) and Corollary 1.23 that the space $\ell^r$ is a subquotient of $L^p(\mathbb{I})$ whenever $1 \leqslant p \leqslant r \leqslant 2$ or $2 \leqslant r \leqslant p \leqslant \infty$.

Although it is not strictly relevant to our work, we note that, for each $r, p \in (1, \infty)$, the space $\ell^r$ embeds in $L^p(\mathbb{I})$ as a complemented subspace if and only if $r = p$ or $r = 2$ [2, Theorem 6.4.21].

We shall also use the following result from [2, Theorem 11.1.8].

PROPOSITION 1.24. *Take* $p$ *with* $1 \leqslant p < \infty$*. Then each separable Banach space that is finitely representable in* $\ell^p$ *is isometrically isomorphic to a closed subspace of* $L^p(\mathbb{I})$*.* ■

We next give in Theorem 1.26 a more general version of Proposition 1.22. We shall use the following remark. Take $p$ with $1 \leqslant p < \infty$, let $(\Omega, \mu)$ be a measure space, and suppose that $E$ is a closed, separable subspace of $L^p(\Omega, \mu)$. Then it is easy to see that $E$ embeds in a space $L^p(\Sigma, \nu)$, where $(\Sigma, \nu)$ is a measure space and $\nu$ is $\sigma$-finite, whence $L^p(\Sigma, \nu)$ is separable. By [33, p. 15] and by [37, p. 128], each infinite-dimensional, separable space of the form $L^p(\Sigma, \nu)$ is isometrically isomorphic to either $\ell^p$ or to $L^p(\mathbb{I})$ or to $\ell^p \oplus_p L^p(\mathbb{I})$ or to $\ell^p_n \oplus_p L^p(\mathbb{I})$ for some $n \in \mathbb{N}$, and hence embeds isometrically in $L^p(\mathbb{I})$.

The first result is close to [2, Proposition 11.17].

PROPOSITION 1.25. *Let* $\Omega$ *be a measure space, and take* $r$ *with* $1 \leqslant r < \infty$*. Then* $L^r(\Omega)$ *is finitely representable in* $\ell^r$*.*

*Proof.* Take a finite-dimensional subspace $X$ of $L^r(\Omega)$ and take $\varepsilon > 0$, say $\{x_1, \dots, x_m\}$ is a basis for $X$, where $m \in \mathbb{N}$. We approximate each $x_i$ by a simple function $f_i$ in $L^r(\Omega)$ in such a way that the linear operator $T : X \to L^r(\Omega)$ with $Tx_i = f_i \ (i \in \mathbb{N}_m)$ is an isomorphism onto $F := \mathrm{lin}\{f_1, \dots, f_m\}$ with $\|T\| \left\|T^{-1}\right\| < 1 + \varepsilon$. Take $\{A_1, \dots, A_n\}$ to be a measurable partition of $\Omega$ such that each function $f_i$ is constant on each set $A_j$, and set $G = \mathrm{lin}\{\chi_{A_1}, \dots, \chi_{A_n}\}$. Then $F \subset G$ and $G \cong \ell^r_n$. We conclude that there is a finite-dimensional subspace $Y$ of $\ell^r$ such that $d(X, Y) < 1 + \varepsilon$, as required. ■

THEOREM 1.26. *Let* $\Omega$ *be a measure space, and take* $p$ *with* $1 \leqslant p < \infty$*.*

(i) *Suppose that* $r$ *is such that* $1 \leqslant p \leqslant r \leqslant 2$ *or that* $p > 2$ *and* $r = 2$ *or* $r = p$*. Then the space* $L^r(\Omega)$ *is finitely representable in* $\ell^p$ *and there is a measure space* $\Sigma$ *such that* $L^r(\Omega)$ *is isometrically isomorphic to a closed subspace of* $L^p(\Sigma)$*.*

(ii) *Suppose that* $2 < p < \infty$ *and* $1 \leqslant r < \infty$ *with* $r \neq 2$ *and* $r \neq p$. *Then* $\ell^r$ *is not isomorphic to a closed subspace of* $L^p(\Omega)$.

(iii) *Suppose that* $1 < p \leqslant r \leqslant 2$ *or* $2 \leqslant r \leqslant p < \infty$. *Then* $L^r(\Omega)$ *belongs to the class* $SQ(p)$.

*Proof.* (i) By Proposition 1.25, $L^r(\Omega)$ is finitely representable in $\ell^r$. By Proposition 1.22, $\ell^r$ embeds isometrically in $L^p(\mathbb{I})$. Again by Proposition 1.25, $L^p(\mathbb{I})$ is finitely representable in $\ell^p$. Thus $L^r(\Omega)$ is finitely representable in $\ell^p$.

By [24, Corollary 8.14(a)], there is a measure space $\Sigma$ such that $L^r(\Omega)$ is isometrically isomorphic to a closed subspace of $L^p(\Sigma)$.

(ii) By Proposition 1.22(ii), $\ell^r$ is not isomorphic to a subspace of $L^p(\mathbb{I})$, and so the result follows from our preliminary remark.

(iii) The case where $1 \leqslant p \leqslant r \leqslant 2$ is covered in (i); the case where $2 \leqslant r \leqslant p < \infty$ follows by duality. ∎

The following theorem implies that $\ell^r$ is isomorphic to a member of the class $SQ(p)$ if and only if $r$ lies between 2 and $p$; it is surely well-known, but we have not found an explicit statement in the literature.

THEOREM 1.27. *Take* $p$ *and* $r$ *with* $1 \leqslant p < \infty$ *and* $1 \leqslant r < \infty$, *and suppose that either* $1 < p \leqslant 2$ *and* $r \notin [p, 2]$ *or* $2 \leqslant p < \infty$ *and* $r \notin [2, p]$.

(i) *For each* $C > 0$, *there exists* $n \in \mathbb{N}$ *such that* $\ell^r_n$ *is not* $C$*–isomorphic to a space in the class* $SQ(p)$.

(ii) *For each measure space* $\Omega$ *such that* $L^r(\Omega)$ *is an infinite-dimensional space, the space* $L^r(\Omega)$ *is not isomorphic to a space in the class* $SQ(p)$.

*Proof.* By duality, it suffices to prove the theorem in the case where $2 \leqslant p < \infty$ and $r \notin [2, p]$, and so we suppose that this is the case.

(i) Assume to the contrary that, for some $C > 0$ and each $n \in \mathbb{N}$, there is an $n$–dimensional subspace $E_n$ of a quotient of the space $L^p(\Sigma)$ with $d(E_n, \ell^r_n) \leqslant C$. By [62, II.E.8] and [40, Corollary 5], respectively, we have

$$d(\ell^r_n, \ell^2_n) = n^{|1/2-1/r|} \quad \text{and} \quad d(E_n, \ell^2_n) \leqslant n^{|1/2-1/p|} \tag{1.6.1}$$

for each $n \in \mathbb{N}$. (Again, the results quoted are given for real-valued spaces, but they extend easily to complex-valued spaces.)

First suppose that $p < r < \infty$. Then, by (1.3.12) and (1.6.1), we have

$$n^{1/2-1/r} \leqslant Cn^{1/2-1/p} \quad (n \in \mathbb{N}),$$

and so $n^{1/p-1/r} \leqslant C \ (n \in \mathbb{N})$, a contradiction.

Next suppose that $1 \leqslant r < 2$. Then we *claim* that

$$d(E_n, \ell^r_n) \geqslant cn^{1/r-1/2} \quad (n \in \mathbb{N}) \tag{1.6.2}$$

for some $c > 0$. Indeed, take $n \in \mathbb{N}$ and closed subspaces $X$ and $Y$ of $L^p(\Sigma)$ such that $Y \subset X \subset L^p(\Sigma)$ and $\dim(X/Y) = n$, with quotient map $Q : X \to X/Y$, and take a contractive isomorphism $T : \ell_n^r \to X/Y$, with inverse $S : X/Y \to \ell_n^r$. We again write $\{\delta_1, \dots, \delta_n\}$ for the canonical basis of $\ell_n^r$. For each $i \in \mathbb{N}_n$, there exists $x_i \in X$ with $\|x_i\| \leqslant 2$ and $Q(x_i) = T\delta_i$. In the following sums, $\varepsilon_1, \dots, \varepsilon_n$ range over all choices of $\pm 1$. We have

$$\frac{1}{2^n}\left\|\sum \varepsilon_i x_i\right\|_X \geqslant \frac{1}{2^n}\left\|\sum \varepsilon_i Q(x_i)\right\|_{X/Y} \geqslant \frac{1}{\|S\|}\frac{1}{2^n}\left\|\sum \varepsilon_i \delta_i\right\|_{\ell_n^r} = \frac{n^{1/r}}{\|S\|}\,.$$

On the other hand, the space $L^p(\Sigma)$ is of type 2 because $p \geqslant 2$ [62, III.A.23], and so there is a constant $M > 0$ such that

$$\frac{1}{2^n}\left\|\sum \varepsilon_i x_i\right\|_X \leqslant M\left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2} \leqslant 2Mn^{1/2}\,.$$

Thus $\|S\| \geqslant cn^{1/r-1/2}$, where $c = 1/2M$, and so $d(E_n, \ell_n^r) \geqslant cn^{1/r-1/2}$ $(n \in \mathbb{N})$, giving the claim (1.6.2).

It follows that $cn^{1/r-1/2} \leqslant C$ $(n \in \mathbb{N})$, again a contradiction.

(ii) Let $\Omega$ be a measure space such that $L^r(\Omega)$ is an infinite-dimensional space, and assume towards a contradiction that $L^r(\Omega)$ is isomorphic to a subquotient $E$ of $L^p(\Sigma)$, where $\Sigma$ is a measure space, say $d(E, L^r(\Omega)) = C$. For each $n \in \mathbb{N}$, the space $\ell_n^r$ is isometrically isomorphic to a closed subspace of $L^r(\Omega)$, and so there is an $n$–dimensional subspace $E_n$ of a quotient of the space $L^p(\Sigma)$ with $d(E_n, \ell_n^r) \leqslant C$. However, by (i), this is not the case for some $n \in \mathbb{N}$, giving the required contradiction.

Thus $L^r(\Omega)$ is not isomorphic to a subquotient of $L^p(\Sigma)$ for any measure space $\Sigma$. ∎

We now present a result about uncomplemented subspaces of the spaces $\ell^p$ that we shall use.

THEOREM 1.28. *Take $p$ with $1 \leqslant p < \infty$ and $p \neq 2$. Then there is a closed subspace $F$ of $\ell^p$ such that $F$ is isomorphic to $\ell^p$ and $F$ is not complemented in $\ell^p$.*

*Proof.* In the case where $p = 1$, this is a theorem of Bourgain [10]. In the two cases where $1 < p < 2$ and $2 < p < \infty$, this is [5, Theorem 3.1] and [54, Corollary to Theorem 6], respectively. ∎

COROLLARY 1.29. *Take $p$ with $1 < p < \infty$ and $p \neq 2$. Then there are a constant $C > 0$, a closed, uncomplemented subspace $F$ of $\ell^p$, and an increasing sequence $(F_n)$ of subspaces of $F$ such that* $\dim F_n = n$, *$d(F_n, \ell_n^p) \leqslant C$, and $\lambda(F_n, F) \leqslant C$ for each $n \in \mathbb{N}$, and further such that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in $F$ and* $\lim_{n\to\infty} \lambda(F_n, \ell^p) = \infty$.

*Proof.* By Theorem 1.28, there is a closed subspace $F$ of $\ell^p$ such that $F$ is not complemented in $\ell^p$ and $F \sim \ell^p$, say $T : \ell^p \to F$ is the specified isomorphism. Set $F_n = T(\ell_n^p)$ $(n \in \mathbb{N})$. We see that $\dim F_n = n$ $(n \in \mathbb{N})$ and that there exists $C > 0$ such that $d(F_n, \ell_n^p) \leqslant C$ and $\lambda(F_n, F) \leqslant C$ for each $n \in \mathbb{N}$, and also that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in $F$. It remains to show that $\lim_{n\to\infty} \lambda(F_n, \ell^p) = \infty$.

Assume towards a contradiction that there is a strictly increasing sequence $(n_k)$ in $\mathbb{N}$ such that each space $F_{n_k}$ is complemented in $\ell^p$ by a projection, say $Q_k \in \mathcal{B}(\ell^p)$, and that $\sup\{\|Q_k\| : k \in \mathbb{N}\} < \infty$. Set $q = p'$, so that $1 < q < \infty$. The space $\mathcal{B}(\ell^p)$ is the dual of the space $G := \ell^p \widehat{\otimes} \ell^q$, and so the sequence $(Q_k)$ has an accumulation point, say $Q$, with respect to the weak* topology $\sigma(\mathcal{B}(\ell^p), G)$ on $\mathcal{B}(\ell^p)$.

Take $f \in \ell^p$. We first *claim* that $Qf \in F$. For otherwise there exists $\lambda \in \ell^q$ such that $\langle Qf, \lambda\rangle = 1$ and $\langle g, \lambda\rangle = 0$ $(g \in F)$. However $\langle Qf, \lambda\rangle = \lim_\alpha \langle g_\alpha, \lambda\rangle = 0$ for a subnet $(g_\alpha)$ of $(Q_k f)$, a contradiction. Thus $Qf \in F$, as claimed.

We next *claim* that $Qf = f$ $(f \in F)$. Indeed, first suppose that $f \in F_{n_k}$ for some $k \in \mathbb{N}$. Then $Q_j f = f$ for each $j \in \mathbb{N}$ with $j \geqslant k$, and so $\langle Qf, \lambda\rangle = \langle f, \lambda\rangle$ for each $\lambda \in \ell^q$, whence $Qf = f$. Since $\bigcup\{F_{n_k} : k \in \mathbb{N}\}$ is dense in $F$, the second claim follows.

We have shown that $Q \in \mathcal{B}(\ell^p)$ is a projection onto $F$, a contradiction of the fact that $F$ is not complemented in $\ell^p$. Thus we conclude that $\lim_{n\to\infty} \lambda(F_n, \ell^p) = \infty$. ∎

A similar result to the above can be obtained in the case where $p = 1$ from results in [10] by somewhat different methods. As we shall not use the case where $p = 1$, we do not provide a proof of this remark.

Example 1.30. Suppose that $F$ is a closed subspace of a Banach space $E$, with the embedding $J_F : F \to E$ and quotient map $Q_F : E \to E/F$. Take $p$ with $1 < p < \infty$ and $p \neq 2$, and take $n \in \mathbb{N}$. Then, as we remarked, it is not necessarily the case that the weak $p$–summing norm $\mu_{p,n}^{E/F}$ on the quotient space $E^n/F^n = (E/F)^n$ is equal to the quotient of the weak $p$–summing norm $\mu_{p,n}^E$ on $E^n$ or that the dual weak $p$–summing norm $\nu_{p,n}^F$ on the subspace $F^n$ of $E^n$ is equal to the restriction to $F^n$ of the dual weak $p$–summing norm $\nu_{p,n}^E$ on $E^n$. Further the relevant norms are not always uniformly equivalent as $n$ varies. Here we present examples to substantiate these remarks.

Denote by $\overline{\mu}_{p,n}^E$ the quotient norm on $(E/F)^n$ of the norm $\mu_{p,n}^E$ on $E^n$. Then we do have

$$\mu_{p,n}^{E/F}(\boldsymbol{x} + F^n) \leqslant \overline{\mu}_{p,n}^E(\boldsymbol{x} + F^n) \quad (\boldsymbol{x} \in E^n)\,, \quad \nu_{p,n}^E(\boldsymbol{x}) \leqslant \nu_{p,n}^F(\boldsymbol{x}) \quad (\boldsymbol{x} \in F^n)\,,$$

and so, for each $n \in \mathbb{N}$, the norms $\mu_{p,n}^{E/F}$ and $\overline{\mu}_{p,n}^E$ are equivalent on $(E/F)^n$ and the norms $\nu_{p,n}^E$ and $\nu_{p,n}^F$ are equivalent on $F^n$. However we shall show that we do not always have uniform equivalence (in $n$) in the two cases.

Set $q = p'$, so that $1 < q < \infty$ and $q \neq 2$, and consider the special case where $E = \ell^q$. By Corollary 1.29, there are a constant $C > 0$, a closed subspace $F$ of $E$, and an increasing sequence $(F_n)$ of subspaces of $F$ such that $\dim F_n = n$, such that $d(F_n, \ell_n^q) \leqslant C$, and such that $\lambda(F_n, F) \leqslant C$ for each $n \in \mathbb{N}$, and further such that $\lim_{n\to\infty} \lambda(F_n, E) = \infty$. For each $n \in \mathbb{N}$, take a projection $Q_n$ of $F$ onto $F_n$ with $\|Q_n\| \leqslant C$, and set

$$c_n = 1/\beta(I_{\ell_n^p} \otimes_\pi J_{F_n})\,,$$

where $J_{F_n} : F_n \to E$ is the inclusion map. Thus, for each $n \in \mathbb{N}$, $c_n$ is the minimum constant such that

$$\nu_{p,n}^{F_n}(\boldsymbol{x}) \leqslant c_n \nu_{p,n}^E(\boldsymbol{x}) \quad (\boldsymbol{x} \in F_n^n)\,.$$

Since $d(F_n, \ell_n^q) \leqslant C$, we have $d(F_n', \ell_n^p) \leqslant C$, and so there is an isomorphism $T_n : F_n' \to \ell_n^p$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \leqslant C$.

Let us combine the commutative diagram

$$\begin{tikzcd} \ell_n^p \widehat{\otimes} F_n \arrow[r, "I_{\ell_n^p} \otimes_\pi J_{F_n}"] \arrow[d, "T_n^{-1} \otimes_\pi I_{F_n}"'] & \ell_n^p \widehat{\otimes} E \\ F_n' \widehat{\otimes} F_n \arrow[r, "I_{F_n'} \otimes_\pi J_{F_n}"] & F_n' \widehat{\otimes} E \arrow[u, "T_n \otimes_\pi I_E"'] \end{tikzcd}$$

with equation (1.3.13) (which applies because $T_n^{-1} \otimes_\pi I_{F_n}$ is an isomorphism), with Theorem 1.13, and with Proposition 1.15. Then we conclude that

$$\frac{1}{c_n} = \beta(I_{\ell_n^p} \otimes_\pi J_{F_n}) \leqslant C\beta(I_{F_n'} \otimes_\pi J_{F_n}) = \frac{C}{\lambda(F_n, E)} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Thus there is a sequence $(\boldsymbol{x}_n)$ such that $\boldsymbol{x}_n \in \ell_n^p \otimes F_n$ with $\nu_{p,n}^{F_n}(\boldsymbol{x}_n) = 1$ for each $n \in \mathbb{N}$ and such that $\nu_{p,n}^E(\boldsymbol{x}_n) \to 0$ as $n \to \infty$.

We now regard $\boldsymbol{x}_n$ as an element of the subspace $F_n^n$ of $F^n$ for $n \in \mathbb{N}$ and use Proposition 1.19 to conclude that

$$\nu_{p,n}^{F_n}(\boldsymbol{x}_n) = \nu_{p,n}^{F_n}(Q_n^{(n)}(\boldsymbol{x}_n)) \leqslant C\nu_{p,n}^F(\boldsymbol{x}_n)\,,$$

and hence that

$$\beta\left(J_F^{(n)} : (F^n, \nu_{p,n}^F) \to (E^n, \nu_{p,n}^E)\right) \to 0 \quad \text{as} \quad n \to \infty\,, \tag{1.6.3}$$

an equation that we shall refer to later.

Recall that $F$ is a closed subspace of $E = \ell^{\,q}$. Since $\ell^{\,p}$ has the approximation property, Theorem 1.13 implies that the map

$$I_{\ell^{\,p}} \otimes_\pi J_F : \ell^{\,p} \widehat{\otimes} F \to \ell^{\,p} \widehat{\otimes} \ell^{\,q}$$

is an injection. However it follows from equation (1.6.3) that it is not an embedding.

Let $U : F' \to E'/F^\perp$ be the inverse of the isometric isomorphism $\overline{J_F'}$ induced by $J_F' : E' \to F'$ as in (1.3.6). Take $n \in \mathbb{N}$, and write $\overline{\mu}_{q,n}^{E'}$ for the quotient norm on the space $(E'/F^\perp)^n$ of the norm $\mu_{q,n}^{E'}$ on $(E')^n$. Then we have a commutative diagram

$$\begin{tikzcd} (E^n, \nu_{p,n}^E)' \arrow[r, "(J_F^{(n)})'"] \arrow[d, equal] & (F^n, \nu_{p,n}^F)' \arrow[d, equal] \\ ((E')^n, \mu_{q,n}^{E'}) \arrow[r, "(J_F')^{(n)}"] \arrow[d, "Q_{F^\perp}^{(n)}"'] & ((F')^n, \mu_{q,n}^{F'}) \arrow[dl, "U^{(n)}"'] \arrow[d, "U^{(n)}"] \\ ((E'/F^\perp)^n, \overline{\mu}_{q,n}^{E'}) & ((E'/F^\perp)^n, \mu_{q,n}^{E'/F^\perp})\,. \arrow[l, "I_{E'/F^\perp}^{(n)}"'] \end{tikzcd}$$

Set

$$d_n = \|I_{E'/F^\perp}^{(n)} : ((E'/F^\perp)^n, \mu_{q,n}^{E'/F^\perp}) \to ((E'/F^\perp)^n, \overline{\mu}_{q,n}^{E'})\|\,,$$

so that $d_n$ is the minimum constant such that

$$\overline{\mu}_{q,n}^{E'}(\boldsymbol{\lambda}+(F^\perp)^n) \leqslant d_n\, \mu_{q,n}^{E'/F^\perp}(\boldsymbol{\lambda}+(F^\perp)^n) \quad (\boldsymbol{\lambda}\in(E')^n)\,.$$

Since $U$ is an isometric isomorphism, Proposition 1.19 (applied to $U$ and its inverse) implies that $U^{(n)}$ is an isometric isomorphism of $((F')^n,\mu_{q,n}^{F'})$ onto $((E'/F^\perp)^n,\mu_{q,n}^{E'/F^\perp})$. Hence, using the above diagram, we see that

$$\begin{aligned} d_n &= \|U^{(n)} : ((F')^n,\mu_{q,n}^{F'}) \to ((E'/F^\perp)^n,\overline{\mu}_{q,n}^{E'})\| \\ &= r((J_F')^{(n)}) = r((J_F^{(n)})') = \frac{1}{\beta(J_F^{(n)})} \to \infty \quad \text{as} \quad n\to\infty\,, \end{aligned} \tag{1.6.4}$$

using (1.3.7), (1.3.14), and (1.6.3). This shows that the norms $\mu_{q,n}^{E'/F^\perp}$ and $\overline{\mu}_{q,n}^{E'}$ on the space $(E'/F^\perp)^n$ are not uniformly equivalent as $n$ varies. ∎

**1.7. Schechtman's space.** In this section, we give a result about quotients of the spaces $L^p(\mathbb{I})$, where $1\leqslant p<\infty$; in the case where $1<p<2$, the result seems to be new, and may have independent interest.

We first describe some Banach spaces $Z_p$ and $S_p$ for $p>1$ that arose in the paper [57] of Schechtman, where a somewhat different notation was used.

Definition 1.31. Take $p$ with $1<p<\infty$. Then $Z_p$ is the Banach space $\ell^{\,p}(\ell^{\,2})$.

Let $\mathbb{M}_\infty$ denote the linear space of all scalar-valued $\mathbb{N}\times\mathbb{N}$–matrices. We may consider the Banach space $Z_p=\ell^{\,p}(\ell^{\,2})$ for $1<p<\infty$ to be a subspace of $\mathbb{M}_\infty$ in the following way. Given $a\in Z_p$, we have $a=(a_j : j\in\mathbb{N})$, where $a_j\in\ell^{\,2}$ $(j\in\mathbb{N})$ with

$$\|a\|_{Z_p} = \left(\sum_{j=1}^\infty \|a_j\|_{\ell^2}^p\right)^{1/p} < \infty\,;$$

we set $a_j=(\alpha_{i,j} : i\in\mathbb{N})$ for $j\in\mathbb{N}$, and identify $a$ with $(\alpha_{i,j})\in\mathbb{M}_\infty$, so that $a_j$ is the $j^{\text{th}}$ column of the matrix $(\alpha_{i,j})$. For later reference, we note that

$$\|(\alpha_{i,j})\|_{Z_p} = \left(\sum_{j=1}^\infty\left(\sum_{i=1}^\infty|\alpha_{i,j}|^2\right)^{p/2}\right)^{1/p}. \tag{1.7.1}$$

The dual space of $Z_p$ is $Z_q$, where $q=p'$; the duality bracket is given by

$$\langle\, (\alpha_{i,j}),\, (\beta_{i,j})\,\rangle = \sum_{i,j=1}^\infty \alpha_{i,j}\beta_{i,j} \quad ((\alpha_{i,j})\in Z_p,\, (\beta_{i,j})\in Z_q)\,.$$

For $a=(\alpha_{i,j})\in\mathbb{M}_\infty$, let $a^t=(\alpha_{j,i})\in\mathbb{M}_\infty$ denote its *transpose*, and consider the subspace

$$S_p=\{b+c^t : b,c\in Z_p\}$$

of $\mathbb{M}_\infty$ and the linear surjection

$$T:(b,c)\mapsto b+c^t\,,\quad Z_p\oplus_1 Z_p\to S_p\,.$$

The kernel of $T$ is clearly a closed subspace of the Banach space $Z_p \oplus_1 Z_p$, and we give $S_p$ the quotient norm, so that

$$\|a\|_{S_p} = \inf\{\|b\|_{Z_p} + \|c\|_{Z_p} : b, c \in Z_p,\ a = b + c^t\} \quad (a \in S_p)\,.$$

Thus $(S_p, \|\cdot\|_{S_p})$ is a Banach space; further, $\|a\|_{S_p} = \|a^t\|_{S_p} \ \ (a \in S_p)$.

In the next lemma, we use 'matrix units' $e_{ij} \in \mathbb{M}_\infty$ for $i, j \in \mathbb{N}$, where $e_{ij}(r,s) = 1$ when $(r,s) = (i,j)$ and $e_{ij}(r,s) = 0$ when $(r,s) \neq (i,j)$, and consider matrices $(\alpha_{i,j})$ with only finitely-many non-zero entries, writing the matrix as $\sum_{i,j} \alpha_{i,j} e_{ij}$. For example, for each sequence $(\alpha_j) \in c_{00}$ and $i \in \mathbb{N}$, the elements

$$\sum_{j=1}^\infty \alpha_j e_{ij} \quad \text{and} \quad \sum_{j=1}^\infty \alpha_j e_{ji}$$

correspond to the $i^{\text{th}}$ row and $i^{\text{th}}$ column, respectively, of $\mathbb{M}_\infty$.

LEMMA 1.32. *Take $p$ with $1 < p < 2$, and suppose that $(\alpha_j) \in c_{00}$. Then:*

(i) *for each $i \in \mathbb{N}$, we have*

$$\left\|\sum_{j=1}^\infty \alpha_j e_{ij}\right\|_{S_p} = \left\|\sum_{j=1}^\infty \alpha_j e_{ji}\right\|_{S_p} = \left(\sum_{j=1}^\infty |\alpha_j|^2\right)^{1/2};$$

(ii) *for each strictly increasing sequences $(i_k)$ and $(j_k)$ in $\mathbb{N}$, we have*

$$\left\|\sum_{k=1}^\infty \alpha_k e_{i_k, j_k}\right\|_{S_p} = \left(\sum_{k=1}^\infty |\alpha_k|^p\right)^{1/p}.$$

*Proof.* (i) Take $i \in \mathbb{N}$.

First consider the row $a = \sum_{j=1}^\infty \alpha_j e_{ij}$, an element of $Z_p \subset S_p \subset \mathbb{M}_\infty$. Then

$$\|a\|_{S_p} = \|a^t\|_{S_p} \leqslant \|a^t\|_{Z_p} = \left(\sum_{j=1}^\infty |\alpha_j|^2\right)^{1/2},$$

where the final equality follows from (1.7.1).

Conversely, given $\varepsilon > 0$, take $b, c \in Z_p$ such that $a = b + c^t$ and

$$\|a\|_{S_p} \geqslant \|b\|_{Z_p} + \|c\|_{Z_p} - \varepsilon\,,$$

say $b = (\beta_{r,s})$ and $c = (\gamma_{r,s})$ as elements of $\mathbb{M}_\infty$. Then $\alpha_j = \beta_{i,j} + \gamma_{j,i}$ $(j \in \mathbb{N})$, so that, by the sub-additivity of the $\ell^2$-norm, we obtain

$$\begin{aligned}
\left(\sum_{j=1}^\infty |\alpha_j|^2\right)^{1/2} &\leqslant \left(\sum_{j=1}^\infty |\beta_{i,j}|^2\right)^{1/2} + \left(\sum_{j=1}^\infty |\gamma_{j,i}|^2\right)^{1/2} \\
&\leqslant \left(\sum_{j=1}^\infty |\beta_{i,j}|^p\right)^{1/p} + \left(\sum_{j=1}^\infty |\gamma_{j,i}|^2\right)^{1/2} \\
&= \left\|\sum_{j=1}^\infty \beta_{i,j} e_{ij}\right\|_{Z_p} + \left\|\sum_{j=1}^\infty \gamma_{j,i} e_{ji}\right\|_{Z_p} \\
&\leqslant \|b\|_{Z_p} + \|c\|_{Z_p} \leqslant \|a\|_{S_p} + \varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\left(\sum_{j=1}^{\infty} |\alpha_j|^2\right)^{1/2} \leqslant \|a\|_{S_p}$.

The claimed equality follows.

(ii) Set $a = \sum_{k=1}^{\infty} \alpha_k e_{i_k,j_k}$. Then

$$\|a\|_{S_p} \leqslant \|a\|_{Z_p} = \left(\sum_{k=1}^{\infty} |\alpha_k|^p\right)^{1/p} .$$

Conversely, given $\varepsilon > 0$, again take $b = (\beta_{r,s})$ and $c = (\gamma_{r,s})$ in $Z_p$ such that $a = b + c^t$ and

$$\|a\|_{S_p} \geqslant \|b\|_{Z_p} + \|c\|_{Z_p} - \varepsilon .$$

Then $\alpha_k = \beta_{i_k,j_k} + \gamma_{j_k,i_k}$ $(k \in \mathbb{N})$, so that, by the sub-additivity of the $\ell^p$-norm, we obtain

$$\left(\sum_{k=1}^{\infty} |\alpha_k|^p\right)^{1/p} \leqslant \left(\sum_{k=1}^{\infty} |\beta_{i_k,j_k}|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |\gamma_{j_k,i_k}|^p\right)^{1/p} \leqslant \|b\|_{Z_p} + \|c\|_{Z_p} \leqslant \|a\|_{S_p} + \varepsilon .$$

As before, this implies that $\left(\sum_{k=1}^{\infty} |\alpha_j|^p\right)^{1/p} \leqslant \|a\|_{S_p}$.

The claimed equality follows. ∎

Theorem 1.33. *Take $p$ with $1 < p < 2$. Then the space $S_p$ is isomorphic to a member of the class $SQ(p)$, but it is not isomorphic to a closed subspace of $L^p(\Omega)$ for any measure space $\Omega$.*

*Proof.* By Proposition 1.22(i), $\ell^2$ embeds in $L^p(\mathbb{I})$, and so $Z_p$ embeds in $L^p(\mathbb{I})$, whence $Z_p \oplus_1 Z_p$ embeds in $L^p(\mathbb{I}) \oplus_1 L^p(\mathbb{I}) \sim L^p(\mathbb{I})$. Since $S_p$ is a quotient of $Z_p \oplus_1 Z_p$, the space $S_p$ is isomorphic to a member of the class $SQ(p)$.

Assume towards a contradiction that there is an embedding $J : S_p \to L^p(\Omega)$ for some measure space $\Omega$, so that $\|Ja\|_{L^p(\Omega)} \geqslant \beta(J)\,\|a\|_{S_p}$ $(a \in S_p)$, where $\beta(J) > 0$, and set

$$f_{i,j} = Je_{ij} \in L^p(\Omega) \quad (i, j \in \mathbb{N}) .$$

It follows from Lemma 1.32(i) that the 'rows' and 'columns' of the array $(f_{i,j})$ each form a basis of the space $\ell^2$, and so it now follows from the main theorem, Theorem 1.1, in [29] that there exist strictly increasing sequences $(i_k)$ and $(j_k)$ in $\mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \left\| \sum_{k=1}^{n} f_{i_k,j_k} \right\|_{L^p(\Omega)} = 0 . \tag{1.7.2}$$

(In fact, the quoted theorem is considerably more general.) Take $n \in \mathbb{N}$. By Lemma 1.32(ii), applied with $\alpha_k = 1$ $(k \leqslant n)$ and $\alpha_k = 0$ $(k > n)$, we see that

$$n^{1/p} = \left\| \sum_{k=1}^{n} e_{i_k,j_k} \right\|_{S_p} \leqslant \frac{1}{\beta(J)} \left\| \sum_{k=1}^{n} f_{i_k,j_k} \right\|_{L^p(\Omega)} ,$$

a contradiction of (1.7.2). Thus $S_p$ is not isomorphic to a closed subspace of $L^p(\Omega)$. ∎

The following theorem will be used in Example 2.31.

THEOREM 1.34. *For each $p$ with $1 \leqslant p < \infty$ and $p \neq 2$, there is a separable Banach space in the class $SQ(p)$ that is not isomorphic to a closed linear subspace of $L^p(\Omega)$ for any measure space $\Omega$.*

*Proof.* First, suppose that $p = 1$. For each measure space $\Omega$, the Banach space $L^1(\Omega)$ has cotype 2 [2, Theorem 6.2.14(i)], and so each closed subspace of $L^1(\Omega)$ has cotype 2. The spaces $E = \ell^q$ for $q > 2$ have cotype $q$ [2, Theorem 6.2.14(ii)], and so these spaces are not isomorphic to a closed linear subspace of $L^1(\Omega)$. Certainly $E \in SQ(1)$. (Indeed, there is a quotient operator from $\ell^1$ onto $E$ [2, Theorem 2.3.1].)

Second, suppose that $p > 2$, and set $q = p'$. Take $r$ with $2 < r < p$, and set $s = r'$. By Proposition 1.22(i), $\ell^s$ embeds isometrically in $L^q(\mathbb{I})$, and hence $\ell^r$ is isometrically isomorphic to a quotient of $L^p(\mathbb{I})$. However, by Theorem 1.26(ii), $\ell^r$ is not isomorphic to a subspace of $L^p(\Omega)$ for any measure space $\Omega$.

Finally, suppose that $1 < p < 2$. Then the result follows from Theorem 1.33. ∎

**1.8. The spaces $L^p(\Omega; E)$ and $p$–spaces.** In this section, we shall define the class of '$p$–spaces'; as a preliminary, we shall recall the definition of the spaces $L^p(\Omega; E)$.

Let $(\Omega, \mu)$ be a measure space, take $p$ with $1 \leqslant p \leqslant \infty$, and suppose that $E$ is a Banach space. Then the space $L^p(\Omega; E)$ consists of the (equivalence classes of) strongly $\mu$–measurable functions $F : \Omega \to E$ such that the function $s \mapsto \|F(s)\|$ on $\Omega$ belongs to $L^p(\Omega, \mu)$; see [25]. Thus $(L^p(\Omega; E), \|\cdot\|)$ is a Banach space with respect to the norm $\|\cdot\|$ specified by

$$\|F\| = \left(\int_\Omega \|F(s)\|^p \, \mathrm{d}\mu(s)\right)^{1/p} \qquad (F \in L^p(\Omega; E))\,,$$

with $\|F\| = \operatorname{ess\,sup}\{\|F(s)\| : s \in \Omega\}$ when $p = \infty$.

The tensor product $L^p(\Omega) \otimes E$ can be identified with a dense subspace of $L^p(\Omega; E)$; indeed, the elementary tensor $f \otimes x \in L^p(\Omega) \otimes E$ corresponds to the function

$$f \otimes x : s \mapsto f(s)x\,, \quad \Omega \to E\,;$$

see [22, Chapter 7], for example. In particular, as before we shall identify $\ell^p_m \otimes E$ with $\ell^p_m(E)$ for $m \in \mathbb{N}$, so that the action of

$$S \otimes I_E : \ell^p_m \otimes E \to \ell^p_n \otimes E$$

(where $m, n \in \mathbb{N}$ and $S \in \mathbb{M}_{n,m}$) corresponds to the action of $S$ as a map from $\ell^p_m(E)$ to $\ell^p_n(E)$; this is consistent with the identification of $(x_1, \ldots, x_n) \in E^n$ with $\sum_{j=1}^n \delta_j \otimes x_j$ in $\mathbb{F}^n \otimes E$ in §1.4 and with equation (1.4.3).

Now suppose that $\Omega$ and $\Sigma$ are measure spaces and that $E$ is a Banach space, and again take $p$ with $1 \leqslant p \leqslant \infty$. For each $S \in \mathcal{B}(L^p(\Omega), L^p(\Sigma))$, there is a linear map

$$S \otimes I_E : L^p(\Omega) \otimes E \to L^p(\Sigma) \otimes E\,,$$

and we consider whether this map is bounded with respect to the relative norms from $L^p(\Omega; E)$ and $L^p(\Sigma; E)$, respectively. (We note in passing the following from [51, §1.2]: An operator $S \in \mathcal{B}(L^p(\Omega), L^p(\Sigma))$ is regular, equivalently, order-bounded (see §4.2) if and only if the above operator $S \otimes I_E$ is bounded for every Banach space $E$.)

The following definition is due to Herz [31, p. 70].

DEFINITION 1.35. Let $E$ be a Banach space, and take $p$ with $1 \leqslant p < \infty$. Then $E$ is a *p–space* whenever

$$\|S \otimes I_E\| \leqslant \|S\| \quad (S \in \mathcal{B}(L^p(\Omega), L^p(\Sigma)))$$

for all measure spaces $\Omega$ and $\Sigma$.

Further, Herz shows the following in [31, Proposition 0].

THEOREM 1.36. *Let $E$ be a Banach space, and take $p$ with $1 \leqslant p < \infty$. Then the following are equivalent:*

(a) *$E$ is a $p$–space;*

(b) $\|S : \ell^p_m(E) \to \ell^p_n(E)\| \leqslant \|S : \ell^p_m \to \ell^p_n\|$ *for each* $S \in \mathcal{B}(\ell^p_m, \ell^p_n)$ *and* $m, n \in \mathbb{N}$*;*

(c) $\|S : \ell^p_m(E) \to \ell^p_m(E)\| \leqslant \|S : \ell^p_m \to \ell^p_m\|$ *for each* $S \in \mathcal{B}(\ell^p_m)$ *and* $m \in \mathbb{N}$*;*

(d) $\|S : \ell^p(E) \to \ell^p(E)\| \leqslant \|S : \ell^p \to \ell^p\|$ *for each* $S \in \mathcal{B}(\ell^p)$*.* ∎

Herz also notes the following; they are easily seen. Take $p$ with $1 \leqslant p < \infty$. Then:

(i) each space $L^p(\Omega)$ for a measure space $\Omega$ is a $p$–space;

(ii) each closed subspace of a $p$–space is a $p$–space;

(iii) each quotient of a $p$–space by a closed subspace is a $p$–space;

(iv) the dual of a $p$–space is a $p'$–space (when $1 < p < \infty$).

It follows that each space in the class $SQ(p)$ is a $p$–space. However, Herz left open the converse to this latter statement; we shall consider this in the next section.

**1.9. Kwapień's theorem.** In this section, we shall characterize the class of $p$–spaces.

In fact, the converse to the above statement of Herz follows from a theorem of Kwapień [36, Theorem 2′]. A generalization of Kwapień's theorem is stated by Pisier in [51, Theorem 4.6]: to obtain Kwapień's result, one must take $C = 1$ and the class $\mathcal{B}$ to be just the singleton $\{\mathbb{F}\}$ in the cited reference. The theorem of Kwapień is important for this memoir and elsewhere, and the original proof is perhaps somewhat inaccessible, and so we wish to present a detailed account; our proof is based on one given by Professor Christian Le Merdy in an unpublished note, and we are grateful to him for agreeing that we could present this proof here.

First, we introduce a further definition; it uses the notation of (1.3.3).

DEFINITION 1.37. Let $(E, \|\cdot\|)$ be a normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Suppose that $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$. Then $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$ if

$$\|\langle \boldsymbol{y}, \lambda\rangle\|_{\ell^p_n} \leqslant \|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell^p_m} \quad (\lambda \in E')\,. \tag{1.9.1}$$

The condition in (1.9.1) is that

$$\left(\sum_{j=1}^{n} |\langle y_j, \lambda\rangle|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m} |\langle x_i, \lambda\rangle|^p\right)^{1/p} \quad (\lambda \in E')\,,$$

where $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$.

Let $E$ be a normed space, take $m \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_m) \in \ell^p_m(E)$, and let $z$ correspond to $\boldsymbol{x}$ in $\ell^p_m \otimes E$, say $z = \sum_{j=1}^m \delta_j \otimes x_j$. Suppose also that

$$z = \sum_{i=1}^k r_i \otimes a_i \,,$$

where $k \in \mathbb{N}$, $r_1, \dots, r_k \in \ell^p_m$ and $a_1, \dots, a_k \in E$. Take $\lambda \in E'$. Then

$$\langle \boldsymbol{x}, \lambda \rangle = (I_m \otimes \lambda)(z) = \sum_{i=1}^k \langle a_i, \lambda \rangle r_i \in \ell^p_m \,. \tag{1.9.2}$$

THEOREM 1.38. *Let $(E, \|\cdot\|)$ be a normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Suppose that $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$ with $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$, and set*

$$Z = \{\langle \boldsymbol{x}, \lambda \rangle : \lambda \in E'\} \,.$$

*Then there is a matrix $A \in \mathbb{M}_{n,m}$ such that $A\boldsymbol{x} = \boldsymbol{y}$, such that $w = (A \otimes I_E)(z)$, where $z \in Z \otimes E$ and $w \in \ell^p_n \otimes E$ correspond to $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and such that the map $A \mid Z : Z \to \ell^p_n$ is a contraction as an element of $\mathcal{B}(Z, \ell^p_n)$, where we regard $Z$ as a subspace of $\ell^p_m$.*

*Proof.* Set $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$, and define

$$z = \sum_{j=1}^m \delta_j \otimes x_j = \sum_{i=1}^k r_i \otimes a_i \quad \text{and} \quad w = \sum_{j=1}^n \delta_j \otimes y_j = \sum_{i=1}^\ell s_i \otimes b_i$$

as elements of $\ell^p_m \otimes E$ and $\ell^p_n \otimes E$, respectively, where we may suppose that $z \neq 0$ and $w \neq 0$, and we specify that $\{r_1, \dots, r_k\}$ and $\{s_1, \dots, s_\ell\}$ are linearly independent subsets of $\ell^p_m$ and $\ell^p_n$, respectively, and that $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_\ell\}$ are linearly independent subsets of $E$. We see from (1.9.2) that $Z = \mathrm{lin}\,\{r_1, \dots, r_k\}$, a linear subspace of $\ell^p_m$, and so $z \in Z \otimes E$.

Take $\lambda \in E'$ with $\langle a_i, \lambda \rangle = 0 \;\; (i \in \mathbb{N}_k)$. By (1.9.2), we have $\langle \boldsymbol{x}, \lambda \rangle = 0$. Thus $\langle \boldsymbol{y}, \lambda \rangle = 0$, and hence $\sum_{i=1}^\ell \langle b_i, \lambda \rangle s_i = 0$. Since $\{s_1, \dots, s_\ell\}$ is a linearly independent set in $\ell^p_n$, it follows that $\langle b_i, \lambda \rangle = 0 \;\; (i \in \mathbb{N}_\ell)$; this implies that $b_i \in \mathrm{lin}\,\{a_1, \dots, a_k\} \;\; (i \in \mathbb{N}_\ell)$, and hence that

$$w = \sum_{i=1}^k t_i \otimes a_i$$

for some $t_1, \dots, t_k \in \ell^p_n$. There is a linear map $A : Z \to \ell^p_n$ such that $Ar_i = t_i \;\; (i \in \mathbb{N}_k)$, and then $w = (A \otimes I_E)(z)$. We extend $A$ (arbitrarily) to a linear map from $\ell^p_m$ to $\ell^p_n$, and regard $A$ as a matrix in $\mathbb{M}_{n,m}$; we have $A\boldsymbol{x} = \boldsymbol{y}$ when we regard $A$ as a map from $E^m$ to $E^n$.

We *claim* that the map $A : Z \to \ell^p_n$ is a contraction. Indeed, take $\zeta_1, \dots, \zeta_k$ in $\mathbb{F}$, and set $r = \sum_{i=1}^k \zeta_i r_i \in Z$. Since $\{a_1, \dots, a_k\}$ is linearly independent, there exists $\lambda \in E'$ with $\langle a_i, \lambda \rangle = \zeta_i \;\; (i \in \mathbb{N}_k)$, and then, by (1.9.2),

$$\|r\|_{\ell^p_m} = \|\langle \boldsymbol{x}, \lambda \rangle\|_{\ell^p_m} \quad \text{and} \quad \|Ar\|_{\ell^p_n} = \left\| \sum_{i=1}^k \zeta_i t_i \right\|_{\ell^p_n} = \|\langle \boldsymbol{y}, \lambda \rangle\|_{\ell^p_n} \,.$$

Since $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$, it follows that $\|Ar\|_{\ell_n^p} \leqslant \|r\|_{\ell_m^p}$, and so $A$ is a contraction in $\mathcal{B}(Z, \ell_n^p)$. ∎

We record a relevant result that we shall use later: it is Lemma 7.7 of [24], taking $C = 1$ and $X = Z = E$ and $Y = F$ for Banach spaces $E$ and $F$ in that result.

THEOREM 1.39. *Let $E$ and $F$ be normed spaces, and take $p$ with $1 \leqslant p < \infty$. Suppose that an operator $T \in \mathcal{B}(E, F)$ has the property that*

$$\left\|T^{(n)}\boldsymbol{y}\right\|_{\ell_n^p(F)} \leqslant \|\boldsymbol{x}\|_{\ell_m^p(E)}$$

*whenever $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, $\boldsymbol{y} \in E^n$, and $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. Then there are a measure space $\Omega$ and a contraction $J : E \to L^p(\Omega)$ such that $\|Tx\| \leqslant \|Jx\|_{L^p(\Omega)}$ $(x \in E)$.* ∎

This theorem says that $T$ 'factors through a subspace of $L^p(\Omega)$, with both factors being contractions'. We obtain the following corollary by taking $F = E$ and $T = I_E$ in the above theorem.

COROLLARY 1.40. *Let $E$ be a normed space, and take $p$ with $1 \leqslant p < \infty$. Suppose that*

$$\|\boldsymbol{y}\|_{\ell_n^p(E)} \leqslant \|\boldsymbol{x}\|_{\ell_m^p(E)}$$

*whenever $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, $\boldsymbol{y} \in E^n$, and $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. Then $E$ embeds isometrically into a space $L^p(\Omega)$ for some measure space $\Omega$.* ∎

Part of the following lemma is exactly [49, Lemma 8.5], with $X_1$ of that reference taken to be the scalar field.

LEMMA 1.41. *Let $E$ be a Banach space, let $\Gamma$ be an index set, and take $Q \in \mathcal{B}(\ell^1(\Gamma), E)$ and $p$ with $1 \leqslant p < \infty$. Suppose that, for each $r, s \in \mathbb{N}$ and each $C \in \mathbb{M}_{r,s}$, we have*

$$\left\|C \otimes Q : \ell_s^p \otimes \ell^1(\Gamma) \to \ell_r^p \otimes E\right\| \leqslant \|C : \ell_s^p \to \ell_r^p\| \,. \tag{1.9.3}$$

*Then*

$$\sum_{j=1}^{n} \|Qg_j\|^p \leqslant \sum_{i=1}^{m} \|f_i\|_1^p \tag{1.9.4}$$

*whenever $m, n \in \mathbb{N}$ and $f_1, \dots, f_m, g_1, \dots, g_n \in \ell^1(\Gamma)$ with $(g_1, \dots, g_n) \preccurlyeq_p (f_1, \dots, f_m)$.*

*Proof.* Set $(X, \|\cdot\|) = (\ell^1(\Gamma), \|\cdot\|_{\ell^1(\Gamma)})$.

We take $m, n \in \mathbb{N}$, $\boldsymbol{f} = (f_1, \dots, f_m) \in X^m$ and $\boldsymbol{g} = (g_1, \dots, g_n) \in X^n$ with $\boldsymbol{g} \preccurlyeq_p \boldsymbol{f}$, and seek to prove inequality (1.9.4).

By reducing $\Gamma$, if necessary, we may suppose that $\max\{|f_1(\gamma)|, \dots, |f_m(\gamma)|\} > 0$ for each $\gamma \in \Gamma$. We may also suppose that $f_i \neq 0$ $(i \in \mathbb{N}_m)$ and that $\sum_{i=1}^m \|f_i\|^p = 1$.

As in Theorem 1.38, set $Z = \{\langle \boldsymbol{f}, \lambda\rangle : \lambda \in X'\}$, regarded as a linear subspace of $\ell_m^p$. Since $\boldsymbol{g} \preccurlyeq_p \boldsymbol{f}$, it follows from Theorem 1.38 that there is a matrix $A \in \mathbb{M}_{n,m}$ such that $A\boldsymbol{f} = \boldsymbol{g}$ and $A \mid Z : Z \to \ell_n^p$ is a contraction as an element of $\mathcal{B}(Z, \ell_n^p)$. We write $A = (a_{j,i} : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n)$.

Define

$$\alpha(\gamma) = \left(\sum_{i=1}^{m} \|f_i\|^{p-1} |f_i(\gamma)|\right)^{1/p} \quad (\gamma \in \Gamma)\,,$$

so that $\alpha(\gamma) > 0 \ (\gamma \in \Gamma)$ and

$$\sum_{\gamma\in\Gamma} \alpha(\gamma)^p = \sum_{i=1}^m \|f_i\|^{p-1} \sum_{\gamma\in\Gamma} |f_i(\gamma)| = 1\,.$$

Thus $\alpha \in \ell^p(\Gamma)$ with $\|\alpha\|_{\ell^p(\Gamma)} = 1$.

Now define $b_{i,\gamma} = f_i(\gamma)/\alpha(\gamma)$ for $i \in \mathbb{N}_m$ and $\gamma \in \Gamma$, so that

$$b_{i,\gamma}\alpha(\gamma) = f_i(\gamma) \quad (i \in \mathbb{N}_m,\ \gamma \in \Gamma)\,.$$

Take $h \in c_{00}(\Gamma)$. Since the function $t \mapsto t^p$ is convex on $\mathbb{R}^+$, we have

$$\left(\sum_{\gamma\in\Gamma} \frac{|f_i(\gamma)|}{\|f_i\|}\frac{|h(\gamma)|}{\alpha(\gamma)}\right)^p \leqslant \sum_{\gamma\in\Gamma} \frac{|f_i(\gamma)|}{\|f_i\|}\frac{|h(\gamma)|^p}{\alpha(\gamma)^p} \quad (i \in \mathbb{N}_m)\,,$$

and so

$$\begin{aligned}\left|\sum_{\gamma\in\Gamma} b_{i,\gamma}h(\gamma)\right|^p &\leqslant \left(\sum_{\gamma\in\Gamma} |f_i(\gamma)|\,\frac{|h(\gamma)|}{\alpha(\gamma)}\right)^p \leqslant \|f_i\|^p \sum_{\gamma\in\Gamma} \frac{|f_i(\gamma)|}{\|f_i\|}\frac{|h(\gamma)|^p}{\alpha(\gamma)^p}\\ &= \|f_i\|^{p-1}\sum_{\gamma\in\Gamma} |f_i(\gamma)|\,\frac{|h(\gamma)|^p}{\alpha(\gamma)^p} \quad (i \in \mathbb{N}_m)\,.\end{aligned}$$

Hence

$$\sum_{i=1}^m \left|\sum_{\gamma\in\Gamma} b_{i,\gamma}h(\gamma)\right|^p \leqslant \sum_{i=1}^m \|f_i\|^{p-1}\sum_{\gamma\in\Gamma} |f_i(\gamma)|\,\frac{|h(\gamma)|^p}{\alpha(\gamma)^p} = \sum_{\gamma\in\Gamma} |h(\gamma)|^p = \|h\|^p_{\ell^p(\Gamma)}\,.$$

This shows that the linear map

$$B : h \mapsto \left(\sum_{\gamma\in\Gamma} b_{i,\gamma}h(\gamma)\right)_{i=1}^m\,, \quad (c_{00}(\Gamma), \|\cdot\|_{\ell^p(\Gamma)}) \to (\ell^p_m, \|\cdot\|_{\ell^p_m})\,,$$

is a contraction. Since $c_{00}(\Gamma)$ is dense in $\ell^p(\Gamma)$, there is a contraction, also denoted by $B$, in $\mathcal{B}(\ell^p(\Gamma), \ell^p_m)$ extending the original $B$. Clearly

$$B\delta_\gamma = (b_{i,\gamma})_{i=1}^m = \frac{1}{\alpha(\gamma)}(f_1(\gamma), \dots, f_m(\gamma)) = \frac{1}{\alpha(\gamma)}\langle \boldsymbol{f}, \varepsilon_\gamma\rangle \quad (\gamma \in \Gamma)\,,$$

where $\varepsilon_\gamma : X \to \mathbb{F}$ is the evaluation functional at $\gamma$. Thus the range of $B$ is contained in the subspace $Z$.

Define

$$C = A \circ B : \ell^p(\Gamma) \to \ell^p_n\,,$$

so that the map $C$ is also a contraction. We set $C = (c_{j,\gamma})$, where

$$c_{j,\gamma} = \sum_{i=1}^m a_{j,i}b_{i,\gamma} \quad (j \in \mathbb{N}_n,\ \gamma \in \Gamma)\,.$$

Thus

$$g_j = \sum_{i=1}^m a_{j,i}f_i = \sum_{i=1}^m \sum_{\gamma\in\Gamma} a_{j,i}b_{i,\gamma}\alpha(\gamma)\delta_\gamma = \sum_{\gamma\in\Gamma} c_{j,\gamma}\alpha(\gamma)\delta_\gamma \quad (j \in \mathbb{N}_n)\,. \tag{1.9.5}$$

Fix $\varepsilon > 0$, and choose a finite subset $\Gamma_0$ of $\Gamma$ such that

$$\sum_{j=1}^{n} \|Q(g_j \mid \Gamma_0)\|^p \geqslant \sum_{j=1}^{n} \|Qg_j\|^p - \varepsilon\,, \tag{1.9.6}$$

say $|\Gamma_0| = k$; we may suppose that $k \geqslant n$. We also write $C$ for the restriction of the original operator $C$ to $\ell^p(\Gamma_0)$, and regard the new map $C$ as a matrix in $\mathbb{M}_{n,k}$. Set $\boldsymbol{x} = (\alpha(\gamma)\delta_\gamma : \gamma \in \Gamma_0) \in \ell^p(\Gamma_0, X)$ and $\boldsymbol{h} = (g_1 \mid \Gamma_0, \dots, g_n \mid \Gamma_0) \in \ell^p_n(X)$. By equation (1.9.5), we have

$$C\boldsymbol{x} = \boldsymbol{h}\,. \tag{1.9.7}$$

As in equation (1.4.3), we can identify the map $C \otimes Q : \ell^p(\Gamma_0) \otimes X \to \ell^p_n \otimes E$ with the map

$$Q^{(n)} \circ C : \ell^p(\Gamma_0, X) \to \ell^p_n(E)\,.$$

Since $C$ is a contraction, the hypothesis (1.9.3) (with $s = k$ and $r = n$) implies that the above map is a contraction, and so, by (1.9.7), we have

$$\sum_{j=1}^{n} \|Q(g_j \mid \Gamma_0)\|^p = \left\|(Q^{(n)} \circ C)(\boldsymbol{x})\right\|^p_{\ell^p_n(E)} \leqslant \|\boldsymbol{x}\|^p_{\ell^p(\Gamma_0,X)} = \sum_{\gamma\in\Gamma_0} |\alpha(\gamma)|^p \leqslant 1\,. \tag{1.9.8}$$

It follows from (1.9.6) and (1.9.8) that

$$\sum_{j=1}^{n} \|Qg_j\|^p \leqslant \sum_{j=1}^{n} \|Q(g_j \mid \Gamma_0)\|^p + \varepsilon \leqslant 1 + \varepsilon\,.$$

This holds true for each $\varepsilon > 0$, and so we obtain the required inequality (1.9.4), where we recall that $\sum_{i=1}^m \|f_i\|^p = 1$. ∎

We can now conclude the proof of Kwapień's theorem.

THEOREM 1.42. *Take $p$ with $1 \leqslant p < \infty$. Then the class $SQ(p)$ coincides with the class of $p$–spaces.*

*Proof.* We have noted, following Herz, that each member of the class $SQ(p)$ is a $p$–space.

Now suppose that $E$ is a Banach space that is a $p$–space. We shall apply Proposition 1.6, Theorem 1.39, and Lemma 1.41.

Take $r, s \in \mathbb{N}$ and $C \in \mathbb{M}_{r,s}$. Since $E$ is a $p$–space, we know that

$$\|C \otimes I_E : \ell^p_s \otimes E \to \ell^p_r \otimes E\| \leqslant \|C : \ell^p_s \to \ell^p_r\|\,.$$

There is an index set $\Gamma$ and a quotient operator $Q : \ell^1(\Gamma) \to E$; by equation (1.5.14), we see that

$$\left\|I_s \otimes Q : \ell^p_s \otimes \ell^1(\Gamma) \to \ell^p_s \otimes E\right\| = \|Q\| = 1\,.$$

Since $C \otimes Q = (C \otimes I_E) \circ (I_s \otimes Q)$, it follows that inequality (1.9.3) of Lemma 1.41 is satisfied, and hence that lemma shows that

$$\sum_{j=1}^{n} \|Qg_j\|^p \leqslant \sum_{i=1}^{m} \|f_i\|^p$$

whenever $m, n \in \mathbb{N}$ and $g_1, \dots, g_n, f_1, \dots, f_m \in \ell^1(\Gamma)$ with $(g_1, \dots, g_n) \preccurlyeq_p (f_1, \dots, f_m)$. By Theorem 1.39 (taken with $E = \ell^1(\Gamma)$, $F = E$, and $T = Q$), there is a contraction $J : \ell^1(\Gamma) \to L^p(\Omega)$ for some measure space $\Omega$ such that

$$\|Qf\| \leqslant \|Jf\|_{L^p(\Omega)} \quad (f \in \ell^1(\Gamma)) .$$

By Proposition 1.6 (taken with $E = \ell^1(\Gamma)$, $F = E$, and $G$ equal to the closure of the range of $J$ in $L^p(\Omega)$), the space $E$ is isometrically isomorphic to a quotient of $G$. Thus $E$ belongs to the class $SQ(p)$. ∎

The above is an 'isometric' version of Kwapień's theorem. There is also an isomorphic version of this theorem; it is proved by a small variation of the above proof.

THEOREM 1.43. *Let $E$ be a Banach space, and take $C \geqslant 1$ and $p$ with $1 \leqslant p < \infty$. Then the following are equivalent:*

(a) *$E$ is $C$–isomorphic to a $p$–space;*

(b) *$\|S \otimes I_E\| \leqslant C\,\|S\| \quad (S \in \mathcal{B}(L^p(\Omega), L^p(\Sigma)))$ for all measure spaces $\Omega$ and $\Sigma$;*

(c) *$\|S : \ell^p_m(E) \to \ell^p_n(E)\| \leqslant C\,\|S : \ell^p_m \to \ell^p_n\|$ for each $S \in \mathcal{B}(\ell^p_m, \ell^p_n)$ and $m, n \in \mathbb{N}$;*

(d) *$\|S : \ell^p_m(E) \to \ell^p_m(E)\| \leqslant C\,\|S : \ell^p_m \to \ell^p_m\|$ for each $S \in \mathcal{B}(\ell^p_m)$ and $m \in \mathbb{N}$.* ∎

COROLLARY 1.44. *Take $p$ with $1 < p < \infty$ and $r$ with $1 \leqslant r < \infty$, and suppose that $\Omega$ is a measure space such that $L^r(\Omega)$ is an infinite-dimensional space. Then the following are equivalent:*

(a) *$L^r(\Omega)$ is a $p$–space;*

(b) *$L^r(\Omega)$ is isomorphic to a $p$–space;*

(c) *either $1 < p \leqslant r \leqslant 2$ or $2 \leqslant r \leqslant p < \infty$.*

*Suppose that $1 < p \leqslant 2$ and $r \notin [p, 2]$ or $2 \leqslant p < \infty$ and $r \notin [2, p]$. Then, for each $C > 0$, there exists $n \in \mathbb{N}$ such that the space $\ell^r_n$ is not $C$–isomorphic to a $p$–space.*

*Proof.* The main part of this result follows immediately from Theorem 1.26(iii), Theorem 1.27(ii), and Theorem 1.42. The final clause follows from Theorems 1.27(i) and 1.42. ∎

**1.10. Interpolation spaces.** We summarize the basics of complex interpolation theory. For details, see [6, §§2.3, 2.4], [28, Chapter 9], and [51]; the seminal paper is that of Calderón [11].

Let $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ be two (real or complex) Banach spaces that are both linear subspaces of a Banach space $(H, \|\cdot\|)$, the *ambient* space, and suppose that the inclusion maps from $(E_j, \|\cdot\|_j)$ into $(H, \|\cdot\|)$ are both continuous. Then the pair

$$\{(E_0, \|\cdot\|_0), (E_1, \|\cdot\|_1)\}$$

is a *compatible couple* (of Banach spaces). It is straightforward to show that, in this case, the spaces $E_0 \cap E_1$ and $E_0 + E_1$ are then Banach spaces under the respective norms defined by:

$$\|x\|_{E_0\cap E_1} = \max\{\|x\|_0, \|x\|_1\} \quad (x \in E_0 \cap E_1);$$
$$\|x\|_{E_0+E_1} = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1,\, x_0 \in E_0,\, x_1 \in E_1\} \quad (x \in E_0 + E_1).$$

A Banach space $(G, \|\cdot\|)$ that contains $E_0 \cap E_1$ and is contained in $E_0 + E_1$ and is such that the two inclusions

$$(E_0 \cap E_1, \|\cdot\|_{E_0\cap E_1}) \to (G, \|\cdot\|) \to (E_0 + E_1, \|\cdot\|_{E_0+E_1})$$

are continuous is then an *intermediate space*.

For details of the following remarks, see [6, Chapter 4], for example. For the remainder of this section, all our Banach spaces are complex Banach spaces.

Suppose that $\{(E_0, \|\cdot\|_0), (E_1, \|\cdot\|_1)\}$ is a compatible couple of Banach spaces. Let $L_0$ and $L_1$ be the lines $\{\mathrm{i}y : y \in \mathbb{R}\}$ and $\{1 + \mathrm{i}y : y \in \mathbb{R}\}$, respectively, in $\mathbb{C}$, and set $S = (0, 1) \times \mathbb{R} \subset \mathbb{C}$, an open strip in $\mathbb{C}$. Take $\mathcal{F}$ to be the linear space of all functions $F$ on $\overline{S}$ taking values in $(E_0 + E_1, \|\cdot\|_{E_0+E_1})$ such that $F$ is bounded and continuous on $\overline{S}$, such that $F$ is analytic on $S$, and such that $F \mid L_j$ is a bounded and continuous map into $(E_j, \|\cdot\|_j)$ for $j = 0, 1$.

We define a norm on $\mathcal{F}$ by setting

$$\|F\|_{\mathcal{F}} = \max_{j=0,1}\{\sup\{\|F(z)\|_j : z \in L_j\}\} \quad (F \in \mathcal{F}).$$

By the Phragmén–Lindelöf theorem,

$$\|F(z)\|_{E_0+E_1} \leqslant \|F\|_{\mathcal{F}} \quad (z \in \overline{S},\, F \in \mathcal{F}).$$

Further $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is a Banach space.

Next take $\theta \in (0, 1)$, and identify $\theta$ with the point $(\theta, 0)$ of $S$. Then the map $F \mapsto F(\theta)$ is a contractive linear map from $\mathcal{F}$ into $(E_0 + E_1, \|\cdot\|_{E_0+E_1})$, and the image of this map is denoted by

$$(E_0, E_1)_\theta = E_{[\theta]};$$

$E_{[\theta]}$ is a Banach space with respect to the quotient norm defined by

$$\|x\|_{[\theta]} = \inf\{\|F\|_{\mathcal{F}} : F \in \mathcal{F},\, F(\theta) = x\} \quad (x \in E_{[\theta]}),$$

so that $\|\cdot\|_{[\theta]}$ is the *interpolation norm*. Further $(E_{[\theta]}, \|\cdot\|_{[\theta]})$ is an intermediate space.

We now note that, in the definition of the family $\mathcal{F}$, we may suppose that $F(\mathrm{i}y)$ and $F(1+\mathrm{i}y)$ tend to 0 in $E_0$ and $E_1$, respectively, as $|y| \to \infty$. Indeed, we can multiply each original function in the family $\mathcal{F}$ by the function

$$z \mapsto \exp(\delta(z^2 - \theta^2)), \quad \overline{S} \to \mathbb{C},$$

for suitable $\delta > 0$ to obtain this without changing the space $(E_{[\theta]}, \|\cdot\|_{[\theta]})$; for this, see [13, p. 1007]. This extra property of $\mathcal{F}$ was assumed by Calderón when he introduced this theory in [11]. We shall suppose throughout that functions in $\mathcal{F}$ have this extra property.

We note that, if we move to norms on $E_0$ and $E_1$ that are equivalent to $\|\cdot\|_0$ and $\|\cdot\|_1$ on $E_0$ and $E_1$, respectively, we do not change the intermediate space $E_{[\theta]}$ (and the interpolation norm is equivalent to the original interpolation norm).

We also note that, in the above situation, the space $E_0 \cap E_1$ is dense in $(E_{[\theta]}, \|\cdot\|_{[\theta]})$; this is [6, Theorem 4.2.2(a)].

A Banach-space-valued form of the famous Riesz–Thorin interpolation theorem is the following; full details are given in [6, Theorem 5.1.2].

THEOREM 1.45. *Let $\Omega$ be a measure space, and let $\{E_0, E_1\}$ be a compatible couple of complex Banach spaces. Take $\theta \in (0,1)$ and $p_0, p_1$ with $1 \leqslant p_0, p_1 < \infty$, and define $p$ by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.$$

*Set $E = (E_0, E_1)_\theta$. Then $\{L^{p_0}(\Omega; E_0), L^{p_1}(\Omega; E_1)\}$ is a compatible couple of Banach spaces, and*

$$(L^{p_0}(\Omega; E_0), L^{p_1}(\Omega; E_1))_\theta = L^p(\Omega; E)$$

*with $\|f\|_{[\theta]} = \|f\|_{L^p(\Omega;E)}$ $(f \in L^p(\Omega; E))$.* ∎

In particular, with the above notation, $\{\ell^{p_0}(E_0), \ell^{p_1}(E_1)\}$ is also a compatible couple of Banach spaces, and

$$(\ell^{p_0}(E_0), \ell^{p_1}(E_1))_\theta = \ell^p(E)\,, \tag{1.10.1}$$

where $E = (E_0, E_1)_\theta$.

Take $n \in \mathbb{N}$. By [6, Theorem 5.1.2], it is also true that $\{\ell_n^{p_0}(E_0), \ell_n^{\infty}(E_1)\}$ is a compatible couple of Banach spaces and that

$$(\ell_n^{p_0}(E_0), \ell_n^{\infty}(E_1))_\theta = \ell_n^p(E)\,, \tag{1.10.2}$$

where $1/p = (1-\theta)/p_0$ and $E = (E_0, E_1)_\theta$.

The fundamental theorem in this context is the following [6, Theorem 4.1.4].

THEOREM 1.46. *Let $\{(E_0, \|\cdot\|_0), (E_1, \|\cdot\|_1)\}$ and $\{(F_0, \|\cdot\|_0), (F_1, \|\cdot\|_1)\}$ be two compatible couples of complex Banach spaces, and suppose that $T : E_0 + E_1 \to F_0 + F_1$ is a linear map such that $T(E_j) \subset F_j$ and $T \mid E_j : E_j \to F_j$ is bounded, with norm $M_j$, for $j = 0, 1$. Take $\theta \in (0,1)$. Then $T(E_{[\theta]}) \subset F_{[\theta]}$ and $\|T \mid E_{[\theta]}\| \leqslant M_0^{1-\theta} M_1^{\theta}$.* ∎

PROPOSITION 1.47. *Let $\{E_0, E_1\}$ be a compatible couple of complex Banach spaces, and take $\theta \in (0,1)$. Suppose that $1 \leqslant p < \infty$ and that $E_0$ and $E_1$ are both $p$–spaces. Then $(E_0, E_1)_\theta$ is also a $p$–space.*

*Proof.* Set $E = (E_0, E_1)_\theta$. By (1.10.1), $(\ell_n^p(E_0), \ell_n^p(E_1))_\theta = \ell_n^p(E)$ $(n \in \mathbb{N})$.

Take $m, n \in \mathbb{N}$ and $T \in \mathcal{B}(\ell_m^p, \ell_n^p)$, and consider $T$ as a map defined on the spaces $E_0^m$ and on $E_1^m$, say

$$M_j = \|T : \ell_m^p(E_j) \to \ell_n^p(E_j)\| \quad (j = 0, 1)\,.$$

Since $E_0$ and $E_1$ are both $p$–spaces, in fact $M_j \leqslant \|T\|$ $(j = 0, 1)$. By Theorem 1.46, $T(\ell_m^p(E)) \subset \ell_n^p(E)$ and

$$\|T : \ell_m^p(E) \to \ell_n^p(E)\| \leqslant M_0^{1-\theta} M_1^{\theta} \leqslant \|T\|^{1-\theta}\, \|T\|^{\theta} = \|T\|\,,$$

and so $E$ is a $p$–space by Theorem 1.36, (b) $\Rightarrow$ (a). ∎

We shall see in Example 2.16, to be given below, that an apparent generalization of the above result to the case where $E_0$ and $E_1$ are $p_0$– and $p_1$–spaces, respectively, and $1/p = (1-\theta)/p_0 + \theta/p_1$ is not necessarily true.

## 2. Power-norms and $p$–multi-norms

**2.1. Power-norms.** We now return to the theory of power-norms. Throughout we continue to consider linear spaces over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Let $(\|\cdot\|_n)$ be a power-norm based on a normed space $E$, as in Definition 1.1. Then it is easy to see [20, Lemma 2.11] that

$$\max_{i=1,\dots,n} \|x_i\| \leqslant \|\boldsymbol{x}\|_n \leqslant \sum_{i=1}^{n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N})\,. \tag{2.1.1}$$

Thus the formulae $\|\boldsymbol{x}\|_n = \max_{i=1,\dots,n} \|x_i\|$ and $\|\boldsymbol{x}\|_n = \sum_{i=1}^n \|x_i\|$ define the minimum and maximum power-norms based on $E$, respectively; the corresponding spaces $E^n$ are just $\ell_n^\infty(E)$ and $\ell_n^1(E)$, respectively.

Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and suppose that $F$ is a subspace of $E$. Then an easy check shows that $(F^n, \|\cdot\|_n)$ is also a power-normed space. In the case where $F$ is a closed subspace of $E$, equation (1.3.10) defines a power-norm based on $E/F$; the latter is called the *quotient power-norm*.

Let $(E^n, \|\cdot\|_n)$ be a power-normed space. Then, by [20, Proposition 2.30], the dual sequence $((E')^n, \|\cdot\|_n')$ is a power-Banach space. We say that $((E')^n, \|\cdot\|_n')$ is the *dual power-Banach space* to $(E^n, \|\cdot\|_n)$. In the case where $(E^n, \|\cdot\|_n)$ is a multi-normed space or a dual multi-normed space, then $((E')^n, \|\cdot\|_n')$ is a dual multi-Banach space or a multi-Banach space, respectively [20, §2.3.2].

The following characterization of power-norms is straightforward.

Proposition 2.1. *Let $E$ be a linear space, and suppose that $\|\cdot\|_n$ is a norm on $E^n$ for each $n \in \mathbb{N}$. Then $(\|\cdot\|_n)$ is a power-norm based on $E$ if and only if*

$$\|T\boldsymbol{x}\|_m \leqslant \max\{|T_{i,j}| : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n\}\, \|\boldsymbol{x}\|_n \quad (\boldsymbol{x} \in E^n) \tag{2.1.2}$$

*for each special matrix $T \in \mathbb{M}_{m,n}$ and each $m, n \in \mathbb{N}$.* ■

In fact, to verify that $(\|\cdot\|_n)$ is a power-norm based on a linear space $E$, it is sufficient to check equation (2.1.2) for a restricted class of special matrices $T$. Indeed, to verify (A1), it is sufficient to consider square matrices $(T_{i,j})$ such that $T_{i,j} = \delta_{i,j}$ save for two specified values $i_0$ and $j_0$ of $i$ and $j$, respectively, and such that $T_{i,j} = 1 - \delta_{i,j}$ when $\{i, j\} = \{i_0, j_0\}$; to verify (A2), it is sufficient to consider diagonal matrices; to verify

(A3), it is sufficient to consider matrices of the form

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{M}_{n+1,n} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{M}_{n,n+1}\,.$$

Definition 2.2. Let $E$ be a linear space, and let $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ be two power-norms based on $E$. Then

$$(\|\cdot\|_n^1) \leqslant (\|\cdot\|_n^2) \quad \text{if} \quad \|\boldsymbol{x}\|_n^1 \leqslant \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N})\,,$$

and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ *dominates* $(\|\cdot\|_n^1 : n \in \mathbb{N})$, written $(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2)$, if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leqslant C\,\|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N})\,; \tag{2.1.3}$$

the two power-norms are *equivalent*, written

$$(\|\cdot\|_n^1 : n \in \mathbb{N}) \cong (\|\cdot\|_n^2 : n \in \mathbb{N}) \quad \text{or} \quad (\|\cdot\|_n^1) \cong (\|\cdot\|_n^2)\,,$$

if each dominates the other.

For discussions of when two multi-norms are equivalent, see [8] and [19].

**2.2. $p$–multi-norms.** We now define the main topic of this memoir, a special class of power-normed spaces.

Definition 2.3. Let $E$ be a linear space, and take $p$ with $1 \leqslant p \leqslant \infty$. A *$p$–multi-norm based on* $E$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on $E^n$ for each $n \in \mathbb{N}$ and such that

$$\|T\boldsymbol{x}\|_m \leqslant \|T : \ell_n^p \to \ell_m^p\|\,\|\boldsymbol{x}\|_n \quad (T \in \mathbb{M}_{m,n},\, \boldsymbol{x} \in E^n,\, m, n \in \mathbb{N})\,, \tag{2.2.1}$$

and then $(E^n, \|\cdot\|_n)$ is a *$p$–multi-normed space.*

In the case where $E$ is a Banach space, we may refer to a *$p$–multi-Banach space.*

This definition was first given by Ramsden in [52], where the term 'type–$p$ multi-norm' was used. As observed in [52, p. 58], it follows from Proposition 2.1 that each $p$–multi-norm is a power-norm.

The motivation for giving this definition is the following. The characterizations given in Theorems 2.35 and 2.36, respectively, of [20] prove that $\infty$–multi-norms and 1–multi-norms in the above sense are exactly the multi-norms and dual multi-norms that were defined in Definition 1.1, and so our new definition generalizes the old one given for the cases $p = 1$ and $p = \infty$.

For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be a class of matrices in $\mathbb{M}_n$ such that

$$\|U : \ell_n^p \to \ell_n^p\| \leqslant 1 \quad (U \in \mathcal{C}_n)$$

and such that the absolutely convex hull of $\mathcal{C}_n$ is the closed unit ball of the space $\mathbb{M}_n$ when this space is identified with $\mathcal{B}(\ell_n^p)$. Then, to verify equation (2.2.1), it suffices to check that axiom (A3) holds and that we have $\|U\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_n \;\; (\boldsymbol{x} \in E^n)$ for each $U \in \mathcal{C}_n$. In particular, in the case where $p = 2$ and $E$ is a complex linear space, the class $\mathcal{U}_n$ of unitary matrices in $\mathbb{M}_n(\mathbb{C})$ satisfies the required condition with $\mathcal{C}_n = \mathcal{U}_n$.

Let $(\|\cdot\|_n : n \in \mathbb{N})$ be a $p$–multi-norm based on a linear space $E$. As noted in [52, Lemmas 4.3.2 and 4.3.4], the following variations of Axioms (A4) and (B4) hold:

$$\|(x_1, \dots, x_{n-1}, \alpha x_n, \beta x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, \gamma_p x_n)\|_n \,, \tag{2.2.2}$$

$$\|(x_1, \dots, x_{n-1}, \alpha x + \beta y)\|_n \leqslant \|(x_1, \dots, x_{n-1}, \gamma_q x, \gamma_q y)\|_{n+1} \,, \tag{2.2.3}$$

for all $\alpha, \beta \in \mathbb{F}$, $x_1, \dots, x_n, x, y \in E$, and $n \in \mathbb{N}$, where $q = p'$ and $\gamma_r = (|\alpha|^r + |\beta|^r)^{1/r}$ for $r = p, q$. In the two cases where $p = 1$ and $p = \infty$, just equation (2.2.2) characterizes a $p$–multi-norm. However, in the case where $1 < p < \infty$, these two equations do not characterize $p$–multi-norms based on $E$, as we shall see in Example 2.7(ii), to be given below. These equations are used by Blasco in [7] to characterize a larger class of power-normed spaces than the $p$–multi-normed spaces.

It follows from (2.2.2) that

$$\|(\alpha_1 x, \dots, \alpha_n x)\|_n = \left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p} \|x\| \quad (\alpha_1, \dots, \alpha_n \in \mathbb{F},\, x \in E,\, n \in \mathbb{N})\,, \tag{2.2.4}$$

and so

$$\|(x, \dots, x)\|_n = n^{1/p} \|x\| \quad (x \in E,\, n \in \mathbb{N})\,. \tag{2.2.5}$$

In particular, for each non-zero normed space $E$, a given power-norm based on $E$ is a $p$–multi-norm for at most one value of $p$.

The following result follows easily from (2.2.3) by induction on $n \in \mathbb{N}$; in particular the given inequality holds for all $p$–multi-norms based on $E$.

PROPOSITION 2.4. *Let $E$ be a normed space, take $p$ with $1 \leqslant p \leqslant \infty$, and suppose that $(\|\cdot\|_n)$ is a power-norm based on $E$ such that inequality* (2.2.3) *is satisfied. Then*

$$\left\|\sum_{i=1}^n \alpha_i x_i\right\| \leqslant \|(\alpha_i)\|_{\ell_n^q} \|\boldsymbol{x}\|_n \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)$$

*for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and $n \in \mathbb{N}$, where $q = p'$.* ■

We note the following standard constructions involving $p$–multi-norms; clause (iv) is [52, Corollary 4.4.12].

PROPOSITION 2.5. *Let $E$ be a normed space, take $p$ with $1 \leqslant p \leqslant \infty$, and suppose that $(\|\cdot\|_n)$ is a $p$–multi-norm based on $E$.*

(i) *For each subspace $F$ of $E$, the power-normed space $(F^n, \|\cdot\|_n)$ is a $p$–multi-normed space.*

(ii) *For $m \in \mathbb{N}$, set $F = E^m$. Then the power-normed space $(F^n, \|\cdot\|_{mn})$ is a $p$–multi-normed space.*

(iii) *For each closed subspace $F$ of $E$, the quotient power-normed space $((E/F)^n, \|\cdot\|_n)$ is a $p$–multi-normed space.*

(iv) *The sequence $(\|\cdot\|'_n)$ of dual norms is a $p'$–multi-norm based on $E'$.*

*Proof.* (i) and (ii) These are easily checked.

(iii) Take $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^n$, and $T \in \mathbb{M}_{m,n}$ with $\|T : \ell_n^p \to \ell_m^p\| \leqslant 1$, and take $\varepsilon > 0$. There exists $\boldsymbol{y} \in F^n$ with $\|\boldsymbol{x} + \boldsymbol{y}\|_n \leqslant \|\boldsymbol{x} + F^n\|_n + \varepsilon$. Since $T\boldsymbol{y} \in F^m$, we have

$$\|T(\boldsymbol{x} + F^n)\|_m \leqslant \|T(\boldsymbol{x} + \boldsymbol{y})\|_m \leqslant \|\boldsymbol{x} + \boldsymbol{y}\|_n \leqslant \|\boldsymbol{x} + F^n\|_n + \varepsilon\,.$$

This holds for each $\varepsilon > 0$, and so $\|T(\boldsymbol{x} + F^n)\|_m \leqslant \|\boldsymbol{x} + F^n\|_n$, as required.

(iv) Set $q = p'$. Take $m, n \in \mathbb{N}$, $T \in \mathbb{M}_{m,n}$, and $\boldsymbol{\lambda} \in (E')^n$. Then, for each $\boldsymbol{x} \in E^m$ with $\|\boldsymbol{x}\|_m \leqslant 1$, we have

$$|\langle \boldsymbol{x},\, T\boldsymbol{\lambda}\rangle| = \left|\langle T^t\boldsymbol{x},\, \boldsymbol{\lambda}\rangle\right| \leqslant \left\|T^t\boldsymbol{x}\right\|_n \|\boldsymbol{\lambda}\|_n' \leqslant \left\|T^t : \ell_m^p \to \ell_n^p\right\| \|\boldsymbol{\lambda}\|_n' = \|T : \ell_n^q \to \ell_m^q\| \, \|\boldsymbol{\lambda}\|_n'\,,$$

and so $\|T\boldsymbol{\lambda}\|_m \leqslant \|T : \ell_n^q \to \ell_m^q\| \, \|\boldsymbol{\lambda}\|_n'$. Thus $(\|\cdot\|_n')$ is $q$–multi-norm based on $E'$. ■

DEFINITION 2.6. Let $(E^n, \|\cdot\|_n)$ be a $p$–multi-normed space, where $1 \leqslant p \leqslant \infty$. Then the sequence $(\|\cdot\|_n')$ of norms is the *dual $p'$–multi-norm* based on $E'$.

EXAMPLES 2.7. Take $p$ with $1 \leqslant p \leqslant \infty$.

(i) For $n \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{F}^n$, set

$$\|\boldsymbol{z}\|_n = \left(\sum_{i=1}^n |z_i|^p\right)^{1/p} = \|\boldsymbol{z}\|_{\ell_n^p}\,.$$

Then $(\|\cdot\|_n)$ is a $p$–multi-norm based on $\mathbb{F}$, and it is immediately checked that it is the unique $p$–multi-norm based on $\mathbb{F}$ such that $\|z\|_1 = |z| \ \ (z \in \mathbb{F})$.

(ii) Let $E$ be a normed space. Then we have defined the $p$–sum norm in Definition 1.7 by the formula

$$\|\boldsymbol{x}\|_{\ell_n^p(E)} = \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \qquad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N})\,. \tag{2.2.6}$$

Set $\|\cdot\|_n = \|\cdot\|_{\ell_n^p(E)}$, so that $(E^n, \|\cdot\|_n) = \ell_n^p(E)$. Then clearly $(\|\cdot\|_n)$ is a power-norm based on $E$; this power-norm is called the *$p$–sum power-norm*. Clearly the sequence $(\|\cdot\|_n)$ satisfies equations (2.2.2) and (2.2.3). In the case where $p = \infty$, we obtain the minimum multi-norm; in the case where $p = 1$, we obtain the maximum dual multi-norm, as in [20].

Now consider the special case in which $E = \ell^p$. Take $m, n \in \mathbb{N}$, $T \in \mathbb{M}_{m,n}$ such that $\|T : \ell_n^p \to \ell_m^p\| \leqslant 1$, and $\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell^p)^n$. For $k \in \mathbb{N}$, set

$$\boldsymbol{\alpha}_k = (x_{1k}, \dots, x_{nk}) \in \mathbb{F}^n\,,$$

where $x_i = (x_{ij} : j \in \mathbb{N})$ for $i \in \mathbb{N}_n$. Then

$$\|T\boldsymbol{x}\|_m^p = \sum_{k=1}^\infty \sum_{i=1}^m |(T\boldsymbol{x})_{ik}|^p = \sum_{k=1}^\infty \|T\boldsymbol{\alpha}_k\|_{\ell_m^p}^p \leqslant \sum_{k=1}^\infty \|\boldsymbol{\alpha}_k\|_{\ell_n^p}^p = \sum_{k=1}^\infty \sum_{j=1}^n |x_{jk}|^p = \|\boldsymbol{x}\|_n^p\,,$$

and so $(\|\cdot\|_n)$ is a $p$–multi-norm based on $\ell^p$. More generally, consider the case where $E = L^p(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space. Then we shall see in Example 2.27(ii), below, that $(\|\cdot\|_n)$ is a strong $p$–multi-norm, and hence that $(\|\cdot\|_n)$ is a $p$–multi-norm.

Next suppose that $1 \leqslant p \leqslant 2$ and that $r \in [p, 2]$. By Proposition 1.22, the spaces $\ell^r$ and $L^r(\mathbb{I})$ are isometrically isomorphic to closed subspaces of $L^p(\mathbb{I})$, and so, by Proposition 2.5(i), the $p$–sum power-norm is a $p$–multi-norm based on the spaces $\ell^r$ and $L^r(\mathbb{I})$. Second, suppose that $2 \leqslant p < \infty$ and that $r \in [2, p]$. Then, by Corollary 1.23, $\ell^r$ and $L^r(\mathbb{I})$ are isometrically isomorphic to quotients of $L^p(\mathbb{I})$, and so, by Proposition 2.5(iii), the $p$–sum power-norm is a $p$–multi-norm on these spaces.

However, we shall see in Theorem 2.8, below, that the $p$–sum norm based on a Banach space is not always a $p$–multi-norm.

(iii) Let $E$ be a normed space. For $n \in \mathbb{N}$, the norm $\mu_{p,n}$ on $E^n$ is the weak $p$–summing norm discussed in §1.5.

It is shown in [20, Theorem 3.16] that $(\mu_{p,n})$ is a $p$–multi-norm based on $E$; we shall prove a stronger result in Example 2.27(iii). It follows that the set of $p$–multi-norms based on an arbitrary normed space $E$ is not empty. In fact, we shall see in Theorem 2.11, below, that $(\mu_{p,n})$ has the property that

$$\mu_{p,n}(\boldsymbol{x}) \leqslant \|\boldsymbol{x}\|_n \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for each $p$–multi-norm $(\|\cdot\|_n)$ based on $E$.

(iv) Let $E$ be a normed space, and set $q = p'$. For $n \in \mathbb{N}$, the dual weak $p$–summing norm $\nu_{p,n}$ on $E^n$ was also discussed in §1.5; indeed, $\nu_{p,n}$ is the restriction to $E^n$ of the dual norm of $\mu_{q,n}^{E'}$ on $(E'')^n$.

Since $(\mu_{q,n})$ is a $q$–multi-norm based on $E'$, it follows that $(\nu_{p,n})$ is a $p$–multi-norm based on $E$. In fact, we shall see in Theorem 2.11, below, that $(\nu_{p,n})$ has the property that

$$\|\boldsymbol{x}\|_n \leqslant \nu_{p,n}(\boldsymbol{x}) \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for each $p$–multi-norm $(\|\cdot\|_n)$ based on $E$. ■

The results concerning $p$–sum power-norms mentioned in Example 2.7(ii), above, are special to the cases mentioned. Indeed, take $p$ with $1 \leqslant p < \infty$. Then it follows from Theorem 1.36, (a) $\Leftrightarrow$ (b), that the $p$–sum power-norm based on a Banach space $E$ is a $p$–multi-norm if and only if $E$ is a $p$–space, and so the following theorem is an immediate consequence of Kwapień's theorem, Theorem 1.42.

Theorem 2.8. *Let $E$ be a Banach space, and take $p$ with $1 \leqslant p < \infty$. Then the following conditions on $E$ are equivalent:*

(a) *the $p$–sum power-norm based on $E$ is a $p$–multi-norm;*

(b) *$E$ is a $p$–space;*

(c) *$E$ belongs to the class $SQ(p)$.* ■

Further, take take $p$ with $1 \leqslant p < \infty$ and $r$ with $1 \leqslant r \leqslant \infty$, and let $\Omega$ be a measure space such that $L^r(\Omega)$ is an infinite-dimensional space. Then, by Theorem 2.8 and Corollary 1.44, the $p$–sum power-norm based on $L^r(\Omega)$ is a $p$–multi-norm if and only if either $1 \leqslant p \leqslant r \leqslant 2$ or $2 \leqslant r \leqslant p < \infty$. In particular, equations (2.2.2) and (2.2.3) do not characterize $p$–multi-norms when $1 < p < \infty$.

EXAMPLE 2.9. We now generalize a construction of [45, p. 17] (using a different terminology).

Fix independent standard normal random variables, $f_1, f_2, \dots$. More specifically, we suppose, in the real case, that each $f_i$ has the probability density function

$$\frac{1}{\sqrt{2\pi}}\exp(-t^2/2) \quad (t \in \mathbb{R}),$$

so that the joint density function of $f_1, \dots, f_n$ on $\mathbb{R}^n$ (for $n \in \mathbb{N}$) is

$$\frac{1}{(2\pi)^{n/2}}\exp(-(t_1^2 + \cdots + t_n^2)/2) \quad (t_1, \dots, t_n \in \mathbb{R}).$$

In the complex case, $f_1, f_2, \dots$ are independent complex standard normal random variables, of the form $(g_i + \mathrm{i}h_i)/\sqrt{2}$, where $g_1, h_1, g_2, h_2, \dots$ are real independent standard normal random variables. For background information, see [41, pp. 148–149].

Now suppose that $E$ is a complex Banach space, take $n \in \mathbb{N}$, and suppose that $U = (U_{i,j}) \in \mathbb{M}_n(\mathbb{C})$ is a unitary matrix. Take $f_1, \dots, f_n$ to be independent complex standard normal random variables, as above. Then the two $n$-tuples $\boldsymbol{f} = (f_1, \dots, f_n)$ and $((U\boldsymbol{f})_1, \dots, (U\boldsymbol{f})_n)$ are equidistributed (see [48, Chapter 2]), and so

$$\mathbb{E}\left\|\sum_{i=1}^n f_i x_i\right\| = \mathbb{E}\left\|\sum_{i=1}^n \left(\sum_{j=1}^n U_{i,j} f_j\right) x_i\right\| \quad (x_1, \dots, x_n \in E). \tag{2.2.7}$$

For $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, define

$$\|\boldsymbol{x}\|_n = \mathbb{E}\left\|\sum_{i=1}^n f_i x_i\right\|, \tag{2.2.8}$$

so that $\|\cdot\|_n$ is a norm on $E^n$.

We *claim* that $(\|\cdot\|_n)$ is a 2–multi-norm based on $E$. Indeed, it is immediate that $(\|\cdot\|_n)$ satisfies axiom (A3). Now take $n \in \mathbb{N}$ and a unitary matrix $V \in \mathbb{M}_n(\mathbb{C})$, and set $U = V^t$, so that $U$ is also a unitary matrix in $\mathbb{M}_n(\mathbb{C})$. It follows from equations (2.2.7) and (2.2.8) that

$$\begin{aligned}\|V\boldsymbol{x}\|_n &= \mathbb{E}\left\|\sum_{i=1}^n f_i (V\boldsymbol{x})_i\right\| = \mathbb{E}\left\|\sum_{i=1}^n f_i \left(\sum_{j=1}^n V_{i,j} x_j\right)\right\| \\ &= \mathbb{E}\left\|\sum_{j=1}^n \left(\sum_{i=1}^n U_{j,i} f_i\right) x_j\right\| = \mathbb{E}\left\|\sum_{i=1}^n f_i x_i\right\| = \|\boldsymbol{x}\|_n,\end{aligned}$$

and so $(\|\cdot\|_n)$ satisfies equation (2.2.1) for each unitary matrix $V$, and hence for all matrices in $\mathbb{M}_n(\mathbb{C})$. It follows that $(\|\cdot\|_n)$ is a 2–multi-norm.

In fact, we could also define

$$\|\boldsymbol{x}\|_{p,n} = \left(\mathbb{E}\left\|\sum_{i=1}^n f_i x_i\right\|^p\right)^{1/p}$$

for each $p$ with $1 \leqslant p < \infty$. This will be a 2–multi-norm based on $E$ by the same reasoning as in the case where $p = 1$. Moreover, all these 2–multi-norms are equivalent: for each such $p$, there is a constant $C_p$ such that

$$\|\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_{p,n} \leqslant C_p\,\|\boldsymbol{x}\|_n \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})\,;$$

the second inequality in the above formula is the Gaussian version of the Khintchine–Kahane inequality [39, §4.2]. ■

EXAMPLE 2.10. As indicated, a number of multi-norms have been introduced in earlier works. Here we recall one of these, from [20, §1.4].

Let $E$ be a normed space, and take $p, q$ with $1 \leqslant p \leqslant q < \infty$. Then the $(p,q)$*–multi-norm* $(\|\cdot\|_n^{(p,q)})$ based on $E$ is defined by

$$\|\boldsymbol{x}\|_n^{(p,q)} = \sup\left\{\left(\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle|^q\right)^{1/q} : \mu_{p,n}(\lambda_1, \dots, \lambda_n) \leqslant 1\right\}$$

for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ and $n \in \mathbb{N}$. By [20, Theorem 4.1], $(\|\cdot\|_n^{(p,q)})$ is indeed a multi-norm based on $E$.

For example, it is shown in [20, Theorem 4.6] that $\|\cdot\|_n^{(1,1)} = \|\cdot\|_n^{\max}$ $(n \in \mathbb{N})$, where $(\|\cdot\|_n^{\max})$ is the maximum multi-norm, defined on page 6.

The theory of when two such multi-norms are equivalent is given in [8]. ■

As in §1.5, the norms $\|\cdot\|_{\varepsilon,n}$ and $\|\cdot\|_{\pi,n}$ are the injective and projective norms, respectively, on $\ell_n^p \otimes E$. The following theorem is similar to results in [52, §4.5].

THEOREM 2.11. *Let $E$ be a normed space, and take $p$ with $1 \leqslant p < \infty$. Suppose that $(\|\cdot\|_n)$ is a $p$–multi-norm based on $E$. Then*

$$\mu_{p,n}(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\varepsilon,n} \leqslant \|\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_{\pi,n} = \nu_{p,n}(\boldsymbol{x}) \quad (\boldsymbol{x} \in E^n\,, n \in \mathbb{N})\,.$$

*Proof.* Set $q = p'$, and take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$. By Proposition 2.4, we have

$$\sup\left\{\left\|\sum_{i=1}^n \alpha_i x_i\right\| : \left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q} \leqslant 1\right\} \leqslant \|\boldsymbol{x}\|_n\ ,$$

and hence, by equations (1.5.3) and (1.5.6),

$$\|\boldsymbol{x}\|_{\varepsilon,n} = \mu_{p,n}(\boldsymbol{x}) = \sup\left\{\left\|\sum_{i=1}^n \alpha_i x_i\right\| : \left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q} \leqslant 1\right\} \leqslant \|\boldsymbol{x}\|_n\ .$$

The dual $q$–multi-norm based on $E'$ is $(\|\cdot\|_n')$. We have $\|\boldsymbol{\lambda}\|_{\varepsilon,n} = \mu_{q,n}(\boldsymbol{\lambda}) \leqslant \|\boldsymbol{\lambda}\|_n'$ for each $\boldsymbol{\lambda} \in (E')^n$, and hence

$$\|\boldsymbol{x}\|_n = \sup\left\{|\langle \boldsymbol{x},\, \boldsymbol{\lambda}\rangle| : \|\boldsymbol{\lambda}\|_n' \leqslant 1\right\} \leqslant \sup\{|\langle \boldsymbol{x},\, \boldsymbol{\lambda}\rangle| : \mu_{q,n}(\boldsymbol{\lambda}) \leqslant 1\} = \nu_{p,n}(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\pi,n}\ .$$

This completes the proof. ■

In particular, for each Banach space $E$ and each $p$ with $1 \leqslant p \leqslant \infty$, there are minimum and maximum $p$–multi-norms based on $E$, namely $(\mu_{p,n})$ and $(\nu_{p,n})$, respectively, as noted in [52]; for $n \in \mathbb{N}$, we have $\mu_{\infty,n} = \|\cdot\|_n^{\min}$ and $\nu_{\infty,n} = \|\cdot\|_n^{\max}$ in the notation of §1.1.

The following remarks are also contained in [52, §4.5]; clauses (i) and (ii) are immediate from Theorem 2.11.

PROPOSITION 2.12. *Let $E$ be a normed space, and take $p$ with $1 \leqslant p \leqslant \infty$.*

(i) *The dual of the maximum $p$–multi-norm based on $E$ is the minimum $p'$–multi-norm based on $E'$.*

(ii) *The dual of the minimum $p$–multi-norm based on $E$ is the maximum $p'$–multi-norm based on $E'$.*

(iii) *The bidual of a $p$–multi-norm based on $E$ is a $p$–multi-norm based on $E''$, and the canonical embedding of $(E^n, \|\cdot\|_n)$ into $((E'')^n, \|\cdot\|''_n)$ is an isometry for each $n \in \mathbb{N}$.* ■

**2.3. Interpolation spaces and $p$–multi-norms.** Let $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ be two (real or complex) Banach spaces such that $\{E_0, E_1\}$ is a compatible couple. Further, suppose that $(\|\cdot\|^0_n)$ and $(\|\cdot\|^1_n)$ are power-norms based on the respective spaces. Then, for each $m \in \mathbb{N}$, we consider the pair

$$\{(E_0^m, \|\cdot\|^0_m), (E_1^m, \|\cdot\|^1_m)\}\,.$$

Since $E_0^m \cap E_1^m = (E_0 \cap E_1)^m$ and $E_0^m + E_1^m = (E_0 + E_1)^m$, it follows that this pair is also a compatible couple of Banach spaces.

Now suppose that $E_0$ and $E_1$ are complex Banach spaces. Take $\theta \in (0,1)$, and set $E = (E_0, E_1)_\theta$, as in §1.10. Then the norms $\|\cdot\|^0_m$ and $\|\cdot\|^1_m$ are equivalent to the norms on $\ell^2_m(E_0)$ and $\ell^2_m(E_1)$, respectively, and so it follows from Theorem 1.45 that the intermediate space $((E_0^m, \|\cdot\|^0_m), (E_1^m, \|\cdot\|^1_m))_\theta$ is isomorphic to $\ell^2_m(E)$; the interpolation norm defined on $E^m$ by using $\|\cdot\|^0_m$ and $\|\cdot\|^1_m$ is denoted by $\|\cdot\|_m$.

THEOREM 2.13. *Let $\{(E_0, \|\cdot\|_0), (E_1, \|\cdot\|_1)\}$ be a compatible couple of complex Banach spaces, and suppose that $(\|\cdot\|^0_n)$ and $(\|\cdot\|^1_n)$ are power-norms based on $E_0$ and $E_1$, respectively. Take $\theta \in (0,1)$, and set $E = (E_0, E_1)_\theta$. Then $(E^n, \|\cdot\|_n)$ is a power-normed space.*

*Proof.* The axioms (A1), (A2), and (A3) are easily checked using Theorem 1.46. ■

DEFINITION 2.14. The pair $(E^n, \|\cdot\|_n)$ is the *interpolation power-normed space* of index $\theta$ defined by the compatible couple of complex Banach spaces $\{(E_0, \|\cdot\|_0), (E_1, \|\cdot\|_1)\}$ and the power-norms $(\|\cdot\|^0_n)$ and $(\|\cdot\|^1_n)$ based on $E_0$ and $E_1$, respectively; the power-norm based on $E$ is the *interpolation power-norm*.

For example, suppose that $(\|\cdot\|^0_n)$ and $(\|\cdot\|^1_n)$ are a $p_0$–sum and a $p_1$–sum power-norm (as in Example 2.7(ii)) based on Banach spaces $E_0$ and $E_1$, respectively, where $1 \leqslant p_0, p_1 < \infty$. Take $\theta \in (0,1)$, and define $p$ by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.$$

Then, by equation (1.10.1), the interpolation norm on $E^m$ is the $p$–sum power-norm based on $E$.

Now suppose that $1 \leqslant p_0, p_1 < \infty$ and that $(\|\cdot\|_n^0)$ is a $p_0$–multi-norm based on a complex Banach space $E_0$ and $(\|\cdot\|_n^1)$ is a $p_1$–multi-norm based on a complex Banach space $E_1$. Take $\theta \in (0,1)$, and define $p$ as above. We ask whether the interpolation power-norm $(\|\cdot\|_n)$ based on $E$ is a $p$–multi-norm. The first theorem shows that this is the case when $p_0 = p_1$; Example 2.16 will show that this may not be the case for certain values of $p_0$ and $p_1$ with $p_0 \neq p_1$, even when $E_0 = E_1$, and Example 2.32 will show that this may not be the case for more general values of $p_0$ and $p_1$.

THEOREM 2.15. *Take $p$ with $1 \leqslant p < \infty$, and suppose that $\{E_0, E_1\}$ is a compatible couple of complex Banach spaces and that there are $p$–multi-norms $(\|\cdot\|_n^0)$ and $(\|\cdot\|_n^1)$ based on $E_0$ and $E_1$, respectively. Take $\theta \in (0,1)$. Then the interpolation power-norm defined from these $p$–multi-norms that is based on $(E_0, E_1)_\theta$ is also a $p$–multi-norm.*

*Proof.* Set $E = (E_0, E_1)_\theta$.

Let $\mathcal{F}$ be the space of functions on the strip $\overline{S}$ taking values in $E_0 + E_1$, as defined in §1.10, and, for $k \in \mathbb{N}$, take $\mathcal{F}_k$ to be the corresponding space of functions on the strip $\overline{S}$ taking values in $E_0^k + E_1^k$, so that the image of the map

$$F \mapsto F(\theta)\,, \quad \mathcal{F}_k \to E_0^k + E_1^k\,,$$

is $E^k$; the space $E^k$ has the interpolation norm, say $\|\cdot\|_k$, determined by $\|\cdot\|_k^0$ and $\|\cdot\|_k^1$.

We need to check inequality (2.2.1) in Definition 2.3 for the interpolation power-norm $(\|\cdot\|_n)$ based on $E$. For this, take $m, n \in \mathbb{N}$, $T \in \mathcal{B}(\ell_m^p, \ell_n^p)$ with $\|T : \ell_m^p \to \ell_n^p\| \leqslant 1$, and $\boldsymbol{x} = (x_1, \ldots, x_m) \in E^m$.

Take $\varepsilon > 0$. Then there exists $F \in \mathcal{F}_m$ with $F(\theta) = \boldsymbol{x}$ and $\|F\|_{\mathcal{F}_m} < \|\boldsymbol{x}\|_m + \varepsilon$. Set

$$G = T \circ F : \overline{S} \to E^n\,.$$

Then it is easily seen that, as a map from $\overline{S}$ into $(E_0^n + E_1^n, \|\cdot\|_{E_0^n + E_1^n})$, the new function $G$ satisfies the conditions for it to belong to the space $\mathcal{F}_n$. For $j = 0, 1$ and $z \in L_j$, we have $\|G(z)\|_{E_j^n} \leqslant \|F(z)\|_{E_j^m}$ because both $E_0$ and $E_1$ are $p$–multi-normed spaces, and so $\|G\|_{\mathcal{F}_n} \leqslant \|F\|_{\mathcal{F}_m}$. Since $G(\theta) = T\boldsymbol{x}$, it follows that $\|T\boldsymbol{x}\|_n < \|\boldsymbol{x}\|_m + \varepsilon$. This holds true for each $\varepsilon > 0$, and so $\|T\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_m$. Thus (2.2.1) holds, as required. ∎

EXAMPLE 2.16. Let $E$ be a complex normed space, and consider the maximum dual multi-norm and minimum multi-norm based on $E$. Take $\theta \in (0,1)$. Then, as in equation (1.10.2), for each $m \in \mathbb{N}$, the interpolation space between $\ell_m^1(E)$ and $\ell_m^\infty(E)$ is $\ell_m^p(E)$, where $p = 1/(1-\theta)$, and so the interpolation power-norm based on $E$ is a $p$–multi-norm if and only if the $p$–sum power-norm based on $E$ of Example 2.7(ii) is a $p$–multi-norm. However this is not the case for suitable Banach spaces $E$. Indeed, suppose that $E = \ell^r$. Then, as stated after Theorem 2.8, the $p$–sum power-norm based on $E$ is not a $p$–multi-norm when $r$ is outside a certain range of values. ∎

**2.4. Characterization of $p$–multi-norms.** We shall now characterize $p$–multi-norms in terms of tensor products.

In [18], it was explained how multi-norms correspond to certain tensor norms. We recall this briefly; details are given in [18, §3].

DEFINITION 2.17. Let $E$ be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a $c_0$*–norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with norm at most $\|T\|$ for each compact operator $T$ on $c_0$.

Suppose that $\|\cdot\|$ is a $c_0$–norm on $c_0 \otimes E$, and set

$$\|(x_1, \dots, x_n)\|_n = \left\| \sum_{i=1}^{n} \delta_i \otimes x_i \right\| \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,. \tag{2.4.1}$$

Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm based on $E$.

A more general and detailed version of the following theorem is given as [18, Theorem 3.4].

THEOREM 2.18. *Let $E$ be a normed space. Then the above construction defines a bijection from the family of $c_0$–norms on $c_0 \otimes E$ onto the family of multi-norms based on $E$. The injective tensor norm and the projective tensor norm on $c_0 \otimes E$ correspond to the minimum and maximum multi-norms, respectively.* ∎

A norm $\|\cdot\|$ on $c_0 \otimes E$ satisfies '*property* (P)' (due to Pisier) [45, §2, p. 12] if

$$\|T \otimes I_E\| \leqslant \|T\| \quad (T \in \mathcal{B}(c_0))\,. \tag{2.4.2}$$

It is shown in [18, Corollary 3.6] that these norms are exactly the $c_0$–norms of Definition 2.17, and so the definition of a multi-normed space corresponds to the theory in the memoir of Marcolino Nhani [45] concerning norms on $c_0 \otimes E$ satisfying property (P). In particular, the word 'compact' is not required in Definition 2.17, as noted in [18]. As we shall explain in §5.1, $c_0$–norms also arise in the thesis [44] of McClaran.

In the paper [18], there is also a notion of an $\ell^1$*–norm* on $\ell^1 \otimes E$, and it is noted in [18, §4.1] that $\ell^1$–norms correspond to dual multi-norms in an analogous way to that defined above. These results will be generalized below.

We have the following analogue of Definition 2.17 and Theorem 2.18.

DEFINITION 2.19. Let $E$ be a normed space, and take $p$ with $1 \leqslant p < \infty$. Then a norm $\|\cdot\|$ on $\ell^p \otimes E$ is an $\ell^p$*–norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(\ell^p \otimes E, \|\cdot\|)$ with norm at most $\|T\|$ for each operator $T$ on $\ell^p$.

It is clear from Theorem 1.13 that the projective tensor norm $\|\cdot\|_\pi$ and the injective tensor norm $\|\cdot\|_\varepsilon$ on $\ell^p \otimes E$ are each $\ell^p$–norms.

Take $p$ with $1 \leqslant p < \infty$, and let $\|\cdot\|$ be an $\ell^p$–norm on $\ell^p \otimes E$. Fix $\alpha \in \ell^p$ and $x \in E$, and define $S\beta = \beta_1 \alpha$ $(\beta \in \ell^p)$. Then $S$ is a finite-rank operator on $\ell^p$ with $\|S\| = \|\alpha\|_{\ell^p}$ and $(S \otimes I_E)(\delta_1 \otimes x) = \alpha \otimes x$. Thus

$$\|\alpha \otimes x\| = \|(S \otimes I_E)(\delta_1 \otimes x)\| \leqslant \|S\|\,\|\delta_1 \otimes x\| = \|\alpha\|_{\ell^p}\,\|x\|\,,$$

and so $\|\cdot\|$ is a sub-cross-norm on $\ell^p \otimes E$. Essentially as in equation (2.4.1), we define

$$\|(x_1, \dots, x_n)\|_n = \left\| \sum_{i=1}^{n} \delta_i \otimes x_i \right\| \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,. \tag{2.4.3}$$

Then $\|\cdot\|_1$ coincides with the given norm on $E$, and it is clear that each $\|\cdot\|_n$ is a norm on $E^n$ and that (2.2.1) is satisfied. Hence $(\|\cdot\|_n : n \in \mathbb{N})$ is a $p$–multi-norm based on $E$.

By Theorem 2.11,

$$\|z\|_{\varepsilon} \leqslant \|z\| \leqslant \|z\|_{\pi} \quad (z \in \ell^p \otimes E),$$

and so it follows from Proposition 1.12 that $\|\cdot\|$ is a reasonable cross-norm on $\ell^p \otimes E$.

The following statement recasts the definition of a $p$–multi-norm in the above notation.

PROPOSITION 2.20. *Let $E$ be a normed space, and take $p$ with $1 \leqslant p < \infty$. Then a sequence $(E^n; \|\cdot\|_n : n \in \mathbb{N})$ corresponds to a $p$–multi-norm based on $E$ if and only if*

$$\|T \otimes I_E : \ell^p_m \otimes E \to \ell^p_n \otimes E\| \leqslant \|T : \ell^p_m \to \ell^p_n\|$$

*for each $m, n \in \mathbb{N}$ and $T \in \mathcal{B}(\ell^p_m, \ell^p_n)$.* ■

THEOREM 2.21. *Let $E$ be a normed space, and take $p$ with $1 \leqslant p < \infty$. Then the construction given in equation (2.4.3) defines a bijection from the family of $\ell^p$–norms on $\ell^p \otimes E$ onto the family of $p$–multi-norms based on $E$.*

*Proof.* Suppose that $\|\cdot\|$ is an $\ell^p$–norm on $\ell^p \otimes E$. Then we have noted that $(\|\cdot\|_n : n \in \mathbb{N})$ is a $p$–multi-norm based on $E$.

Conversely, suppose that $(\|\cdot\|_n : n \in \mathbb{N})$ is a $p$–multi-norm based on $E$.

First note that each element $z$ of $c_{00} \otimes E$ can be expressed 'essentially uniquely' in the form $z = \sum_{j=1}^n \delta_j \otimes x_j$ for some $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, in the sense that the representation is unique up to the addition of some zero vectors $x_j$. In this case, we define

$$\|z\| = \|(x_1, \dots, x_n)\|_n \,.$$

That $\|z\|$ is uniquely defined follows because $(\|\cdot\|_n)$ satisfies Axiom (A3). It is clear that $\|\cdot\|$ is a norm on $c_{00} \otimes E$.

We *claim* that $\|\cdot\|$ is a cross-norm on $c_{00} \otimes E$ with respect to the norm $\|\cdot\|_{\ell^p}$ on $c_{00}$. Indeed, take $\alpha = (\alpha_1, \dots, \alpha_n)$ in $c_{00}$ and $x \in E$. Then

$$\|\alpha \otimes x\| = \left\|\sum_{j=1}^n \delta_j \otimes \alpha_j x\right\| = \|(\alpha_1 x, \dots, \alpha_n x)\|_n = \|\alpha\|_{\ell^p} \|x\|$$

by equation (2.2.4), and this gives the claim.

Next, take $m, n \in \mathbb{N}$, and consider $z = \sum_{j=1}^n \delta_j \otimes x_j \in c_{00} \otimes E$ and $T = (T_{i,j}) \in \mathbb{M}_{m,n}$. Then

$$(T \otimes I_E)(z) = \sum_{j=1}^n \sum_{i=1}^m T_{i,j}\delta_i \otimes x_j = \sum_{i=1}^m \delta_i \otimes \left(\sum_{j=1}^n T_{i,j} x_j\right) = \sum_{i=1}^m \delta_i \otimes (T\boldsymbol{x})_i \,, \tag{2.4.4}$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$, and so

$$\|(T \otimes I_E)(z)\| = \|T\boldsymbol{x}\|_m \leqslant \|T : \ell^p_n \to \ell^p_m\| \, \|\boldsymbol{x}\|_n = \|T\| \, \|z\| \,.$$

Thus

$$\|T \otimes I_E\| \leqslant \|T\| \,. \tag{2.4.5}$$

We shall now extend the above norm $\|\cdot\|$ from $c_{00} \otimes E$ to $\ell^p \otimes E$.

Take $z \in \ell^p \otimes E$, so that

$$z = \sum_{j=1}^{k} u_j \otimes x_j \quad \text{for some} \quad k \in \mathbb{N}\,, \quad u_1, \dots, u_k \in \ell^p\,, \;\text{ and }\; x_1, \dots, x_k \in E\,; \tag{2.4.6}$$

we may suppose that $z \neq 0$ and that the sets $\{u_1, \dots, u_k\}$ and $\{x_1, \dots, x_k\}$ are linearly independent in $\ell^p$ and $E$, respectively. Define

$$z_n = (P_n \otimes I_E)(z) \in c_{00} \otimes E\,, \quad t_n = \|z_n\| \quad (n \in \mathbb{N})\,.$$

Then we have

$$t_n = \|(P_n \otimes I_E)(z)\| = \|(P_n \otimes I_E)(P_{n+1} \otimes I_E)(z)\| \leqslant \|P_n\|\, t_{n+1} \quad (n \in \mathbb{N})$$

by (2.4.5). Since $\|P_n\| = 1$, the sequence $(t_n)$ is increasing in $\mathbb{R}$. Further,

$$t_n = \left\| \sum_{j=1}^{k} P_n u_j \otimes x_j \right\| \leqslant \sum_{j=1}^{k} \|P_n u_j \otimes x_j\| = \sum_{j=1}^{k} \|P_n u_j\|\, \|x_j\| \leqslant \sum_{j=1}^{k} \|u_j\|\, \|x_j\| \quad (n \in \mathbb{N})\,,$$

where we are using the fact that $\|\cdot\|$ is a cross-norm on $c_{00} \otimes E$, and so $(t_n)$ is bounded above. Hence $(t_n)$ converges, and so we may define

$$\|z\| = \lim_{n \to \infty} t_n = \sup\{\|(P_n \otimes I_E)(z)\| : n \in \mathbb{N}\}\,.$$

In the case where $z \in c_{00} \otimes E$, the new definition is consistent with the existing definition.

Clearly the map $\|\cdot\| : z \mapsto \|z\|$ is a semi-norm on $\ell^p \otimes E$. Now take $z \in \ell^p \otimes E$ with $z \neq 0$, and express $z$ in the form (2.4.6). Since $\{u_1, \dots, u_k\}$ is linearly independent in $\ell^p$, it follows from Proposition 1.10 that there exists $n \in \mathbb{N}$ such that $\{P_n u_1, \dots, P_n u_k\}$ is linearly independent in $c_{00}$. Thus $z_n = \sum_{j=1}^{k} P_n u_j \otimes x_j \neq 0$. This implies that $\|z\| \geqslant \|z_n\| > 0$, and so $\|\cdot\|$ is a norm on $\ell^p \otimes E$. This norm extends the specified norm on $c_{00} \otimes E$, and also $z = \lim_{n \to \infty} z_n$ with respect to $\|\cdot\|$ for each $z \in \ell^p \otimes E$, so that $c_{00} \otimes E$ is dense in $(\ell^p \otimes E, \|\cdot\|)$.

Take $T$ to be an operator on $\ell^p$, say with $\|T\| = 1$, and take $z \in \ell^p \otimes E$ to be of the form in equation (2.4.6). Then, for each $m, n \in \mathbb{N}$, we see that

$$\begin{aligned} \|(P_n T \otimes I_E)(z)\| &\leqslant \|(P_n T \otimes I_E)(z - z_m)\| + \|(P_n T \otimes I_E)(z_m)\| \\ &\leqslant \sum_{j=1}^{k} \|P_n T\|\, \|(I_{\ell^p} - P_m)(u_j)\|\, \|x_j\| + \|(P_n T P_m \otimes I_E)(z_m)\|\,. \end{aligned}$$

We have $\lim_{m \to \infty} \|(I_{\ell^p} - P_m)(u)\| = 0$ for each $u \in \ell^p$. Also, by (2.4.5), we have

$$\|(P_n T P_m \otimes I_E)(z_m)\| \leqslant \|P_n T P_m\|\, \|z_m\| \leqslant \|z_m\| \leqslant \|z\| \quad (m, n \in \mathbb{N})\,,$$

and so $\|(P_n T \otimes I_E)(z)\| \leqslant \|z\| \;\; (n \in \mathbb{N})$. Hence $\|(T \otimes I_E)(z)\| \leqslant \|z\|$, and so $\|\cdot\|$ is an $\ell^p$–norm on $\ell^p \otimes E$.

The correspondence that we have described is clearly a bijection. ∎

The above proof also establishes Theorem 2.18 by replacing '$\ell^p$' by '$c_0$' throughout. As such the proof seems to be simpler than the one of this specific fact given in [18].

**2.5. Strong $p$–multi-norms.** There are strengthenings of the concept of a $p$–multi-norm that we shall describe in the next two sections. The rôle of these strengthenings will become apparent later, in the representation theorems of Chapter 5. We recall that the notation $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$ was introduced in Definition 1.37.

Definition 2.22. Let $E$ be a linear space, and take $p$ with $1 \leqslant p \leqslant \infty$. A *strong $p$–multi-norm based on $E$* is a sequence $(\|\cdot\|_n)$ such that $\|\cdot\|_n$ is a norm on $E^n$ for each $n \in \mathbb{N}$ and such that $\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$ whenever $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, $\boldsymbol{y} \in E^n$, and $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. In this case, $(E^n, \|\cdot\|_n)$ is a *strong $p$–multi-normed space*.

It is clear that each strong $p$–multi-norm is a power-norm. The following result shows that it is indeed a $p$–multi-norm.

Proposition 2.23. *Let $E$ be a linear space, and take $p$ with $1 \leqslant p \leqslant \infty$. Suppose that $(E^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space. Then $(\|\cdot\|_n)$ is a $p$–multi-norm based on $E$.*

*Proof.* Take $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^n$, and $T \in \mathbb{M}_{m,n}$ with $\|T : \ell_n^p \to \ell_m^p\| \leqslant 1$. Then

$$\|\langle T\boldsymbol{x},\, \lambda\rangle\|_{\ell_m^p} = \|T(\langle \boldsymbol{x},\, \lambda\rangle)\|_{\ell_m^p} \leqslant \|\langle \boldsymbol{x},\, \lambda\rangle\|_{\ell_n^p} \quad (\lambda \in E')\,,$$

and so $\boldsymbol{T}\boldsymbol{x} \preccurlyeq_p \boldsymbol{x}$. Hence $\|\boldsymbol{T}\boldsymbol{x}\|_m \leqslant \|\boldsymbol{x}\|_n$ by the defining condition of a strong $p$–multi-norm. This shows that $(\|\cdot\|_n)$ is a $p$–multi-norm. ∎

The following result is immediately checked.

Proposition 2.24. *Let $E$ be a linear space, take $p$ with $1 \leqslant p \leqslant \infty$, and let $(\|\cdot\|_n)$ be a strong $p$–multi-norm based on $E$.*

(i) *Suppose that $F$ is a subspace of $E$. Then $(F^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space.*

(ii) *Suppose that $m \in \mathbb{N}$, and set $F = E^m$. Then $(F^n, \|\cdot\|_{mn})$ is a strong $p$–multi-normed space.* ∎

We shall now see that the converse of Proposition 2.23 is true in the special cases where $p = 2$ or $p = \infty$; we recall that the latter case corresponds to multi-norms themselves. In Example 2.31, we shall show that the converse holds for all Banach spaces *only* when $p = 2$ or $p = \infty$.

Theorem 2.25. *Let $p = 2$ or $p = \infty$, and suppose that $(E^n, \|\cdot\|_n)$ is a $p$–multi-normed space. Then $(\|\cdot\|_n)$ is a strong $p$–multi-norm.*

*Proof.* Take $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$ such that $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$, and set

$$Z = \{\langle \boldsymbol{x},\, \lambda\rangle : \lambda \in E'\}\,.$$

By Theorem 1.38, there is a matrix $A \in \mathbb{M}_{n,m}$ such that $A\boldsymbol{x} = \boldsymbol{y}$ and $A \mid Z : Z \to \ell_n^p$ is a contraction as an element of $\mathcal{B}(Z, \ell_n^p)$, where the norm on $Z$ is the restriction of the norm on $\ell_m^p$.

In the case where $p = 2$, there is an orthogonal projection $P$ of $\ell_m^p$ onto $Z$ with $\|P\| = 1$, and we set $T = (A \mid Z) \circ P : \ell_m^p \to \ell_n^p$. In the case where $p = \infty$, the space $\ell_n^p$ is a 1–injective space, and so there is an extension $T : \ell^p_m \to \ell^p_n$ of $A \mid Z$ with $\|T\| = \|A \mid Z\|$. In both cases $T$ is a contraction.

For each $\lambda \in E'$, we have

$$\langle T\boldsymbol{x},\, \lambda\rangle = (\langle A\boldsymbol{x},\, \lambda\rangle) = \langle \boldsymbol{y},\, \lambda\rangle\,,$$

and so $\boldsymbol{y} = T\boldsymbol{x}$. Since $(\|\cdot\|_n)$ is a $p$–multi-norm, inequality (2.2.1) holds, and so we have $\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$, as required. ∎

In particular, each multi-norm is a strong multi-norm.

Recall that the quotient of a $p$–multi-norm is a $p$–multi-norm. However, it is not generally true that the quotient of a strong $p$–multi-norm is necessarily a strong $p$–multi-norm. (By Theorem 2.25, this is true for $p = 2$ and $p = \infty$.) An example to demonstrate this when $2 < p < \infty$ will be given within Example 2.30, below, and a counter-example for each $p$ with $1 \leqslant p < \infty$ and $p \neq 2$ will be given in Example 2.31. The example within Example 2.30 will also show that, for $1 \leqslant p < 2$, the dual of a strong $p$-multi-norm, which is a $p'$-multi-norm, is not necessarily a strong $p'$–multi-norm; Corollary 2.38 will show the stronger result that this holds for each $p$ with $1 < p < \infty$ and $p \neq 2$.

THEOREM 2.26. *Let $E$ and $F$ be infinite-dimensional Banach spaces such that $E$ is finitely representable in $F$, and take $p$ with $1 \leqslant p \leqslant \infty$. Suppose that the $p$–sum power-norm based on $F$ is a strong $p$–multi-norm. Then the $p$–sum power-norm based on $E$ is also a strong $p$–multi-norm.*

*Proof.* Take $m, n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$, and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$ such that $\boldsymbol{y} \leqslant_p \boldsymbol{x}$, so that

$$\sum_{j=1}^{n} |\langle y_j,\, \lambda\rangle|^p \leqslant \sum_{i=1}^{m} |\langle x_i,\, \lambda\rangle|^p \quad (\lambda \in E')\,.$$

Set $X = \mathrm{lin}\,\{x_1, \dots, x_m, y_1, \dots, y_n\}$, a finite-dimensional subspace of $E$, and take $\varepsilon > 0$. Then there is a finite-dimensional subspace $Y$ of $F$ and an isomorphism $T : X \to Y$ such that $\|T\|\,\|T^{-1}\| < 1 + \varepsilon$.

Take $\mu \in F'$. Then $T'(\mu \mid Y)$ belongs to $X'$, and so has a norm-preserving extension, say $\lambda$, to $E'$. Thus $\langle Tz,\, \mu\rangle = \langle z,\, T'(\mu \mid Y)\rangle = \langle z, \lambda\rangle$ $(z \in X)$, and so

$$\sum_{j=1}^{n} |\langle Ty_j,\, \mu\rangle|^p \leqslant \sum_{i=1}^{m} |\langle Tx_i,\, \mu\rangle|^p \quad \text{and} \quad \sum_{j=1}^{n} \|Ty_j\|^p \leqslant \sum_{i=1}^{m} \|Tx_i\|^p$$

because the $p$–sum power-norm based on $F$ is a strong $p$–multi-norm. Hence

$$\begin{aligned}\sum_{j=1}^{n} \|y_j\|^p &\leqslant \|T^{-1}\|^p \sum_{j=1}^{n} \|Ty_j\|^p \leqslant \|T^{-1}\|^p \sum_{i=1}^{m} \|Tx_i\|^p \\ &\leqslant \|T\|^p\,\|T^{-1}\|^p \sum_{i=1}^{m} \|x_i\|^p \leqslant (1+\varepsilon)^p \sum_{i=1}^{m} \|x_i\|^p\,.\end{aligned}$$

This holds true for each $\varepsilon > 0$, and so $\sum_{j=1}^{n} \|y_j\|^p \leqslant \sum_{i=1}^{m} \|x_i\|^p$.

We have shown that the $p$–sum power-norm based on $E$ is a strong $p$–multi-norm. ∎

We consider again some examples of $p$–multi-norms that were given above in Example 2.7.

EXAMPLES 2.27. Take $p$ with $1 \leqslant p \leqslant \infty$.

(i) The unique $p$–multi-norm based on $\mathbb{F}$ is obviously a strong $p$–multi-norm.

(ii) Let $E$ be a normed space, and again consider the $p$–sum power-norm based on $E$ given by

$$\|\boldsymbol{x}\|_n := \|\boldsymbol{x}\|_{\ell^p_n(E)} = \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \qquad (x_1, \dots, x_n \in E,\, n \in \mathbb{N}).$$

Certainly this power-norm is a strong $p$–multi-norm when $p = \infty$, so we now suppose that $1 \leqslant p < \infty$.

We know that this power-norm is a $p$–multi-norm in the special case that $E = \ell^p$. In fact, it is a strong $p$–multi-norm in this case. To see this, fix $m, n \in \mathbb{N}$, and take $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$, say $x_i = (x_{ik})$ for $i \in \mathbb{N}_m$ and $y_j = (y_{jk})$ for $j \in \mathbb{N}_n$. Suppose that $\|\langle \boldsymbol{y}, \lambda\rangle\|_{\ell^p_n} \leqslant \|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell^p_m}$ just for each $\lambda \in E' = \ell^{p'}$ of the form $\delta_k$ for $k \in \mathbb{N}$. Then

$$\sum_{j=1}^n |y_{jk}|^p \leqslant \sum_{i=1}^m |x_{ik}|^p$$

for each $k \in \mathbb{N}$, and so

$$\sum_{j=1}^n \sum_{k=1}^\infty |y_{jk}|^p = \sum_{k=1}^\infty \sum_{j=1}^n |y_{jk}|^p \leqslant \sum_{k=1}^\infty \sum_{i=1}^m |x_{ik}|^p = \sum_{i=1}^m \sum_{k=1}^\infty |x_{ik}|^p \,.$$

Thus

$$\sum_{j=1}^n \|y_j\|^p \leqslant \sum_{i=1}^m \|x_i\|^p \,,$$

and hence $\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$, as required.

By Theorem 2.26, the $p$–sum power-norm is a strong $p$–multi-norm when based on any Banach space $E$ that is finitely representable in $\ell^p$.

Let $\Omega$ be a measure space. Suppose that either $1 \leqslant p \leqslant r \leqslant 2$ or $p > 2$ and $r = 2$ or $r = p$. Then, by Theorem 1.26(i), the space $L^r(\Omega)$ is finitely representable in $\ell^p$, and so the $p$–sum power-norm based on $L^r(\Omega)$ is a strong $p$–multi-norm. In particular, the $p$–sum power-norm based on $L^p(\Omega)$ is a strong $p$–multi-norm.

We shall see shortly that the $p$–sum power-norm based on a Banach space $E$ may be a $p$–multi-norm that is not a strong $p$–multi-norm.

(iii) Let $E$ be a normed space, and consider the weak $p$–summing norm $(\mu_{p,n})$ based on $E$. Take $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$ with $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. Since

$$\mu_{p,m}(\boldsymbol{x}) = \sup\{\|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell^p_m} : \lambda \in B_{E'}\}\,,$$

it is immediate that $\mu_{p,n}(\boldsymbol{y}) \leqslant \mu_{p,m}(\boldsymbol{x})$, and so $(\mu_{p,n})$ is a strong $p$–multi-norm.

However it is not necessarily the case that each quotient of the weak $p$–summing norm is a strong $p$–multi-norm; we shall see this in Example 2.39.

(iv) Let $E$ be a normed space, and consider the dual weak $p$–summing norm $(\nu_{p,n})$ based on $E$.

There are Banach spaces $E$ such that $(\nu_{p,n})$, when based on $E$, is and is not a strong $p$–multi-norm. Indeed, by (i), $(\nu_{p,n})$, when based on $\mathbb{F}$, is a strong $p$–multi-norm. However Theorem 2.37 will show that this is not necessarily the case when $1 \leqslant p < \infty$ and $p \neq 2$, even for certain finite-dimensional spaces $E$.

(v) The 2–multi-norm defined in Example 2.9 is a strong 2–multi-norm. ∎

Let $E$ be a Banach space. We showed in Theorem 2.8 that the $p$–sum power-norm based on $E$ is a $p$–multi-norm if and only if $E$ belongs to the class $SQ(p)$ if and only if $E$ is a $p$–space. In contrast, we obtain the following theorem; it is an immediate consequence of Corollary 1.40 and the above remarks.

THEOREM 2.28. *Let $E$ be a Banach space, and take $p$ with $1 \leqslant p \leqslant \infty$. Then the following conditions on $E$ are equivalent:*

(a) *the $p$–sum power-norm based on $E$ is a strong $p$–multi-norm;*

(b) *$E$ is isometrically isomorphic to a closed subspace of $L^p(\Omega, \mu)$ for some measure space $(\Omega, \mu)$.* ∎

COROLLARY 2.29. *Take $p$ and $r$ with $2 < p < \infty$ and $1 \leqslant r < \infty$. Then:*

(i) *the $p$–sum power-norm based on $\ell^r$ is a $p$–multi-norm if and only if $2 \leqslant r \leqslant p$;*

(ii) *the $p$–sum power-norm based on $\ell^r$ is a strong $p$–multi-norm if and only if $r = 2$ or $r = p$;*

(iii) *the $p$–sum power-norm based on $\ell^r_n$ is a strong $p$–multi-norm for each $n \in \mathbb{N}$ if and only if $r = 2$ or $r = p$.*

*Proof.* (i) This is noted on page 46.

(ii) It follows from Theorem 2.28 that the $p$–sum power-norm based on $\ell^r$ is a strong $p$–multi-norm if and only if $\ell^r$ is isometrically isomorphic to a closed subspace of $L^p(\Omega, \mu)$ for some measure space $(\Omega, \mu)$; by Proposition 1.22 and Theorem 1.26(ii), this holds if and only if $r = 2$ or $r = p$.

(iii) Suppose that $r = 2$ or $r = p$. By (ii), the $p$–sum power-norm based on $\ell^r$ is a strong $p$–multi-norm, and so the same is true for the $p$–sum power-norm based on $\ell^r_n$ for each $n \in \mathbb{N}$.

Suppose that the $p$–sum power-norm based on $\ell^r_n$ is a strong $p$–multi-norm for each $n \in \mathbb{N}$. By Theorem 2.28 and the remarks above Theorem 1.26, $\ell^r_n$ embeds isometrically in $L^p(\mathbb{I})$ for each $n \in \mathbb{N}$. It follows that $\ell^r$ is finitely representable in $L^p(\mathbb{I})$, and so, by Proposition 1.24, $\ell^r$ is isometrically isomorphic to a closed subspace of $L^p(\mathbb{I})$. Again, this implies that $r = 2$ or $r = p$. ∎

EXAMPLE 2.30. Take $p$ and $r$ with $2 < r < p < \infty$. Then the $p$–sum power-norm based on $L^p(\mathbb{I})$ is certainly a strong $p$–multi-norm. By Corollary 1.23, $\ell^r$ is isometrically isomorphic to a quotient of $L^p(\mathbb{I})$. The quotient multi-norm based on $\ell^r$ is also the $p$–sum power-norm, but, by Corollary 2.29(ii), this is not a strong $p$–multi-norm. Thus the quotient of a strong $p$–multi-norm is not necessarily a strong $p$–multi-norm.

Now suppose that $1 < p < s < 2$. Then the $p$–sum power-norm based on $\ell^s$ is a strong $p$–multi-norm, but the dual $p'$–multi-norm based on $\ell^r$, where $r = s'$, is a $p'$–multi-norm that is not a strong $p'$–multi-norm. Thus the dual of a strong $p$–multi-norm is not necessarily a strong $p'$–multi-norm. ∎

Example 2.31. This example will extend the previous one by showing that, for each $p$ with $1 \leqslant p < \infty$ and $p \neq 2$, there is a Banach space $E$, a strong $p$–multi-norm based on $E$, and a closed subspace $F$ of $E$ such that the quotient power-norm based on $E/F$ is not a strong $p$–multi-norm. In particular, this shows that, for each such $p$, there is a $p$–multi-norm that is not strong.

Indeed, for each $p$ with $1 \leqslant p < \infty$ and $p \neq 2$, it follows from Theorem 1.34 that there is a closed subspace $E$ of a space $L^p(\Omega)$ that has a quotient $F$ which is not isomorphic to a closed linear subspace of any space $L^p(\Sigma)$. By Theorem 2.28, the $p$–sum power-norm based on $E$ is a strong $p$–multi-norm. The quotient of this power-norm is the $p$–sum power-norm based on $F$; by Theorem 2.28 again, this latter $p$–multi-norm is not strong. ∎

In summary, the class of $p$–multi-normed spaces is closed under taking quotients, but this is not true for the class of strong $p$–multi-normed spaces when $1 \leqslant p < \infty$ and $p \neq 2$.

We now consider when interpolation preserves strong $p$–multi-norms. The first example given below shows that the interpolation space between two strong $p_0$– and $p_1$–multi-normed spaces (with $p_0 \neq p_1$) need not be a $p$–multi-normed space and, even in the special case that $p_0 = p_1 = p$, so that the interpolation space is a $p$–multi-normed space, it is not necessarily a strong $p$–multi-normed space.

Example 2.32. Let $E_0$ and $E_1$ be two complex Banach spaces, take $p_0$ and $p_1$ with $1 \leqslant p_0, p_1 < \infty$, and take $\theta \in (0,1)$. As usual, define $p$ by the formula

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.$$

As in equation (1.10.1), $(\ell^{p_0}(E_0), \ell^{p_1}(E_1))_\theta = \ell^p(E)$, where $E = (E_0, E_1)_\theta$, and so, as before, the interpolated norm on $\ell^p(E)$ from the $p_0$– and $p_1$–sum power-norms on $E_0$ and $E_1$, respectively, is the $p$–sum power-norm based on $E$.

Suppose that

$$1 \leqslant p_0 < 2 < p_1 < \infty\,,$$

and take $E_0 = \ell^{p_0}(\mathbb{C})$ and $E_1 = \ell^2(\mathbb{C})$. Now take $j$ to be 0 or 1. In both cases, it follows from Proposition 1.22 that the space $E_j$ embeds isometrically into $L^{p_j}(\mathbb{I}, \mathbb{C})$, and so, by Theorem 2.28, the $p_j$–sum power-norm based on $E_j$ is a strong $p_j$–multi-norm. However, $(E_0, E_1)_\theta = \ell^q(\mathbb{C})$, where

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{2}\,,$$

and clearly $q < \min\{p, 2\}$. By remarks on page 46, the $p$–sum power-norm based on $\ell^{\,q}(\mathbb{C})$ is not a $p$–multi-norm.

Now suppose that $2 < p < \infty$ and take $E_0 = \ell^{\,2}(\mathbb{C})$ and $E_1 = \ell^{\,p}(\mathbb{C})$. By Proposition 1.22, both the spaces $E_0$ and $E_1$ embed isometrically into $L^p(\mathbb{I}, \mathbb{C})$, and so the $p$–sum power-norm on both $E_0$ and $E_1$ is a strong $p$–multi-norm. We have $(E_0, E_1)_\theta = \ell^{\,r}(\mathbb{C})$, where

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{p}\,,$$

so that $2 < r < p$. By Corollary 2.29(i), $\ell^{\,r}(\mathbb{C})$ is in the class $SQ(p)$, and so the interpolated $p$–sum power-norm on $(E_0, E_1)_\theta$ is a $p$–multi-norm; this also follows from Theorem 2.15. However, by Corollary 2.29(ii), this $p$–multi-norm is not a strong $p$–multi-norm. ■

We now exhibit a finite-dimensional Banach space and a 1–multi-norm (i.e., a dual multi-norm) based on this space such that the 1–multi-norm is not a strong 1–multi-norm. The example also shows that the dual of a multi-norm, which is a 1–multi-norm, is not necessarily a strong 1–multi-norm. A more general example will be given in Corollary 2.38, but the present calculation is elementary and avoids an appeal to deep theorems contained within Theorem 1.28.

Example 2.33. Fix $n \in \mathbb{N}$, and consider the finite-dimensional Banach space $E = \ell^{\,\infty}_n$, with dual space $E' = \ell^{\,1}_n$. We define $\boldsymbol{y} = c_n(\delta_1, \dots, \delta_n) \in E^n$, where $c_n > 0$ is to be determined. Set $m = 2^n$, and let $x_1, \dots, x_m$ be the vectors in $E$ of the form $(\varepsilon_1, \dots, \varepsilon_n)$, where each $\varepsilon_i$ is equal to $\pm 1$ and each choice of $(\varepsilon_1, \dots, \varepsilon_n)$ is taken exactly once, so that $\|x_j\|_E = 1 \ (j \in \mathbb{N}_m)$; set $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$.

Now take $\lambda = (\lambda_1, \dots, \lambda_n) \in E'$, say with $\|\lambda\|_{\ell^{\,1}_n} = \sum_{j=1}^n |\lambda_j| = 1$. Then we have

$$\|\langle \boldsymbol{y}, \lambda\rangle\|_{\ell^{\,1}_n} = c_n \sum_{j=1}^n |\lambda_j| = c_n\,.$$

Also

$$\|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell^{\,1}_m} = \sum \left\{ \left| \sum_{j=1}^n \varepsilon_j \lambda_j \right| : \varepsilon_j = \pm 1,\ j \in \mathbb{N}_n \right\} \geqslant A_1 m \cdot \left( \sum_{j=1}^n |\lambda_j|^2 \right)^{1/2}$$

by Khintchine's inequality; here $A_1$ is an absolute constant. In fact, by [58], $A_1 = 1/\sqrt{2}$. By Hölder's inequality, we have

$$1 = \sum_{j=1}^n |\lambda_j| \leqslant n^{1/2} \cdot \left( \sum_{j=1}^n |\lambda_j|^2 \right)^{1/2},$$

and so $\|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell^{\,1}_m} \geqslant 2^n/(2n)^{1/2}$. Thus $\boldsymbol{y} \preccurlyeq_1 \boldsymbol{x}$ when we make the choice $c_n = 2^n/(2n)^{1/2}$ for $n \in \mathbb{N}$.

We consider the 1–multi-norm based on $E$ that is defined by

$$\|(z_1, \dots, z_k)\|_k = \sum_{j=1}^k \|z_j\|_E \quad (z_1, \dots, z_k \in E,\ k \in \mathbb{N})\,;$$

this is the maximum dual multi-norm based on $E$. We have

$$\|\boldsymbol{y}\|_n = c_n \sum_{j=1}^{n} \|\delta_j\|_E = nc_n = \frac{n^{1/2}2^n}{\sqrt{2}}\,.$$

Moreover, $\|\boldsymbol{x}\|_m = m = 2^n$, and so the inequality '$\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$' fails whenever $n^{1/2} > \sqrt{2}$, i.e., whenever $n \geqslant 3$.

We conclude that there is a 1–multi-norm based on a finite-dimensional space $\ell^\infty_n$ that is not a strong 1–multi-norm.

Now take $F = E' = \ell^1_n$. Then the corresponding dual of the prescribed maximum 1–multi-norm based on $E$ is the minimum $\infty$–multi-norm based on $F$. By Theorem 2.25, each $\infty$–multi-norm is a strong $\infty$–multi-norm. But of course the dual of this strong $\infty$–multi-norm based on $F$ is the 1–multi-norm based on $E$ that was defined above, and this is not a strong 1-multi-norm. ∎

We wish now to determine when the maximum $p$–multi-norm $(\nu_{p,n})$ when based on various spaces is a strong $p$–multi-norm. We first give an equivalent condition for a $p$–multi-norm to be strong; in the following theorem, the norm on $\ell^p_n \otimes E$, for $n \in \mathbb{N}$, is that specified by equation (2.4.3).

THEOREM 2.34. *Let $E$ be a linear space, and take $p$ with $1 \leqslant p \leqslant \infty$. Suppose that $(\|\cdot\|_n)$ is a sequence such that $\|\cdot\|_n$ is a norm on $E^n$ for each $n \in \mathbb{N}$. Then $(\|\cdot\|_n)$ is a strong $p$–multi-norm if and only if, for each $m, n \in \mathbb{N}$, for any subspaces $Z$ and $W$ of $\ell^p_m$ and $\ell^p_n$, respectively, and any contraction $T$ in $\mathcal{B}(Z,W)$, the map*

$$T \otimes I_E : Z \otimes E \to W \otimes E$$

*is also a contraction with respect to the associated norms on $\ell^p_m \otimes E$ and $\ell^p_n \otimes E$, respectively.*

*Proof.* Suppose that $(\|\cdot\|_n)$ is a strong $p$–multi-norm, and take $m, n \in \mathbb{N}$, subspaces $Z$ and $W$ of $\ell^p_m$ and $\ell^p_n$, respectively, and a contraction $T$ in $\mathcal{B}(Z,W)$.

Let $z \in Z \otimes E$, say

$$z = \sum_{j=1}^{m} \delta_j \otimes x_j = \sum_{i=1}^{k} r_i \otimes a_i\,,$$

where $x_1, \dots, x_m \in E$, $k \in \mathbb{N}$, and $\{r_1, \dots, r_k\}$ and $\{a_1, \dots, a_k\}$ are subsets of $Z$ and $E$, respectively. Take $\lambda \in E'$. Then, by (1.9.2), we have

$$\left(\sum_{j=1}^{m} |\langle x_j,\, \lambda\rangle|^p\right)^{1/p} = \left\|\sum_{i=1}^{k} \langle a_i,\, \lambda\rangle r_i\right\|_{\ell^p_m}\,.$$

Now set $w = (T \otimes I_E)(z) \in W \otimes E \subset \ell^p_n \otimes E$, so that $w = \sum_{j=1}^{n} \delta_j \otimes y_j$, say, where

$y_1, \ldots, y_n \in E$. Then, by another application of (1.9.2), we have

$$\begin{aligned}\sum_{j=1}^{n} |\langle y_j,\, \lambda\rangle|^p &= \left\|\sum_{i=1}^{k}\langle a_i,\, \lambda\rangle T r_i\right\|_{\ell_n^p}^p = \left\|T\left(\sum_{i=1}^{k}\langle a_i,\, \lambda\rangle r_i\right)\right\|_{\ell_n^p}^p \\ &\leqslant \left\|\sum_{i=1}^{k}\langle a_i,\, \lambda\rangle r_i\right\|_{\ell_m^p}^p = \sum_{j=1}^{m} |\langle x_j,\, \lambda\rangle|^p\,.\end{aligned}$$

Set $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$. Then we have shown that $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$, and so, by hypothesis, $\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$, i.e., $\|w\| \leqslant \|z\|$. Thus $T \otimes I_E$ is a contraction.

Conversely, suppose that the stated condition holds. Take $m, n \in \mathbb{N}$, and then take $\boldsymbol{x} = (x_1, \ldots, x_m) \in E^m$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in E^n$ such that $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. Set $z = \sum_{j=1}^{m} \delta_j \otimes x_j$ and $w = \sum_{j=1}^{n} \delta_j \otimes y_j$.

By Theorem 1.38, there are a subspace $Z$ of $\ell_m^p$ and a contraction $T$ in $\mathcal{B}(Z, \ell_n^p)$ with $(T \otimes I_E)(z) = w$. By hypothesis, $T \otimes I_E : Z \otimes E \to \ell_n^p \otimes E$ is also a contraction, and so

$$\|\boldsymbol{y}\|_n = \|w\| = \|(T \otimes I_E)z\| \leqslant \|z\| = \|\boldsymbol{x}\|_m\,.$$

This shows that $(\|\cdot\|_n)$ is a strong $p$–multi-norm. ∎

Example 2.35. Take $p$ with $1 < p < \infty$. We shall now exhibit some further Banach spaces $E$ such that the maximum $p$–multi-norm $(\nu_{p,n})$ of Example 2.7(iv), when based on $E$, is a strong $p$–multi-norm. We recall from equation (1.5.10) that $\nu_{p,n}$ corresponds to the projective tensor norm on $\ell_n^p \otimes E$ for $n \in \mathbb{N}$.

Indeed, take $E$ to be $L^1(\Omega, \mu)$ for a measure space $(\Omega, \mu)$. In particular, consider the case where $E = \ell^{\,1}(I)$ for an index set $I$. By Proposition 1.14(iii), $X \widehat{\otimes} E$ is isometrically a closed subspace of $Y \widehat{\otimes} E$ whenever $X$ is a closed subspace of a Banach space $Y$. Now take $m, n \in \mathbb{N}$ and subspaces $Z$ and $W$ of $\ell_m^p$ and $\ell_n^p$, respectively, and let $T$ be a contraction in $\mathcal{B}(Z, W)$. Then, by Theorem 1.13,

$$T \otimes I_E : Z \widehat{\otimes} E \to W \widehat{\otimes} E$$

is also a contraction with respect to the projective norms on $\ell_m^p \otimes E$ and $\ell_n^p \otimes E$, respectively. By Theorem 2.34, $(\nu_{p,n})$ is a strong $p$–multi-norm based on $E$. ∎

The spaces $E = L^1(\Omega, \mu)$ for a measure space $(\Omega, \mu)$ are the only Banach spaces that we know to have the property that the maximum $p$–multi-norm $(\nu_{p,n})$, when based on $E$, is strong.

Next we shall describe a criterion that will enable us to see that certain maximum $p$–multi-norms $(\nu_{p,n})$ are not strong; the projection constant $\lambda(F, E)$ was defined on page 13.

Theorem 2.36. *Take $p$ with $1 \leqslant p \leqslant \infty$, and suppose that $Z$ and $W$ are two subspaces $\ell^{\,p}$, of the same finite dimension, such that*

$$d(Z, W)\lambda(W, \ell^{\,p}) < \lambda(Z, \ell^{\,p})\,. \tag{2.5.1}$$

*Then the maximum $p$–multi-norm $(\nu_{p,n})$ based on the dual space $Z'$ is not strong.*

*Proof.* By a small perturbation, we may suppose that both $Z$ and $W$ are subspaces of $\ell^p_m$ for some $m \in \mathbb{N}$, and then $\lambda(W, \ell^p_m) = \lambda(W, \ell^p)$ and $\lambda(Z, \ell^p_m) = \lambda(Z, \ell^p)$.

Set $k = \dim Z = \dim W$ and $c = 1/\beta(J_Z \otimes_\pi I_{Z'})$, where $J_Z : Z \to \ell^p_m$ is the natural embedding, so that, by Proposition 1.15, we have $c = \lambda(Z, \ell^p_m)$. Set

$$\mathbf{W} = W \widehat{\otimes} Z', \quad \mathbf{Z} = Z \widehat{\otimes} Z', \quad \text{and} \quad \mathbf{L} = \ell^p_m \widehat{\otimes} Z'.$$

By the definition of $c$, there exists $z \in \mathbf{Z}$ with $\|z\|_{\mathbf{L}} = 1$ and $\|z\|_{\mathbf{Z}} = c$, taking the corresponding projective norms on $\mathbf{L}$ and $\mathbf{Z}$.

There is a linear bijection $T : Z \to W$ such that $\|T\| = 1$ and $\|T^{-1}\| = d(Z, W)$; set $w = (T \otimes I_{Z'})(z) \in \mathbf{W} \subset \mathbf{L}$. Using Theorem 1.13, we have the calculation that

$$c = \|z\|_{\mathbf{Z}} = \left\|(T^{-1} \otimes I_{Z'})(w)\right\|_{\mathbf{Z}} \leqslant \left\|T^{-1}\right\| \|I_{Z'}\| \|w\|_{\mathbf{W}} = d(Z, W) \|w\|_{\mathbf{W}} .$$

Also $\lambda(\mathbf{W}, \mathbf{L}) \leqslant \lambda(W, \ell^p_m)$, and so $\|w\|_{\mathbf{W}} \leqslant \lambda(W, \ell^p_m) \|w\|_{\mathbf{L}}$. Hence

$$\lambda(Z, \ell^p_m) = c \leqslant d(Z, W)\lambda(W, \ell^p_m) \|w\|_{\mathbf{L}} . \tag{2.5.2}$$

Assume that the maximum $p$–multi-norm $(\nu_{p,n})$ based on $Z'$ is strong. Since $\nu_{p,n}$ corresponds to the projective tensor norm on $\ell^p_n \widehat{\otimes} Z'$ for $n \in \mathbb{N}$, it follows from Theorem 2.34 that the map $T \otimes I_{Z'} : \mathbf{Z} \to \mathbf{W}$ is also a contraction with respect to the norm $\|\cdot\|_{\mathbf{L}}$ on $\mathbf{Z}$ and $\mathbf{W}$. Thus we see that $\|w\|_{\mathbf{L}} \leqslant \|z\|_{\mathbf{L}} = 1$, and so it follows from equation (2.5.2) that $\lambda(Z, \ell^p_m) \leqslant d(Z, W)\lambda(W, \ell^p_m)$. Hence inequality (2.5.1) does not hold, a contradiction.

This completes the proof. ∎

Theorem 2.37. *Take $p$ with $1 \leqslant p < \infty$ and $p \neq 2$. Then there is a finite-dimensional Banach space $E$ such that the maximum $p$–multi-norm based on $E$ is not strong.*

*Proof.* In the case where $p = 1$, an appropriate example (with dimension 3) is given in Example 2.33, and so we now suppose that $p > 1$. We shall apply Theorem 2.36.

By Corollary 1.29, there are a constant $C > 0$ and an increasing sequence $(F_n)$ of subspaces of $\ell^p$ such that $d(F_n, \ell^p_n) \leqslant C \ (n \in \mathbb{N})$ and $\lim_{n\to\infty} \lambda(F_n, \ell^p) = \infty$. Take $n \in \mathbb{N}$ with $\lambda(F_n, \ell^p) > C$, and set $Z = F_n$ and $W = \ell^p_n$. Then $d(Z, W) \leqslant C$, $\lambda(W, \ell^p) = 1$, and $\lambda(Z, \ell^p) > C$, and so inequality (2.5.1) holds. By Theorem 2.36, the maximum $p$–multi-norm on the dual space $Z' = F'_n$ is not strong. ∎

Corollary 2.38. *Take $q$ with $1 < q < \infty$ and $q \neq 2$. Then there is a finite-dimensional Banach space $F$ such that the minimum $q$–multi-norm $(\mu_{q,n})$ based on $F$ is strong, but such that the dual $q'$–multi-norm $(\nu_{q',n})$ based on $F'$ is not strong.*

*Proof.* By Example 2.27(iii), the minimum $q$–multi-norm based on the above space $Z$ is strong, but, as stated, the dual $q'$–multi-norm based on $Z'$, which is the maximum $q'$–multi-norm $(\nu_{q',n})$, is not strong. ∎

Example 2.39. We finally exhibit a quotient of a weak $p$–summing norm that is not a strong $p$–multi-norm. To see this, we shall again use the example given in Corollary 1.29 and the characterization of strong $p$–multi-norms given in Theorem 2.34.

In this example, we suppose that $1 < p < \infty$ and $p \neq 2$; a variation of Corollary 1.29 that holds in the case where $p = 1$ would give an analogous example for the case where

$p = 1$. However we shall give an easier example of the same phenomenon in this case in Example 5.12.

Thus take $p$ with $1 < p < \infty$ and $p \neq 2$, set $q = p'$, and let $C \geqslant 1$ be the constant specified in Corollary 1.29. Then there are $n, N \in \mathbb{N}$, a closed subspace $Z$ of $\ell_N^p$ with $\dim Z = n$ and $\lambda(Z, \ell_N^p) > C$, and an isomorphism $T\colon Z \to \ell_n^p$ with $\|T\| = 1$ and $\|T^{-1}\| \leqslant C$.

Set $E = \ell_N^q$, so that $E' = \ell_N^p$, and $F = Z^\perp \subset E$. Let $Q_F\colon E \to E/F$ be the quotient map. Then $Q_F'\colon (E/F)' \to E'$ is an isometry onto the subspace $F^\perp = Z$, and so the map $U\colon (E/F)' \to Z$ given by

$$U\lambda = Q_F'(\lambda) \quad (\lambda \in (E/F)')$$

is an isometric isomorphism.

We consider the weak $p$–summing norm $(\mu_{p,m})$ based on $E$. As in Example 2.27(iii), this $p$–multi-norm is strong. The purpose of this example is to show that the induced quotient $p$–multi-norm based on $E/F$ is not strong.

Take $m \in \mathbb{N}$. We recall that

$$(E^m, \mu_{p,m}) \cong \ell_m^p \check{\otimes} E \cong \mathcal{B}(E', \ell_m^p)\,.$$

We shall again write $\overline{\mu}_{p,m}$ for the quotient norm on $(E/F)^m = \ell_m^p \otimes (E/F)$ induced by the norm $\mu_{p,m}$ on $E^m = \ell_m^p \otimes E$. As usual, $(\delta_i)_{i=1}^m$ denotes the standard basis for $\ell_m^p$; we shall denote by $(\delta_i')_{i=1}^m$ the corresponding sequence of biorthogonal functionals, which is equal to the standard basis for $\ell_m^q$ under our identification of $\ell_m^q$ with the dual of $\ell_m^p$.

Define

$$\boldsymbol{y} = \sum_{i=1}^n \delta_i \otimes U'T'\delta_i' \in \ell_n^p \otimes (E/F) \tag{2.5.3}$$

and

$$\boldsymbol{x} = (T^{-1} \otimes I_{E/F})\boldsymbol{y} = \sum_{i=1}^n T^{-1}\delta_i \otimes U'T'\delta_i' \in Z \otimes (E/F) \subset \ell_N^p \otimes (E/F)\,.$$

For each $\lambda \in (E/F)'$, equation (1.9.2) implies that

$$\langle \boldsymbol{y}, \lambda\rangle = \sum_{i=1}^n \langle \lambda, U'T'\delta_i'\rangle \delta_i = \sum_{i=1}^n \langle TU\lambda, \delta_i'\rangle\, \delta_i = TU\lambda\,.$$

A similar calculation shows that $\langle \boldsymbol{x}, \lambda\rangle = U\lambda$, and hence we have

$$\|\langle \boldsymbol{y}, \lambda\rangle\|_{\ell_n^p} = \|TU\lambda\|_{\ell_n^p} \leqslant \|U\lambda\|_{\ell_N^p} = \|\langle \boldsymbol{x}, \lambda\rangle\|_{\ell_N^p}$$

because $\|T\| = 1$. This shows that $\boldsymbol{y} \leqslant_p \boldsymbol{x}$.

Let $\mu \in \ell_N^q$ and $\lambda \in (E/F)'$. By applying the functional $\mu \otimes \lambda$, which is given by equation (1.4.1), to the element $\boldsymbol{x} \in \ell_N^p \otimes (E/F)$, we obtain

$$\begin{aligned}\langle \boldsymbol{x},\, \mu \otimes \lambda\rangle &= \sum_{i=1}^n \langle T^{-1}\delta_i,\, \mu\rangle\langle U'T'\delta_i',\, \lambda\rangle = \left\langle \sum_{i=1}^n \langle TU\lambda, \delta_i'\rangle\, \delta_i,\, (T^{-1})'\mu \right\rangle \\ &= \langle TU\lambda,\, (T^{-1})'\mu\rangle = \langle U\lambda,\, \mu\rangle = \langle Q_F'\lambda,\, \mu\rangle = \left\langle \sum_{i=1}^N \langle Q_F'\lambda, \delta_i'\rangle\, \delta_i,\, \mu \right\rangle \\ &= \sum_{i=1}^N \langle \delta_i,\, \mu\rangle\langle Q_F\delta_i',\, \lambda\rangle = \left\langle (I_N \otimes Q_F)\left(\sum_{i=1}^N \delta_i \otimes \delta_i'\right),\, \mu \otimes \lambda \right\rangle.\end{aligned}$$

Since the functionals of the form $\mu \otimes \lambda$ span the space $(\ell_N^p \otimes (E/F))'$, it follows that

$$\boldsymbol{x} = (I_N \otimes Q_F)\left(\sum_{i=1}^N \delta_i \otimes \delta_i'\right),$$

and hence $\overline{\mu}_{p,N}(\boldsymbol{x}) \leqslant \left\|\sum_{i=1}^N \delta_i \otimes \delta_i'\right\|_{\varepsilon,N} = 1$.

We shall now assume towards a contradiction that the $p$–multi-norm $(\overline{\mu}_{p,m})$ based on $E/F$ is strong. Note that $\overline{\mu}_{p,n}(\boldsymbol{y}) \leqslant \overline{\mu}_{p,N}(\boldsymbol{x}) \leqslant 1$. The quotient norm of $\boldsymbol{y}$ is attained because $\ell_n^p \otimes E$ is a finite-dimensional space, and so we can find an element

$$\boldsymbol{v} = \sum_{i=1}^n \delta_i \otimes v_i \in \ell_n^p \otimes E$$

such that $\|\boldsymbol{v}\|_{\varepsilon,n} \leqslant 1$ and $(I_n \otimes Q_F)(\boldsymbol{v}) = \boldsymbol{y}$. Comparing the definition (2.5.3) of $\boldsymbol{y}$ with the expression

$$(I_n \otimes Q_F)(\boldsymbol{v}) = \sum_{i=1}^n \delta_i \otimes Q_F v_i\,,$$

we deduce that

$$U'T'\delta_i' = Q_F v_i \quad (i \in \mathbb{N}_n)\,. \tag{2.5.4}$$

Define $V\colon \ell_N^p \to \ell_n^p$ by setting

$$Vz = \sum_{i=1}^n \langle z, v_i\rangle\, \delta_i \quad (z \in \ell_N^p)\,,$$

so that $V$ is the operator corresponding to the element $\boldsymbol{v}$, and hence $\|V\| = \|\boldsymbol{v}\|_{\varepsilon,n} \leqslant 1$. We observe that $V \mid Z = T$. Indeed, for $z \in Z$, set $\lambda = U^{-1}z \in (E/F)'$, so that $z = U\lambda = Q_F'\lambda$; using (2.5.4), we obtain

$$Vz = \sum_{i=1}^n \langle \lambda, Q_F v_i\rangle \delta_i = \sum_{i=1}^n \langle \lambda, U'T'\delta_i'\rangle \delta_i = TU\lambda = Tz\,,$$

as required. This implies that the operator $P := T^{-1}V \in \mathcal{B}(\ell_N^p)$ is a projection with image $Z$, and consequently $\lambda(Z, \ell_N^p) \leqslant \|P\| \leqslant C$, which contradicts our choice of $Z$.

Thus we have shown that the $p$–multi-norm $(\overline{\mu}_{p,m})$ based on $E/F$ is not strong, as required. ∎

**2.6. Convex and concave power-norms.** The second strengthening of the concept of a $p$–multi-norm that we shall consider involves convexity.

DEFINITION 2.40. Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Then $(E^n, \|\cdot\|_n)$ is *$p$–convex* if

$$\|(\boldsymbol{x}, \boldsymbol{y})\|_{m+n} \leqslant \left(\|\boldsymbol{x}\|_m^p + \|\boldsymbol{y}\|_n^p\right)^{1/p} \tag{2.6.1}$$

and *$p$–concave* if

$$\|(\boldsymbol{x}, \boldsymbol{y})\|_{m+n} \geqslant \left(\|\boldsymbol{x}\|_m^p + \|\boldsymbol{y}\|_n^p\right)^{1/p}, \tag{2.6.2}$$

in both cases for each $m, n \in \mathbb{N}$, each $\boldsymbol{x} \in E^m$, and each $\boldsymbol{y} \in E^n$.

Each power-norm is obviously 1–convex and $\infty$–concave. Suppose that a power-norm is $p$–convex, respectively, $p$–concave. Then it is also $r$–convex, respectively, $r$–concave, for each $r \in [1, p]$, respectively $r \in [p, \infty]$.

For example, take $p, q$ with $1 \leqslant p \leqslant q < \infty$, and let $(\|\cdot\|_n^{(p,q)})$ be the $(p, q)$–multi-norm defined in Example 2.10. Then $(\|\cdot\|_n^{(p,q)})$ is $r$–convex for $r \geqslant 1$ if and only if $r \in [1, q]$. We shall see in Theorem 4.26 that, for each $p$ with $1 < p < \infty$, there are $p$–multi-norms that are not $p$–convex.

A 2–convex 2–multi-norm based on a Banach space $E$ is exactly what is termed a *sequential norm* in [38, Definition 2.1], and the corresponding space $(E^n, \|\cdot\|_n)$ is an *operator sequence space*. A related notion of a *$p$–operator space* (for $1 \leqslant p \leqslant \infty$) was introduced by Daws in [21]. One could say that our theory of $p$–multi-normed spaces is 'half-way' between that of classical Banach space theory and operator space theory; our hope is that it sheds some light on both of these topics and their connections.

The main texts on operator space theory are those of Blecher and Le Merdy [9], of Effros and Ruan [27], of Helemskii [30], and of Pisier [50].

Let $(E^n, \|\cdot\|_n)$ be a $p$–convex or $p$–concave power-normed space, and suppose that $F$ is a subspace of $E$. Then the corresponding power-norms based on $F$ and, in the case where $F$ is closed, on the quotient $E/F$ are both $p$–convex or $p$–concave, respectively.

For $m, n \in \mathbb{N}$, consider the linear bijection $J_{m,n}$ that takes the element $\boldsymbol{x} + \boldsymbol{y}$ in $E^m \oplus E^n$ to the concatenation $(\boldsymbol{x}, \boldsymbol{y})$ in $E^{m+n}$. Then $(E^n, \|\cdot\|_n)$ is $p$–convex if and only if

$$J_{m,n} : (E^m, \|\cdot\|_m) \oplus_p (E^n, \|\cdot\|_n) \to (E^{m+n}, \|\cdot\|_{m+n}) \tag{2.6.3}$$

is a contraction for each $m, n \in \mathbb{N}$. Similarly, $(E^n, \|\cdot\|_n)$ is $p$–concave if and only if the inverse $J_{m,n}^{-1}$ of $J_{m,n}$ is a contraction for each $m, n \in \mathbb{N}$.

PROPOSITION 2.41. *Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Then $(E^n, \|\cdot\|_n)$ is $p$–concave if and only if $((E')^n, \|\cdot\|_n')$ is $p'$–convex, and $(E^n, \|\cdot\|_n)$ is $p$–convex if and only if $((E')^n, \|\cdot\|_n')$ is $p'$–concave.*

*Proof.* For notational convenience, set $q = p'$ and $F = E'$.

Suppose that $(E^n, \|\cdot\|_n)$ is $p$–convex, so that the above map $J_{m,n}$ is a contraction for each $m, n \in \mathbb{N}$. The dual $J_{m,n}'$ of $J_{m,n}$ is the linear bijection taking $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ in the space $F^{m+n}$ to $\boldsymbol{\lambda} + \boldsymbol{\mu}$ in $F^m \oplus_q F^n = (E^m \oplus_p E^n)'$, and this map is also a contraction.

But the map $J'_{m,n}$ is exactly the map corresponding to $J_{m,n}^{-1}$ on $(F^{m+n}, \|\cdot\|'_{m+n})$, and so $(F^n, \|\cdot\|'_n)$ is $q$–concave.

Similarly, we see that $(F^n, \|\cdot\|'_n)$ is $q$–convex whenever $(E^n, \|\cdot\|_n)$ is $p$–concave.

Now suppose that $(F^n, \|\cdot\|'_n)$ is $q$–convex, respectively, $q$–concave. Then we have shown that the bidual $((E'')^n, \|\cdot\|''_n)$ is $p$–concave, respectively, $p$–convex, and hence $(E^n, \|\cdot\|_n)$ is $p$–concave, respectively, $p$–convex. ∎

EXAMPLES 2.42. Take $p$ with $1 \leqslant p \leqslant \infty$.

(i) The unique $p$–multi-norm based on $\mathbb{F}$ is obviously $p$–convex and $p$–concave, as is the $p$–sum power-norm based on a normed space.

(ii) It is easy to see that the $p$–sum power-norm is the maximum $p$–convex power-norm, in the sense that, for each normed space $E$ and each $p$–convex power-norm $(\|\cdot\|_n)$ based on $E$, we have

$$\|\boldsymbol{x}\|_n \leqslant \|\boldsymbol{x}\|_{\ell_n^p(E)} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})\,. \tag{2.6.4}$$

(iii) The weak $p$–summing norm $(\mu_{p,n})$ based on a normed space $E$ is a $p$–multi-norm, and it is $p$–convex. For take $m, n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$, and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$. For each $\lambda \in B_{E'}$, we have

$$\left(\sum_{i=1}^m |\langle x_i,\, \lambda\rangle|^p + \sum_{i=1}^n |\langle y_i,\, \lambda\rangle|^p\right)^{1/p} \leqslant (\mu_{p,m}(\boldsymbol{x})^p + \mu_{p,n}(\boldsymbol{y})^p)^{1/p}\ ,$$

and so $\mu_{p,m}((\boldsymbol{x}, \boldsymbol{y})) \leqslant (\mu_{p,m}(\boldsymbol{x})^p + \mu_{p,n}(\boldsymbol{y})^p)^{1/p}$. Thus $(\mu_{p,n})$ is $p$–convex.

In particular, $(\mu_{2,n})$ based on a Banach space $E$ is a sequential norm, and in fact $(E^n, \mu_{2,n})$ is the minimum operator sequence space based on $E$, in the language of [38, p. 250].

(iv) Let $E$ be a normed space, and consider the maximum $p$–multi-norm $(\nu_{p,n})$ based on $E$. The dual of this $p$–multi-norm is the $p'$–multi-norm $(\mu_{p',n})$ based on $E'$, and so, by (ii) and Proposition 2.41, $(\nu_{p,n})$ is $p$–concave. ∎

PROPOSITION 2.43. *Take $p$ with $1 \leqslant p < \infty$. Let $\{E_0, E_1\}$ be a compatible couple of complex Banach spaces, and suppose that $(E_0^n, \|\cdot\|_n^0)$ and $(E_1^n, \|\cdot\|_n^1)$ are $p$–convex power-normed spaces based on $E_0$ and $E_1$, respectively. Take $\theta \in (0,1)$, and set $E = (E_0, E_1)_\theta$. Then the power-normed space $(E^n, \|\cdot\|_n)$ is $p$–convex.*

*Proof.* Take $m, n \in \mathbb{N}$. Then, essentially as in equation (1.10.1), $\{E_0^m \oplus_p E_0^n,\, E_1^m \oplus_p E_1^n\}$ is a compatible couple of complex Banach spaces and

$$(E_0^m \oplus_p E_0^n,\, E_1^m \oplus_p E_1^n)_\theta = E^m \oplus_p E^n\,.$$

The maps $J_{m,n}^{(0)}$ and $J_{m,n}^{(1)}$ associated with $E_0$ and $E_1$, respectively, are both contractions, and so, by Theorem 1.46, the map $J_{m,n}$ associated with $E$ is also a contraction. Thus $(E^n, \|\cdot\|_n)$ is $p$–convex. ∎

# 3. Multi-bounded operators

We obtain preliminary results on multi-bounded operators.

**3.1. Definitions and basic results.** We recall that the $n^{\text{th}}$ amplification $T^{(n)}$ of a linear mapping $T$ between linear spaces $E$ and $F$ was defined for $n \in \mathbb{N}$ in Definition 1.2; indeed, $T^{(n)}$ is specified by the formula

$$T^{(n)} : (x_1, \dots, x_n) \mapsto (Tx_1, \dots, Tx_n)\,, \quad E^n \to F^n\,.$$

Suppose that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are two power-normed spaces, and that $T \in \mathcal{B}(E, F)$. It follows from (2.1.1) that the $n^{\text{th}}$ amplification of $T$ is bounded as a linear map from $(E^n, \|\cdot\|_n)$ to $(F^n, \|\cdot\|_n)$ (with $\|T\| \leqslant \left\|T^{(n)}\right\| \leqslant n\,\|T\|$) for each $n \in \mathbb{N}$. However, in general, the norms $\left\|T^{(n)}\right\|$ will not be *uniformly* bounded in $n \in \mathbb{N}$. The following generalizes definitions given in [20, §6.1.3]. Recall that $\beta(S)$ is the embedding constant of an operator $S$, as on page 12.

Definition 3.1. Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be power-normed spaces, and suppose that $T \in \mathcal{B}(E, F)$. Then $T$ is *multi-bounded*, with norm $\|T\|_{mb}$, if

$$\|T\|_{mb} := \sup\left\{\left\|T^{(n)}\right\| : n \in \mathbb{N}\right\} < \infty\,.$$

The map $T$ is a *multi-contraction*, respectively, a *multi-isometry*, if the map

$$T^{(n)} : (E^n, \|\cdot\|_n) \to (F^n, \|\cdot\|_n)$$

is a contraction, respectively, an isometry, for each $n \in \mathbb{N}$. Further, $T$ is a *multi-isomorphism* if it is a bijection and if both $T : E \to F$ and $T^{-1} : F \to E$ are multi-bounded, and $T$ is a *multi-embedding* if it is an embedding and if $\inf\{\beta(T^{(n)}) : n \in \mathbb{N}\} > 0$.

The spaces $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are *multi-isomorphic*, respectively, *multi-isometric*, if there is a multi-isomorphism, respectively, a bijective multi-isometry from $E$ onto $F$.

The collection of multi-bounded maps from $E$ to $F$ is denoted by $\mathcal{M}(E, F)$.

In particular, in the case where $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are $p$–multi-normed spaces for some $p$ with $1 \leqslant p \leqslant \infty$, we shall sometimes say that $T$ is *$p$–multi-bounded* if it is multi-bounded with respect to the two $p$–multi-norms, and we shall write $\mathcal{M}_p(E, F)$ for the collection of $p$–multi-bounded maps from $E$ to $F$. In this case, the norm of a $p$–multi-bounded operator $T \in \mathcal{M}_p(E, F)$ is sometimes denoted by $\|T\|_{p-\mathrm{mb}}$.

In the case where $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are operator sequence spaces, our definitions coincide with those of *sequentially bounded* maps, *sequential contractions*, and *sequential isometries* given in [38, Definition 2.2].

For a study of $\mathcal{M}_\infty(E,F)$ and $\mathcal{M}_\infty(E) = \mathcal{M}_\infty(E,E)$ (in the setting of multi-bounded spaces), see [20, Chapter 6].

Let $E$ and $F$ be Banach spaces such that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are power-normed spaces. Then $(\mathcal{M}(E,F), \|\cdot\|_{mb})$ is easily seen to be a Banach space; *cf.* [20, Theorem 6.15].

EXAMPLE 3.2. Take $p, q$ such that $1 \leqslant p, q \leqslant \infty$, suppose that $(E^n, \|\cdot\|_n)$ is a $p$–multi-normed space and that $(F^n, \|\cdot\|_n)$ is a $q$–multi-normed space, and consider the space $(\mathcal{M}(E,F), \|\cdot\|_{mb})$. We suppose that $E, F \neq \{0\}$.

First suppose that $p \leqslant q$. Take $y \in F$ and $\lambda \in E'$, and consider $T := y \otimes \lambda \in \mathcal{F}(E,F)$. Then, for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\begin{aligned}
\left\|T^{(n)}\boldsymbol{x}\right\|_n &= \|(\langle x_1, \lambda\rangle y, \dots, \langle x_n, \lambda\rangle y)\|_n \\
&= \left(\sum_{j=1}^n |\langle x_j, \lambda\rangle|^q\right)^{1/q} \|y\| \quad \text{by equation (2.2.4)} \\
&\leqslant \left(\sum_{j=1}^n |\langle x_j, \lambda\rangle|^p\right)^{1/p} \|y\| \\
&\leqslant \mu_{p,n}(\boldsymbol{x}) \|\lambda\| \|y\| \leqslant \|\boldsymbol{x}\|_n \|\lambda\| \|y\| \quad \text{by Theorem 2.11},
\end{aligned}$$

and so $T \in \mathcal{M}(E,F)$ with $\|T\|_{mb} = \|\lambda\| \|y\|$. It follows that $\mathcal{F}(E,F) \subset \mathcal{M}(E,F)$. In particular, $\mathcal{M}(E,F) \neq \{0\}$.

Second suppose that $p > q$. Take $T \in \mathcal{B}(E,F)$ with $T \neq 0$, and then take $x \in E$ with $\|x\| = 1$ and $Tx \neq 0$. For $n \in \mathbb{N}$, set $\boldsymbol{x} = (x, \dots, x) \in E^n$. By (2.2.5), $\|\boldsymbol{x}\|_n = n^{1/p}$ and $\left\|T^{(n)}\boldsymbol{x}\right\|_n = \|Tx\| n^{1/q}$, and so $\left\|T^{(n)}\right\| \geqslant \|Tx\| n^{1/q - 1/p} \to \infty$ as $n \to \infty$. It follows that $T \notin \mathcal{M}(E,F)$, and so $\mathcal{M}(E,F) = \{0\}$. ■

EXAMPLE 3.3. Let $E$ and $F$ be Banach spaces, and take $p, q$ with $1 \leqslant p \leqslant q < \infty$. Consider the weak $p$–summing norm $(\mu_{p,n})$ based on $E$, so that $(\mu_{p,n})$ is a strong $p$–multi-norm, and the $q$–sum power-norm $(\|\cdot\|_{\ell_n^q(F)})$ based on $F$, so that $(\|\cdot\|_{\ell_n^q(F)})$ is a power-norm that is sometimes a (strong) $q$–multi-norm. Then the space of multi-bounded operators from $(E^n, \mu_{p,n})$ to $(F^n, \|\cdot\|_{\ell_n^q(F)})$ with the multi-bounded norm $\|\cdot\|_{mb}$ is exactly the space $(\Pi_{q,p}(E,F), \pi_{q,p})$ of $(q,p)$–summing operators from $E$ to $F$, and so

$$\|T\|_{mb} = \pi_{q,p}(T) \quad (T \in \Pi_{q,p}(E,F)).$$

Consider the special case when $F = E$ and $q = p$; we shall write $\Pi_p(E)$ for $\Pi_{p,p}(E,E)$, $\pi_p$ for $\pi_{p,p}$, and $\pi_p(E)$ for $\pi_p(I_E)$, as is standard. Thus

$$\left\|I_E : (E^n, \mu_{p,n}) \to (E^n, \|\cdot\|_{\ell_n^p(E)})\right\|_{mb} = \pi_p(E). \tag{3.1.1}$$

See [20, §3.4.2], for example, for background on $(q,p)$–summing operators. ■

Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and let $F$ be a linear subspace of $E$. Then the inclusion $J_F : F \to E$ is a multi-isometry. Suppose that $F$ is closed in $E$. As we remarked after equation (1.3.5), for each $n \in \mathbb{N}$, we identify the $n^{\text{th}}$ amplification of the quotient mapping $Q_F : E \to E/F$ with the quotient mapping of $E^n$ onto $E^n/F^n$, and so $Q_F$ is a multi-contraction.

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two power-normed spaces, and take $T \in \mathcal{B}(E, F)$ and $n \in \mathbb{N}$. Recall from equation (1.3.20) that we have identified the dual of the $n^{\text{th}}$ amplification of $T$ with the $n^{\text{th}}$ amplification of the dual $T'$ of $T$, so that $(T^{(n)})' = (T')^{(n)}$. Moreover, we have identified the $n^{\text{th}}$ amplification of the canonical embedding of $E$ into its bidual $E''$ with the canonical embedding of $E^n$ into its bidual $(E^n)''$. Since the latter operator is an isometry, we see that the canonical embedding of a power-normed space into its bidual is a multi-isometry.

Proposition 3.4. *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be power-normed spaces, and take $T \in \mathcal{B}(E, F)$. Then $T$ is multi-bounded if and only if $T' : F' \to E'$ is multi-bounded with respect to the dual power-norms based on $F'$ and $E'$, respectively, and, in this case, $\|T'\|_{mb} = \|T\|_{mb}$. In the case where $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are $p$–multi-normed spaces for some $p$ with $1 \leqslant p \leqslant \infty$ and $T \in \mathcal{M}_p(E, F)$, we have $T' \in \mathcal{M}_{p'}(F', E')$.*

*Proof.* Take $n \in \mathbb{N}$. Then $\|(T')^{(n)}\| = \|(T^{(n)})'\| = \|T^{(n)}\|$. Thus $T'$ is multi-bounded if and only if $T$ is multi-bounded; in this case, $\|T'\|_{mb} = \|T\|_{mb}$. ∎

The following remarks are contained in [20, Chapter 6] in the setting of multi-norms, but they apply in the setting of power-norms and, in particular, for $p$–multi-norms.

Definition 3.5. Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $(x_i) \in E^{\mathbb{N}}$. Then $(x_i)$ is a *multi-null sequence* in $E$ if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_{n+1}, \dots, x_{n+k})\|_k < \varepsilon \quad (n \geqslant n_0)\,.$$

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two power-normed spaces, and take $T \in \mathcal{B}(E, F)$. Then $T$ is *multi-continuous* if $(Tx_i)$ is a multi-null sequence in $F$ whenever $(x_i)$ is a multi-null sequence in $E$.

The following result has the same proof as [20, Theorem 6.14].

Theorem 3.6. *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two power-normed spaces, and take $T \in \mathcal{B}(E, F)$. Then $T$ is multi-continuous if and only if $T$ is multi-bounded.* ∎

We shall next prove the power-normed analogue of the theorem on quotient operators stated as Proposition 1.4(i). This result will be used later.

Proposition 3.7. *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be power-normed spaces, and suppose that $T \in \mathcal{B}(E, F)$. Then the operator $\overline{T}\colon E/\ker T \to F$ induced by $T$ is a multi-isometry if and only if $T^{(n)}$ is a quotient operator for each $n \in \mathbb{N}$.*

*Proof.* We have identified the two spaces $(E/\ker T)^n$ and $E^n/(\ker T)^n$ isometrically for each $n \in \mathbb{N}$. Hence the diagram (1.3.7) implies that $\overline{T}$ is a multi-isometry if and only if $\overline{T^{(n)}}$ is an isometry for each $n \in \mathbb{N}$, and, by Proposition 1.4(i), the latter happens if and only if $T^{(n)}$ is a quotient operator for each $n \in \mathbb{N}$. ∎

Corollary 3.8. *Let $F$ be a closed subspace of a power-normed space $E$. Then the isomorphism $\overline{J_F'} : E'/F^{\perp} \to F'$ induced by the dual of the inclusion $J_F : F \to E$ is a multi-isometry.*

*Proof.* Take $n \in \mathbb{N}$. By Proposition 3.7, we must show that $(J'_F)^{(n)}$ is a quotient operator. Since $(J'_F)^{(n)} = (J_F^{(n)})'$ and $J_F^{(n)}$ is an isometry, this follows from Proposition 1.4(ii). ∎

PROPOSITION 3.9. *Let $E$ be an infinite-dimensional Banach space. Then:*

(i) *for $1 \leqslant p < \infty$, the power-normed spaces $(E^n, \mu_{p,n})$ and $(E^n, \|\cdot\|_{\ell_n^p(E)})$ are not multi-isomorphic;*

(ii) *for $1 < p \leqslant \infty$, the power-normed spaces $(E^n, \|\cdot\|_{\ell_n^p(E)})$ and $(E^n, \nu_{p,n})$ are not multi-isomorphic.*

*Proof.* (i) Assume towards a contradiction that there is a multi-isomorphism $T \in \mathcal{B}(E)$, where $T^{(n)}$ maps $(E^n, \mu_{p,n})$ onto $(E^n, \|\cdot\|_{\ell_n^p(E)})$ for each $n \in \mathbb{N}$. Then we have

$$\sup\left\{\left\|T^{(n)} : (E^n, \mu_{p,n}) \to (E^n, \|\cdot\|_{\ell_n^p(E)})\right\| : n \in \mathbb{N}\right\} < \infty .$$

By Example 3.3 (in the case where $p = q$), this means that $T$ is a $p$–summing operator, which contradicts the fact that $T$ is an isomorphism on an infinite-dimensional Banach space. Indeed, the composition of any two $p$–summing operators is compact (see [24, p. 50]), and hence the isomorphism $T^2$ would be compact if $T$ were $p$–summing.

(ii) This follows easily by duality. ∎

Note that, for each $n \in \mathbb{N}$, we have the equalities $\mu_{\infty,n} = \|\cdot\|_{\ell_n^\infty(E)}$ by equation (1.5.1) and $\|\cdot\|_{\ell_n^1(E)} = \nu_{1,n}$ by equation (1.5.9).

COROLLARY 3.10. *Let $E$ be an infinite-dimensional Banach space, and take $p$ with $1 \leqslant p \leqslant \infty$. Then the $p$–multi-normed spaces $(E^n, \mu_{p,n})$ and $(E^n, \nu_{p,n})$ are not multi-isomorphic.*

*Proof.* For $1 \leqslant p < \infty$, this follows immediately by combining Proposition 3.9(i) with the inequality (1.5.12), while the case where $p = \infty$ follows from equation (1.5.1) and Proposition 3.9(ii). ∎

There is a quantitative version of Proposition 3.9 and Corollary 3.10 in the case where $E$ is a finite-dimensional space. Indeed, suppose that $\dim E = k$. Then

$$\sqrt{k} \leqslant \|I_E : (E^n, \mu_{p,n}) \to (E^n, \nu_{p,n})\|_{mb} \leqslant k .$$

The upper bound follows from equation (1.4.4). The lower bound follows from equation (3.1.1) in the case where $1 \leqslant p \leqslant 2$ because $\pi_p(E) \geqslant \pi_2(E) = \sqrt{k}$; in the case where $2 \leqslant p \leqslant \infty$, it follows by duality. It can be shown that both these bounds are optimal to within a multiplicative constant.

EXAMPLE 3.11. We shall show that, for each $p$ with $1 \leqslant p \leqslant \infty$, the inverse of a bijective multi-contraction need not be multi-bounded, and hence there is no analogue of the Banach isomorphism theorem for multi-bounded operators.

(We remark that, in the setting of multi-norms themselves, several examples of the failure of the Banach isomorphism theorem were given in [20]. For example, Example 6.25 of [20] shows that there are multi-norms based on infinite-dimensional Banach spaces $E$ and $F$ such that $\mathcal{M}(E,F) = \mathcal{B}(E,F)$, but $\mathcal{M}(F,E) = \mathcal{N}(F,E)$, the nuclear operators from $F$ to $E$, and Example 6.30 of [20] shows that the analogue of the Banach isomorphism theorem may fail even when there is one multi-norm based on a Banach space $E$ and we consider operators in $\mathcal{B}(E)$. See [20, Example 6.39] for a further example.)

In the present situation, take $p$ with $1 \leqslant p \leqslant \infty$ and take any infinite-dimensional Banach space $E$, and consider the identity operator $I_E$ on $E$. Equation (1.5.13) shows that

$$\left\| I_E^{(n)} : (E^n, \nu_{p,n}) \to (E^n, \mu_{p,n}) \right\| \leqslant 1 \quad (n \in \mathbb{N}),$$

but Corollary 3.10 implies that its inverse is not multi-bounded, so that

$$\left\| I_E^{(n)} : (E^n, \mu_{p,n}) \to (E^n, \nu_{p,n}) \right\| \to \infty \quad \text{as} \quad n \to \infty .$$

With a little more work, we can present a similar example for *strong* $p$–multi-normed spaces. In the case where either $p = 2$ or $p = \infty$, this follows immediately from the above example by Theorem 2.25. Otherwise, take $E = \ell^p$. By Example 2.27, (ii) and (iii), the $p$–sum norm $(\|\cdot\|_{\ell^p_n(E)})$ and the weak $p$–summing norm $(\mu_{p,n})$ are strong $p$–multi-norms based on $E$. Equation (1.5.2) shows that

$$\left\| I_E^{(n)} : (E^n, \|\cdot\|_{\ell^p_n(E)}) \to (E^n, \mu_{p,n}) \right\| \leqslant 1 \quad (n \in \mathbb{N}),$$

but, by Proposition 3.9(i), its inverse is not multi-bounded, so that

$$\left\| I_E^{(n)} : (E^n, \mu_{p,n}) \to (E^n, \|\cdot\|_{\ell^p_n(E)}) \right\| \to \infty \quad \text{as} \quad n \to \infty .$$

This provides the required example. ∎

Example 3.12. Let $F$ be a closed subspace of a Banach space $E$, and take $p$ with $1 \leqslant p \leqslant \infty$. Equation (1.5.8) shows that the inclusion $J_F : F \to E$ is a multi-isometry with respect to the minimum $p$–multi-norms $(\mu^F_{p,n})$ and $(\mu^E_{p,n})$ based on $F$ and $E$, respectively.

In contrast, suppose that $F$ and $E$ are endowed with their maximum $p$–multi-norms $(\nu^F_{p,n})$ and $(\nu^E_{p,n})$, respectively. Proposition 1.19 implies that $J_F$ is a multi-contraction, but it is not always a multi-embedding. Indeed, suppose that $1 < p < \infty$ and $p \neq 2$ and that $E$ and $F$ have been chosen as in Example 1.30. Then equation (1.6.3) shows that $J_F$ is not a multi-embedding of $(F^n, \nu^F_{p,n})$ into $(E^n, \nu^E_{p,n})$. ∎

Example 3.13. Again, let $F$ be a closed subspace of a Banach space $E$, and take $p$ with $1 \leqslant p \leqslant \infty$. We observe that, by Propositions 1.20 and 3.7, $\nu^{E/F}_{p,n}$ is equal to the quotient norm on $(E/F)^n$ of the norm $\nu^E_{p,n}$ on $E^n$ for each $n \in \mathbb{N}$.

However, the analogous result may fail for the minimum $p$–multi-norm. To see this, take $q$ with $1 < q < \infty$ and $q \neq 2$, and choose $E$ and $F$ as in Example 1.30. Then it follows from equation (1.6.4) that the $q$–multi-normed space $((E'/F^\perp)^n, \mu^{E'/F^\perp}_{q,n})$ is not multi-isomorphic to the $q$–multi-normed space $((E'/F^\perp)^n, \overline{\mu}^{E'}_{q,n})$, where $\overline{\mu}^{E'}_{q,n}$ denotes the quotient norm on $(E'/F^\perp)^n$ of the norm $\mu^{E'}_{q,n}$ on $(E')^n$ for $n \in \mathbb{N}$. ∎

We have noted in Theorem 2.18 that multi-norms correspond to $c_0$–norms on $c_0 \otimes E$. Suppose that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are multi-normed spaces. Then $T \in \mathcal{B}(E, F)$ is multi-bounded if and only if $I_{c_0} \otimes T$ is bounded as a map from $c_0 \otimes E$ to $c_0 \otimes F$, and further $\|T\|_{mb} = \|I_{c_0} \otimes T\|$. Thus, in this case, our multi-bounded operators are the same as the 'opérateurs réguliers' of [45, Définition 3.2] (where they are defined in the special case that $E$ and $F$ are Banach lattices). More generally, take $p$ with $1 \leqslant p < \infty$ and suppose that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are $p$–multi-normed spaces. Then $p$–multi-norms based on $E$ correspond to $\ell^p$–norms on $\ell^p \otimes E$, where the correspondence is given in equation (2.4.3). Thus the following theorem follows from Theorem 2.21.

THEOREM 3.14. *Take $p$ with $1 \leqslant p < \infty$, and suppose that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are $p$–multi-normed spaces. Take $T \in \mathcal{B}(E,F)$. Then $T$ is $p$–multi-bounded if and only if $I_{\ell^p} \otimes T$ is bounded as a map from $\ell^p \otimes E$ to $\ell^p \otimes F$; in this case, $\|T\|_{mb} = \|I_{\ell^p} \otimes T\|$.* ■

Let $\{E_0, E_1\}$ and $\{F_0, F_1\}$ be two compatible couples of complex Banach spaces, and suppose that $T : E_0 + E_1 \to F_0 + F_1$ is a linear map such that $T(E_j) \subset F_j$ and $T \mid E_j : E_j \to F_j$ is bounded for $j = 0, 1$. Take $\theta \in (0,1)$, and set

$$E = (E_0, E_1)_\theta \quad \text{and} \quad F = (F_0, F_1)_\theta\,.$$

Then, as in Theorem 1.46, $T(E) \subset F$ and $T \mid E \in \mathcal{B}(E,F)$. Now take $n \in \mathbb{N}$. Then $T^{(n)}$ is a linear map from $(E_0 + E_1)^n$ to $(F_0 + F_1)^n$ such that $T^{(n)}(E_j^n) \subset F_j^n$ and $T^{(n)} \mid E_j^n \in \mathcal{B}(E_j^n, F_j^n)$ for $j = 0, 1$. Take $p$ with $1 \leqslant p < \infty$, and suppose that there are $p$–multi-norms based on all of the spaces $E_0$, $E_1$, $F_0$, and $F_1$. By Theorem 2.15, the two interpolation spaces $E$ and $F$ are such that both the interpolation power-norms based on these two spaces are also $p$–multi-norms. As in Theorem 2.15, $(E_0^n, E_1^n)_\theta = E^n$ and $(F_0^n, F_1^n)_\theta = F^n$ for each $n \in \mathbb{N}$.

We use the above notation in the following theorem.

THEOREM 3.15. *Let $\{E_0, E_1\}$ and $\{F_0, F_1\}$ be two compatible couples of complex Banach spaces, and take $p$ with $1 \leqslant p < \infty$ and $\theta \in (0,1)$. Suppose that there is a $p$–multi-norm based on each of these spaces and that $T : E_0 + E_1 \to F_0 + F_1$ is a linear map such that $T \mid E_j \in \mathcal{M}_p(E_j, F_j)$ for $j = 0$ and $j = 1$. Then $T(E) \subset F$ and $T \mid E \in \mathcal{M}_p(E,F)$.*

*Proof.* The $p$–multi-norms based on each space are all denoted by $(\|\cdot\|_n)$.

There exist constants $M_0$ and $M_1$ such that

$$\left\| T^{(n)} : (E_j^n, \|\cdot\|_n) \to (F_j^n, \|\cdot\|_n) \right\| \leqslant M_j \quad (n \in \mathbb{N})$$

for $j = 0$ and $j = 1$. By Theorem 1.46, $T^{(n)}(E^n) \subset F^n$ and

$$\left\| T^{(n)} : E^n \to F^n \right\| \leqslant M_0^{1-\theta} M_1^{\theta} \quad (n \in \mathbb{N})\,,$$

and so $T(E) \subset F$ and $T \mid E \in \mathcal{M}_p(E,F)$, giving the result. ■

**3.2. Multi-norms on spaces of multi-bounded operators.** We consider how to recognize the space $\mathcal{M}(E,F)$ as a power-normed space.

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be power-normed spaces. Then we saw in Proposition 1.11(i) that the map

$$(T_1, \ldots, T_m) \mapsto \Delta_{(T_1,\ldots,T_m)}\,, \quad \mathcal{B}(E,F)^m \to \mathcal{B}(E,F^m)\,,$$

is a linear isomorphism for each $m \in \mathbb{N}$.

Now suppose that $m \in \mathbb{N}$ and that $T_1, \dots, T_m \in \mathcal{M}(E, F)$; set $T = \Delta_{(T_1,\dots,T_m)}$. Then

$$T^{(n)}(x_1, \dots, x_n) = (T_i x_j : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n) \quad (x_1, \dots, x_n \in E)\,,$$

and so

$$\left\| T^{(n)}(x_1, \dots, x_n) \right\|_{mn} \leqslant \sum_{i=1}^{m} \left\| T_i^{(n)}(x_1, \dots, x_n) \right\|_n \quad (x_1, \dots, x_n \in E)\,.$$

This shows that $T \in \mathcal{M}(E, F^m)$ with $\|T\|_{mb} \leqslant \sum_{i=1}^m \|T_i\|_{mb}$, and so we have a linear map

$$\Psi_m : (T_1, \dots, T_m) \mapsto \Delta_{(T_1,\dots,T_m)}\,, \quad \mathcal{M}(E, F)^m \to \mathcal{M}(E, F^m)$$

for each $m \in \mathbb{N}$. Now take $T \in \mathcal{M}(E, F^m)$, and set $T_i = \pi_i \circ T \in \mathcal{B}(E, F)$ for $i \in \mathbb{N}_m$, as in Proposition 1.11(i). Then $\left\| T_i^{(k)} \right\| \leqslant \left\| T^{(k)} \right\|$ $(k \in \mathbb{N})$, and so $T_i \in \mathcal{M}(E, F)$ $(i \in \mathbb{N}_m)$. Thus $\Psi_m$ is a surjection, and hence a linear bijection.

We denote by $\|\cdot\|_m^\dagger$ the norm on $\mathcal{M}(E, F)^m$ induced by this identification, so that

$$\|(T_1, \dots, T_m)\|_m^\dagger = \left\|\Delta_{(T_1,\dots,T_m)}\right\|_{mb} \quad (T_1, \dots, T_m \in \mathcal{M}(E, F),\ m \in \mathbb{N})\,.$$

Thus

$$\|(T_1, \dots, T_m)\|_m^\dagger = \sup\left\{\|(T_i x_j : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n)\|_{mn} : \|(x_1, \dots, x_n)\|_n \leqslant 1,\ n \in \mathbb{N}\right\} \quad (3.2.1)$$

for $T_1, \dots, T_m \in \mathcal{M}(E, F)$, essentially as in [20, Proposition 6.19]. We see easily that

$$(\mathcal{M}(E, F)^m, \|\cdot\|_m^\dagger)$$

is a power-normed space.

Clause (i) of the following result was given in [52, Proposition 4.4.7].

THEOREM 3.16. *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be power-normed spaces, take $p$ such that $1 \leqslant p \leqslant \infty$, and set $\mathcal{M} = (\mathcal{M}(E, F)^m, \|\cdot\|_m^\dagger)$.*

(i) *Suppose that $(F^n, \|\cdot\|_n)$ is a $p$–multi-normed space. Then $\mathcal{M}$ is a $p$–multi-normed space.*

(ii) *Suppose that $(F^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space. Then $\mathcal{M}$ is a strong $p$–multi-normed space.*

(iii) *Suppose that $(F^n, \|\cdot\|_n)$ is a $p$–convex power-normed space. Then $\mathcal{M}$ is a $p$–convex power-normed space.*

*Proof.* (i) Take $m, n \in \mathbb{N}$, $S \in \mathbb{M}_{m,n}$, and $T_1, \dots, T_n \in \mathcal{M}(E, F)$; set $\boldsymbol{T} = (T_1, \dots, T_n)$. We have

$$(\Psi_m(S(\boldsymbol{T})))^{(k)}(\boldsymbol{x}) = S^{(k)}\left((\Psi_n(\boldsymbol{T}))^{(k)}(\boldsymbol{x})\right) \quad (\boldsymbol{x} \in E^k,\ k \in \mathbb{N})\,,$$

and so

$$\left\|(\Psi_m(S(\boldsymbol{T})))^{(k)}\right\| \leqslant \|S : \ell_n^p \to \ell_m^p\| \left\|(\Psi_n(\boldsymbol{T}))^{(k)}\right\| \quad (k \in \mathbb{N})$$

because, by Proposition 2.5(ii), $((F^{kj}, \|\cdot\|_{kj}) : j \in \mathbb{N})$ is a $p$–multi-normed space. It follows that

$$\|S(\boldsymbol{T})\|_m^\dagger = \|(\Psi_m(S(\boldsymbol{T}))\|_{mb} \leqslant \|S : \ell_n^p \to \ell_m^p\|\ \|\Psi_n(\boldsymbol{T})\|_{mb} = \|S : \ell_n^p \to \ell_m^p\|\ \|\boldsymbol{T}\|_n^\dagger\,,$$

and this shows that $(\mathcal{M}(E, F)^m, \|\cdot\|_m^\dagger)$ is a $p$–multi-normed space.

(ii) Take $m,n \in \mathbb{N}$, $(S_1,\dots,S_m) \in \mathcal{M}(E,F)^m$, and $(T_1,\dots,T_n) \in \mathcal{M}(E,F)^n$ such that

$$(T_1,\dots,T_n) \leqslant_p (S_1,\dots,S_m)\,.$$

For each $x \in E$ and $\lambda \in F'$, the map $T \mapsto \langle Tx,\, \lambda\rangle$, $\mathcal{M}(E,F) \to \mathbb{F}$, is a continuous linear functional, and so

$$\left(\sum_{j=1}^{n} |\langle T_j x,\, \lambda\rangle|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m} |\langle S_i x,\, \lambda\rangle|^p\right)^{1/p}.$$

Now take $k \in \mathbb{N}$ and $x_1,\dots,x_k \in E$. Then

$$\left(\sum_{j=1}^{n}\sum_{r=1}^{k} |\langle T_j x_r,\, \lambda\rangle|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m}\sum_{r=1}^{k} |\langle S_i x_r,\, \lambda\rangle|^p\right)^{1/p} \qquad (\lambda \in F')\,.$$

Since the power-norm based on $F$ is a strong $p$–multi-norm, it follows that

$$\|(T_j x_r : j \in \mathbb{N}_n,\, r \in \mathbb{N}_k)\|_{nk} \leqslant \|(S_i x_r : i \in \mathbb{N}_m,\, r \in \mathbb{N}_k)\|_{mk}\,.$$

By equation (3.2.1),

$$\|(T_1,\dots,T_n)\|_n^{\dagger} \leqslant \|(S_1,\dots,S_m)\|_m^{\dagger}\,.$$

This shows that $(\|\cdot\|_m^{\dagger})$ is a strong $p$–multi-norm based on $\mathcal{M}$.

(iii) Take $m,n \in \mathbb{N}$, $S_1,\dots,S_m \in \mathcal{M}(E,F)$, and $T_1,\dots,T_n \in \mathcal{M}(E,F)$, and set $\boldsymbol{S} = (S_1,\dots,S_m)$ and $\boldsymbol{T} = (T_1,\dots,T_n)$. For each $k \in \mathbb{N}$ and $\boldsymbol{x} = (x_1,\dots,x_k) \in E^k$, we have

$$\begin{aligned}&\|((S_i x_r : i \in \mathbb{N}_m,\, r \in \mathbb{N}_k),\, (T_j x_r : j \in \mathbb{N}_n,\, r \in \mathbb{N}_k))\|_{(m+n)k}\\ &\qquad\leqslant \left(\|(S_i x_r : i \in \mathbb{N}_m,\, r \in \mathbb{N}_k)\|_{mk}^p + \|(T_j x_r : j \in \mathbb{N}_n,\, r \in \mathbb{N}_k)\|_{nk}^p\right)^{1/p}\end{aligned}$$

because the power-norm based on $F$ is $p$–convex, and so, by (3.2.1),

$$\|(\boldsymbol{S},\boldsymbol{T})\|_{m+n}^{\dagger} \leqslant \left(\left(\|\boldsymbol{S}\|_m^{\dagger}\right)^p + \left(\|\boldsymbol{T}\|_n^{\dagger}\right)^p\right)^{1/p}.$$

This shows that $(\mathcal{M}(E,F)^m, \|\cdot\|_m^{\dagger})$ is $p$–convex. ■

We remark that one can also identify $\mathcal{M}(E,F)^m$ with $\mathcal{M}(E^m,F)$, following Proposition 1.11(ii), so obtaining another power-norm, say $(\|\cdot\|_n^{\times})$, based on $\mathcal{M}(E,F)$ when $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are power-normed spaces. In the case where $1 \leqslant p \leqslant \infty$ and $(E^n, \|\cdot\|_n)$ is a $p$–multi-normed space, $(\mathcal{M}(E,F)^m, \|\cdot\|_n^{\times})$ is a $q$–multi-normed space, where $q = p'$. Similar results to those in Theorem 3.16 hold; see [52].

# 4. Banach lattices

**4.1. Background on Banach lattices.** We now consider how the theory of $p$–multi-norms described above applies in the special case where they are based on a Banach lattice. In particular we shall introduce the canonical lattice $p$–multi-norm associated with a Banach lattice.

Background on Banach lattice theory which is relevant to the theory of multi-norms is given in [20, §1.3]. For example, the spaces $C(K)$, $\ell^r$ and $L^r(\Omega)$ (for each $r$ with $1 \leqslant r \leqslant \infty$) are Banach lattices in the usual way.

In most texts (for example, see [43]), a 'Banach lattice' is based on a real Banach space; we shall call this a *real Banach lattice*, and the complexification of a real Banach lattice is what we shall term a *complex Banach lattice*, as in [20]. We shall use the term *Banach lattice* for a real or complex Banach lattice.

The lattice operations in a real Banach lattice $E$ are denoted by $\vee$ and $\wedge$, and we shall use standard notation; for example,

$$x^+ = x \vee 0\,, \quad x^- = (-x) \vee 0\,, \quad |x| = x \vee (-x) = x^+ + x^-\,,$$

for $x \in E$.

We recall the standard construction of the complexification of a real Banach lattice. Indeed, suppose that $E$ is a (complex) linear space such that $E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$ for a real Banach lattice $(E_{\mathbb{R}}, \|\cdot\|)$. The positive cone of $E_{\mathbb{R}}$ is denoted by $E^+$; it is the *positive cone of* $E$. Take $z \in E$, say $z = x + \mathrm{i}y$, where $x, y \in E_{\mathbb{R}}$, so that $x = \Re z$ and $y = \Im z$, and first define the *modulus* $|z| \in E^+$ of $z$ by

$$|z| = \left(|x|^2 + |y|^2\right)^{1/2}$$

(the right-hand side is well-defined in $E^+$ by the 'Youdine–Krivine functional calculus', given below), and then define

$$\|z\| = \|\,|z|\,\| \quad (z \in E)\,.$$

Alternatively, we can set

$$|z| = |x + \mathrm{i}y| = \sup\{x\,\cos\theta + y\,\sin\theta : 0 \leqslant \theta \leqslant 2\pi\}\,; \tag{4.1.1}$$

the supremum always exists in $E^+$ and the two definitions of $|z|$ are consistent. Then $(E, \|\cdot\|)$ is a complex Banach lattice; the space $E_{\mathbb{R}}$ is the *underlying real lattice*. For details of these remarks, see [1, §3.2], [3], [20], [43, §1.d], [46, §2.2], and [56, Chapter II, §11].

Let $E$ be a Banach lattice. For $x \in E^+$, we set

$$\Delta_x = \{z \in E : |z| \leqslant x\}\,.$$

A functional $\lambda \in E'$ is *positive* if

$$\langle x,\, \lambda\rangle \geqslant 0 \quad (x \in E^+)\,,$$

and these positive linear functionals form the positive cone $(E')^+$ in $E'$, so that $E'$ is the dual Banach lattice. In fact, take $\lambda, \mu \in E'_{\mathbb{R}}$. Then $\langle x,\, \lambda \vee \mu\rangle$ and $\langle x,\, \lambda \wedge \mu\rangle$ are defined for $x \in E^+$ by the following Riesz–Kantorovich formulae:

$$\left\{\begin{array}{lcl} \langle x,\, \lambda \vee \mu\rangle &=& \sup\{\langle y,\, \lambda\rangle + \langle z,\, \mu\rangle : y, z \in E^+,\, y + z = x\}\,, \\ \langle x,\, \lambda \wedge \mu\rangle &=& \inf\{\langle y,\, \lambda\rangle + \langle z,\, \mu\rangle : y, z \in E^+,\, y + z = x\}\,, \end{array}\right. \tag{4.1.2}$$

and then $\lambda \vee \mu$ and $\lambda \wedge \mu$ are extended in the obvious way to be defined on $E_{\mathbb{R}}$.

Now take $F = E \oplus \mathrm{i}E$ to be the complexification of a real Banach lattice $E$. Let $\lambda$ be a continuous, real-linear functional on $E$. Then $\lambda$ extends uniquely to a continuous, complex-linear functional on $F$: indeed, we define

$$\lambda(x + \mathrm{i}y) = \lambda(x) + \mathrm{i}\lambda(y) \quad (x, y \in E)\,,$$

and so we may regard $E'$ as a real-linear subspace of $F'$. For each $\lambda$ in $F'$, there exist $\lambda_1$ and $\lambda_2$ in $E'$ such that $\lambda(x) = \lambda_1(x) + \mathrm{i}\lambda_2(x)$ $(x \in E)$, and so $F'$ is isomorphic as a complex Banach space to the complexification $E' \oplus \mathrm{i}E'$. In fact, this identification is isometric; the details of this are given in [1, Corollary 3.26] and [45, Proposition 2.2.6], for example. Thus we obtain the *dual Banach lattice* of a (complex) Banach lattice.

Similarly, given a bounded operator $T : E \to F$ between two real Banach lattices, one can define the *complexification* $T_{\mathbb{C}}$ of $T$ by

$$T_{\mathbb{C}} : x + \mathrm{i}y \mapsto Tx + \mathrm{i}Ty\,, \quad E \oplus \mathrm{i}E \to F \oplus \mathrm{i}F\,.$$

It is easy to see that $T_{\mathbb{C}}$ is again a bounded operator with $\|T\| \leqslant \|T_{\mathbb{C}}\| \leqslant 2\,\|T\|$; see [1, p. 106], for example.

A linear subspace $F$ of a real Banach lattice $E$ is a *sublattice* if $x \vee y, x \wedge y \in F$ whenever $x, y \in F$; a linear subspace $F$ of a complex Banach lattice $E$ is a *sublattice* if $F$ is the complexification of a sublattice of $E_{\mathbb{R}}$. The lattice operations in a real Banach lattice are continuous, and so, for example, the closure of a sublattice in a Banach lattice is a sublattice. A linear subspace $F$ of a Banach lattice $E$ is an *order-ideal* in $E$ if $x \in F$ whenever $x \in E$ and $|x| \leqslant |y|$ for some $y \in F$; clearly each order-ideal in $E$ is a sublattice of $E$.

Let $F$ be a norm-closed order-ideal in a Banach lattice $E$, and let $Q_F : E \to E/F$ be the quotient map. Then the quotient space $E/F$, taken with the positive cone $Q_F(E^+)$, is a Banach lattice.

Let $E$ be a Banach lattice. We set

$$B_E^+ = B_E \cap E^+\,.$$

We shall use the following easy fact. Suppose that $x, y \in E^+$ and $\langle y,\, \lambda\rangle \leqslant \langle x,\, \lambda\rangle$ for each positive linear functional $\lambda$ on $E$. Then $y \leqslant x$ in $E^+$. Also, for each $\lambda \in (E')^+$, we have

$$\|\lambda\| = \sup\{\langle x, \lambda\rangle : x \in B_E^+\}\,. \tag{4.1.3}$$

We shall often use the following *Riesz decomposition property* of Banach lattices; see [43, p. 2] or [46, Theorem 1.1.1], for example.

PROPOSITION 4.1. *Let $E$ be a Banach lattice. Suppose that $x_1, x_2, y \in E^+$ are such that $y \leqslant x_1+x_2$. Then there are $y_1, y_2 \in E^+$ with $y_1 \leqslant x_1$, with $y_2 \leqslant x_2$, and with $y = y_1+y_2$.* ∎

DEFINITION 4.2. A Banach lattice $(E, \|\cdot\|)$ is *monotonically bounded* if every increasing net in $B_E^+$ is bounded above; it is *Dedekind complete* if every non-empty subset of $E^+$ which is bounded above has a supremum; it has the *Fatou property* if, for every increasing net $(x_\alpha : \alpha \in A)$ in $E^+$ that has a supremum $x \in E^+$, necessarily

$$\|x\| = \sup\{\|x_\alpha\| : \alpha \in A\}\,. \tag{4.1.4}$$

For example, suppose that $K$ is a compact space. Then the Banach lattice $C(K)$ is Dedekind complete if and only if $K$ is extremely disconnected [17, Theorem 2.3.3].

A Dedekind complete Banach lattice has the Fatou property if and only if it has the Nakano property, in the sense of [20, Definition 1.22(v)]. A dual Banach lattice is always Dedekind complete and has the Fatou property [46, Proposition 2.4.19].

DEFINITION 4.3. A Banach lattice $(E, \|\cdot\|)$ is an *AL-space* if

$$\|x+y\| = \|x\| + \|y\| \quad \text{whenever} \quad x, y \in E^+ \quad \text{with} \quad x \wedge y = 0\,,$$

and an *AM-space* if

$$\|x \vee y\| = \max\{\|x\|\,, \|y\|\} \quad \text{whenever} \quad x, y \in E^+ \quad \text{with} \quad x \wedge y = 0\,.$$

We shall use the following terminology.

Let $E$ and $F$ be real Banach lattices. A linear map $T : E \to F$ is a *lattice homomorphism* if

$$T(x \vee y) = Tx \vee Ty \quad (x, y \in E)\,.$$

Let $E$ and $F$ be complex Banach lattices that are the complexifications of the real Banach lattices $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$, respectively. A linear map $T : E \to F$ is a *lattice homomorphism* if $T(x + \mathrm{i}y) = Sx + \mathrm{i}Sy$ $(x, y \in E_{\mathbb{R}})$, where $S$ is a lattice homomorphism from $E_{\mathbb{R}}$ to $F_{\mathbb{R}}$.

Now suppose that $E$ and $F$ are Banach lattices and that $T \in \mathcal{B}(E, F)$. Then $T$ is a *lattice isomorphism* if it is a bijective lattice homomorphism; one can easily see that, in this case, the inverse map $T^{-1}$ is also a lattice homomorphism. The map $T$ is a *lattice isometry* if $T$ is a lattice homomorphism that is an isometry; the two lattices $E$ and $F$ are *lattice isomorphic*, respectively, *lattice isometric*, if there is a lattice isomorphism, respectively, a lattice isometry, from $E$ onto $F$. A *lattice embedding* from $E$ to $F$ is an embedding that is a lattice isomorphism onto its range. For example, the canonical embedding $\kappa_E : E \to E''$ is a lattice isometry [3, Theorem I.5.4].

Let $E$ and $F$ be complex Banach lattices, and suppose that $T \in \mathcal{B}(E, F)$ is a lattice isomorphism such that $\|Tx\| = \|x\|$ $(x \in E^+)$. One can easily check (using equation (4.1.1)) that $T : E \to F$ is an isometry.

The following central representation theorem is proved in [1, Theorems 3.5 and 3.6], [3, Theorems 4.27 and 4.29], [43, §1.b], and [46, Theorems 2.1.3 and 2.7.1], for example; we shall call it '*Kakutani's theorem*'; detailed attributions for the various statements are given in [1]. The proofs in the above sources are for real Banach lattices; the complex version is given in [1, Theorem 3.20].

DEFINITION 4.4. Let $E$ be a Banach lattice. Then $e \in E^+$ is an *$AM$-unit* for $E$ if, for each $x \in E$, we have $\|x\| \leqslant 1$ if and only if $|x| \leqslant e$.

THEOREM 4.5. (i) *A Banach lattice is an $AL$-space if and only if it is lattice isometric to a Banach lattice of the form $L^1(\Omega)$ for some measure space $\Omega$.*

(ii) *A Banach lattice is an $AM$-space if and only if it is lattice isometric to a closed sublattice of a space $C(K)$ for some compact space $K$.*

(iii) *A Banach lattice with an $AM$-unit is lattice isometric to a space $C(K)$ for some compact space $K$.* ■

We recall one standard construction concerning Banach lattices; see [43, §1.d] for details.

Let $E$ be a Banach lattice, and take $e > 0$ in $E$. We denote by $I_e$ the principal order-ideal in $E$ generated by $e$, so that

$$I_e = \{x \in E : |x| \leqslant \zeta e \quad \text{for some} \quad \zeta \geqslant 0\}\,.$$

For $x \in I_e$, set

$$\|x\|_e = \inf\{\zeta \geqslant 0 : |x| \leqslant \zeta e\}\,.$$

Then $(I_e, \|\cdot\|_e)$ is a Banach lattice that is an $AM$-space, and $e$ is an $AM$-unit for $I_e$, and so, by Theorem 4.5(iii), $I_e$ is lattice isometric to $C(K)$ for some compact space $K$.

DEFINITION 4.6. Let $E$ be a Banach lattice. An element $e$ with $e > 0$ is a *strong unit* if $I_e = E$.

Thus $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent norms on $E$ when $e$ is a strong unit, and we have the following result.

PROPOSITION 4.7. *Let $(E, \|\cdot\|)$ be a Banach lattice with a strong unit, $e$. Then $\|\cdot\|_e$ is equivalent to the given norm $\|\cdot\|$, and $(E, \|\cdot\|_e)$ is lattice isometric to $C(K)$ for a certain compact space $K$.* ■

Let $n \in \mathbb{N}$. A function $F : \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if

$$F(\alpha t_1, \dots, \alpha t_n) = \alpha F(t_1, \dots, t_n) \quad (\alpha \in \mathbb{R}^+,\, t_1, \dots, t_n \in \mathbb{R})\,.$$

Now let $E$ be a real Banach lattice, take $x_1, \dots, x_n \in E$, and choose an element $e \in E^+$ such that $|x_i| \leqslant e$ $(i \in \mathbb{N}_n)$; for example, take $e = |x_1| \vee \cdots \vee |x_n|$ in $E$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous, positively homogeneous function. Then, identifying $I_e$ with $C(K, \mathbb{R})$ for some compact space $K$, we can set

$$F(x_1, \dots, x_n)(t) = F(x_1(t), \dots, x_n(t)) \quad (t \in K)\,,$$

and so

$$F(x_1, \dots, x_n) \in I_e \subset E\,;$$

in fact, the element $F(x_1, \dots, x_n)$ is independent of the choice of $e$. The map that takes $F$ to $F(x_1, \dots, x_n)$ is the *Youdine–Krivine calculus* [63, 35]; for details of this construction, see [43, §1.d], for example. In particular, for each $p$ with $1 \leqslant p \leqslant \infty$ and each Banach lattice $E$, we can define the element

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \in E^+$$

for $x_1, \ldots, x_n \in E$, where we interpret this element as $|x_1| \vee \cdots \vee |x_n|$ in the case where $p = \infty$. Similarly, for each $\theta \in (0,1)$, we can define the element $|x|^{1-\theta} |y|^{\theta}$ for $x, y \in E$.

The Youdine–Krivine functional calculus as above is indeed usually given for real Banach lattices. There is an extension to the complex setting; this is given in [34, Section 3], for example.

Let $E$ and $G$ be real Banach lattices, and suppose that $T \in \mathcal{B}(E,G)$ is a lattice homomorphism. Take $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$, and a continuous, positively homogeneous function $F : \mathbb{R}^n \to \mathbb{R}$. Then

$$T(F(x_1, \ldots, x_n)) = F(Tx_1, \ldots, Tx_n)\,. \tag{4.1.5}$$

Let $E$ be a real Banach lattice, let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous, positively homogeneous function, and suppose that $F(t_1, \ldots, t_n) \geqslant 0$ $(t_1, \ldots, t_n \in \mathbb{R})$. Then we see that $F(x_1, \ldots, x_n) \geqslant 0$ for each $x_1, \ldots, x_n \in E$. Thus, in order to verify an inequality (or an equality) that involves only continuous, positively homogeneous functions of finitely-many variables (and, in particular, any lattice operations) in an arbitrary real Banach lattice, it suffices to verify the inequality for real numbers.

Take $p$ with $1 \leqslant p \leqslant \infty$. We recall from [43, p. 42] that, for a real Banach lattice $E$, $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in E$, we have

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \sup\left\{\sum_{i=1}^{n} \alpha_i x_i : (\alpha_1, \ldots, \alpha_n) \in B_{\ell_n^q(\mathbb{R})}\right\}, \tag{4.1.6}$$

where $q = p'$. The same proof as that in [43] shows that, for a complex Banach lattice $E$, $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in E$, we have

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \sup\left\{\left|\sum_{i=1}^{n} \alpha_i x_i\right| : (\alpha_1, \ldots, \alpha_n) \in B_{\ell_n^q(\mathbb{C})}\right\}, \tag{4.1.7}$$

where again $q = p'$. It follows that

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \sup\left\{\Re\left(\sum_{i=1}^{n} \alpha_i x_i\right) : (\alpha_1, \ldots, \alpha_n) \in B_{\ell_n^q(\mathbb{C})}\right\}. \tag{4.1.8}$$

Indeed, these equalities hold in $C(K,\mathbb{C})$ for each compact space $K$, and hence in an arbitrary Banach lattice.

We have the following generalized versions of Hölder's inequality.

Proposition 4.8. *Let $E$ be a Banach lattice.*

(i) *Take $p_0, p_1$ with $1 \leqslant p_0 \leqslant p_1 < \infty$ and take $\theta$ with $0 < \theta < 1$, and define $p$ by*

$1/p = (1-\theta)/p_0 + \theta/p_1$. *Then*

$$\left(\sum_{i=1}^n \alpha_i\,|x_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n \alpha_i\,|x_i|^{p_0}\right)^{(1-\theta)/p_0} \left(\sum_{i=1}^n \alpha_i\,|x_i|^{p_1}\right)^{\theta/p_1}$$

*for each* $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$, *and* $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$, *and*

$$\left(\sum_{i=1}^n (|x_i|^{1-\theta}\,|y_i|^{\theta})^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n |x_i|^{p_0}\right)^{(1-\theta)/p_0} \left(\sum_{i=1}^n |y_i|^{p_1}\right)^{\theta/p_1} \tag{4.1.9}$$

*for each* $n \in \mathbb{N}$ *and* $x_1, \dots, x_n, y_1, \dots, y_n \in E$.

(ii) *Take* $p$ *with* $1 \leqslant p \leqslant \infty$. *Then*

$$\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle| \leqslant \left\langle \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\rangle$$

*for each* $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$, *and* $\lambda_1, \dots, \lambda_n \in E'$, *where* $q = p'$.

*Proof.* The first part of clause (i) and clause (ii) are given in [43, Proposition 1.d.2, (ii) and (iii)], for example.

For the second part of clause (i), recall that the following generalization of Hölder's inequality holds for each $n \in \mathbb{N}$, each $q_0, q_1 \in (1, \infty)$, and each $s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{R}^+$, where $1/p = 1/q_0 + 1/q_1$:

$$\left(\sum_{i=1}^n s_i^p t_i^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n s_i^{q_0}\right)^{1/q_0} \left(\sum_{i=1}^n t_i^{q_1}\right)^{1/q_1}. \tag{4.1.10}$$

Now take $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{F}$, and set $s_i = |x_i|^{1-\theta}$ and $t_i = |y_i|^{\theta}$ for $i \in \mathbb{N}_n$, and set $q_0 = p_0/(1-\theta)$ and $q_1 = p_1/\theta$. Then we see that our two definitions of $p$ coincide and that inequality (4.1.9) holds with this interpretation of the symbols.

Define

$$F : (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \left(\sum_{i=1}^n (|x_i|^{1-\theta}\,|y_i|^{\theta})^p\right)^{1/p}, \quad \mathbb{R}^{2n} \to \mathbb{R},$$

and

$$G : (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \left(\sum_{i=1}^n |x_i|^{p_0}\right)^{(1-\theta)/p_0} \left(\sum_{i=1}^n |y_i|^{p_1}\right)^{\theta/p_1}, \quad \mathbb{R}^{2n} \to \mathbb{R}.$$

Then $F$ and $G$ are continuous and positively homogeneous functions on $\mathbb{R}^{2n}$ such that

$$F(x_1, \dots, x_n, y_1, \dots, y_n) \leqslant G(x_1, \dots, x_n, y_1, \dots, y_n) \quad (x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}).$$

By the Youdine–Krivine calculus described above, the same inequality holds whenever $x_1, \dots, x_n, y_1, \dots, y_n \in E$, where we note that all terms are in $E^+$, and so inequality (4.1.9) holds in this case. ∎

In the next result, we shall use the following form of the Riesz–Kantorovich formula for complex Banach lattices specifically given in [1, Corollary 3.26].

Let $E$ be a complex Banach lattice, and take $\lambda \in E'$. Then

$$\langle x,\, |\lambda| \rangle = \sup\{ |\langle z,\, \lambda \rangle| : z \in \Delta_x \} \quad (x \in E^+)\,. \tag{4.1.11}$$

It follows that, for each $\lambda \in (E')^+$, we have

$$\langle |x|\,,\, \lambda \rangle = \sup\{ |\langle x,\, \mu \rangle| : \mu \in \Delta_\lambda \} \quad (x \in E)\,. \tag{4.1.12}$$

PROPOSITION 4.9. *Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p < \infty$. Then*

$$\left\langle x,\, \left( \sum_{i=1}^n |\lambda_i|^q \right)^{1/q} \right\rangle = \sup \left\{ \left| \sum_{i=1}^n \langle x_i,\, \lambda_i \rangle \right| : x_1, \dots, x_n \in E,\ \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leqslant x \right\}$$

*for each $x \in E^+$, $n \in \mathbb{N}$, and $\lambda_1, \dots, \lambda_n \in E'$, where $q = p'$.*

*Proof.* This result in the case where $E$ is a real Banach lattice is given in [43, p. 48].

Now suppose that $E$ is a complex Banach lattice with underlying real Banach lattice $E_{\mathbb{R}}$, and take $x \in E^+$, $n \in \mathbb{N}$, and $\lambda_1, \dots, \lambda_n \in E'$. By the real case, we have

$$\left\langle x,\, \left( \sum_{i=1}^n |\lambda_i|^q \right)^{1/q} \right\rangle = \sup \left\{ \sum_{i=1}^n \langle x_i,\, |\lambda_i| \,\rangle : x_1, \dots, x_n \in E_{\mathbb{R}},\ \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leqslant x \right\}.$$

Fix $\varepsilon > 0$, and take $x_1, \dots, x_n \in E^+$ with

$$\left( \sum_{i=1}^n x_i^p \right)^{1/p} \leqslant x \quad \text{and} \quad \left\langle x,\, \left( \sum_{i=1}^n |\lambda_i|^q \right)^{1/q} \right\rangle \leqslant \sum_{i=1}^n \langle x_i,\, |\lambda_i| \rangle + \varepsilon\,.$$

By (4.1.11), there exist $z_1, \dots, z_n \in E$ such that

$$|z_i| \leqslant x_i \quad \text{and} \quad \langle x_i,\, |\lambda_i| \rangle \leqslant |\langle z_i,\, \lambda_i \rangle| + \varepsilon$$

for each $i \in \mathbb{N}_n$. By multiplying each $z_i$ by a complex number of modulus 1, we may suppose that $\langle z_i,\, \lambda_i \rangle \in \mathbb{R}^+$. It follows that

$$\left( \sum_{i=1}^n |z_i|^p \right)^{1/p} \leqslant x \quad \text{and} \quad \left\langle x,\, \left( \sum_{i=1}^n |\lambda_i|^q \right)^{1/q} \right\rangle \leqslant \sum_{i=1}^n \langle z_i,\, \lambda_i \rangle + \varepsilon(n+1)\,.$$

This holds true for each $\varepsilon > 0$, and so

$$\left\langle x,\, \left( \sum_{i=1}^n |\lambda_i|^q \right)^{1/q} \right\rangle \leqslant \sup \left\{ \left| \sum_{i=1}^n \langle z_i,\, \lambda_i \rangle \right| : z_1, \dots, z_n \in E,\ \left( \sum_{i=1}^n |z_i|^p \right)^{1/p} \leqslant x \right\}.$$

The opposite inequality follows immediately from Proposition 4.8(ii), and so the result is proved. ∎

The following is Khintchine's inequality for Banach lattices; it follows easily from the same inequality for scalars and the Youdine–Krivine calculus.

PROPOSITION 4.10. *Let $E$ be a real Banach lattice. Then*

$$\frac{1}{\sqrt{2}} \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \leqslant \frac{1}{2^n} \sum \left| \sum_{i=1}^n \varepsilon_i x_i \right|$$

*for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, where the outer sum on the right-hand side is taken over all choices of $\varepsilon_i = \pm 1$ for $i \in \mathbb{N}_n$.* ∎

The following deep theorem of Krivine is taken from [43, Proposition 1.f.14]; here $K_G$ denotes Grothendieck's constant.

THEOREM 4.11. *Let $E$ and $F$ be Banach lattices, and take $T \in \mathcal{B}(E, F)$. Then*

$$\left\| \left( \sum_{i=1}^{n} |Tx_i|^2 \right)^{1/2} \right\| \leqslant K_G \, \|T\| \left\| \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|$$

*for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$.* ∎

The following definition is taken from [43, Definition 1.d.3].

DEFINITION 4.12. Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p \leqslant \infty$. Then $E$ is *$p$–convex* (with constant 1) if

$$\left\| (|x|^p + |y|^p)^{1/p} \right\| \leqslant (\|x\|^p + \|y\|^p)^{1/p} \quad (x, y \in E)$$

and *$p$–concave* (with constant 1) if

$$\left\| (|x|^p + |y|^p)^{1/p} \right\| \geqslant (\|x\|^p + \|y\|^p)^{1/p} \quad (x, y \in E)\,.$$

For example, for a space $E = L^p(\Omega)$, where $1 \leqslant p \leqslant \infty$ and $\Omega$ is a measure space, we have

$$\left\| \left( \sum_{i=1}^{n} |f_i|^p \right)^{1/p} \right\|_E = \left( \sum_{i=1}^{n} \|f_i\|_E^p \right)^{1/p} \qquad (f_1, \dots, f_n \in E,\ n \in \mathbb{N})\,, \tag{4.1.13}$$

and so $L^p(\Omega)$ is both $p$–convex and $p$–concave. Conversely, it is shown in [43, p. 59] that each Banach lattice that is both $p$–convex and $p$–concave is lattice isometric to a Banach lattice of the form $L^p(\Omega)$. More generally, a calculation shows that, for $r$ with $1 \leqslant r \leqslant \infty$, the Banach lattice $L^r(\Omega)$ is $p$–convex if and only if $r \in [p, \infty]$ and is $p$–concave if and only if $r \in [1, p]$.

Take $p$ with $1 \leqslant p \leqslant \infty$. It is noted in [43, Proposition 1.d.4] that a Banach lattice is $p$–convex, respectively, $p$–concave, if and only if the dual Banach lattice is $p'$–concave, respectively, $p'$–convex.

**4.2. Regular and order-bounded operators.** We first recall the definitions of two Banach spaces $\mathcal{B}_r(E, F)$ and $\mathcal{B}_b(E, F)$.

Let $E$ be a Banach lattice. A subset $B$ of $E$ is *order-bounded* if there exists $x \in E^+$ such that $B \subset \Delta_x$. Let $E$ and $F$ be real Banach lattices, and let $S$ and $T$ be linear operators from $E$ to $F$. Then

$$S \leqslant T \quad \text{if} \quad Sx \leqslant Tx \quad (x \in E^+)\,.$$

Clearly $(\mathcal{L}(E, F), \leqslant)$ is an ordered linear space.

Definition 4.13. Let $E$ and $F$ be real Banach lattices, and consider a linear operator $T$ from $E$ to $F$. Then:

(i) $T$ is *positive* if $T \geqslant 0$;

(ii) $T$ is *regular* if $T = T_1 - T_2$, where $T_1$ and $T_2$ are positive operators;

(iii) $T$ is *order-bounded* if $T(B)$ is an order-bounded subset of $F$ for each order-bounded subset $B$ of $E$.

The set of positive operators from $E$ to $F$ is closed under addition and multiplication by $\alpha \in \mathbb{R}^+$, and so it is a cone. Each regular operator is order-bounded. The book [3] is devoted to positive operators on real Banach lattices (and more general spaces).

Now suppose that $E$ and $F$ are complex Banach lattices, with underlying real Banach lattices $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$, respectively. Then $T \in \mathcal{L}(E, F)$ is *positive* if $T(E_{\mathbb{R}}) \subset F_{\mathbb{R}}$ and the map $T \mid E_{\mathbb{R}} : E_{\mathbb{R}} \to F_{\mathbb{R}}$ is positive. For a positive operator, we have $|Tx| \leqslant T(|x|)$ $(x \in E)$. Each operator in $\mathcal{L}(E, F)$ has a unique expression in the form $S + \mathrm{i}T$, where $S$ and $T$ belong to $\mathcal{L}(E_{\mathbb{R}}, F_{\mathbb{R}})$ and

$$(S + \mathrm{i}T)(x + \mathrm{i}y) = Sx - Ty + \mathrm{i}(Sy + Tx) \quad (x, y \in E_{\mathbb{R}})\,;$$

such an operator is *regular* or *order-bounded* if both $S$ and $T$ are regular or order-bounded, respectively.

Let $E$ and $F$ be Banach lattices. Each order-bounded operator is continuous, and so we denote the spaces of all positive, all regular, and all order-bounded operators from $E$ to $F$ by $\mathcal{B}(E, F)^+$, $\mathcal{B}_r(E, F)$, and $\mathcal{B}_b(E, F)$, respectively. Thus we have

$$\mathcal{B}(E, F)^+ \subset \mathcal{B}_r(E, F) \subset \mathcal{B}_b(E, F) \subset \mathcal{B}(E, F)\,.$$

We write $\mathcal{B}_r(E)$ and $\mathcal{B}_b(E)$ for $\mathcal{B}_r(E, E)$ and $\mathcal{B}_b(E, E)$, respectively. Take $T \in \mathcal{B}(E, F)^+$. Then

$$\|T\| = \sup\{\|Tx\| : x \in B_E^+\}\,. \tag{4.2.1}$$

Proposition 4.14. *Let $E$ and $F$ be Banach lattices, and take $p$ with $1 \leqslant p \leqslant \infty$. For each $T \in \mathcal{B}(E, F)^+$, we have*

$$\left(\sum_{i=1}^{n} |Tx_i|^p\right)^{1/p} \leqslant T\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,.$$

*Proof.* We may suppose that $x_1, \dots, x_n \in E^+$ and that we are working in $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$. Set $q = p'$.

By equation (4.1.6), we have $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \sup A$, where

$$A = \left\{\sum_{i=1}^{n} \alpha_i x_i : (\alpha_1, \dots, \alpha_n) \in B_{\ell_n^q(\mathbb{R})}\right\}\,.$$

Further,

$$\left(\sum_{i=1}^{n} |Tx_i|^p\right)^{1/p} = \sup T(A)\,.$$

Since $\sup T(A) \leqslant T(\sup A)$, the result follows. ■

It follows that

$$\left\| \left( \sum_{i=1}^{n} |Tx_i|^p \right)^{1/p} \right\| \leqslant \|T\| \left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}) \tag{4.2.2}$$

for each $T \in \mathcal{B}(E, F)^+$, a result of Krivine [43, Proposition 1.d.9].

In particular, Proposition 4.14 implies that

$$\left( \sum_{i=1}^{n} |\langle x_i,\ \lambda\rangle|^p \right)^{1/p} \leqslant \left\langle \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \lambda \right\rangle \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}) \tag{4.2.3}$$

for each $\lambda \in (E')^+$.

Let $E$ and $F$ be Banach lattices. We now describe the norms on $\mathcal{B}_r(E, F)$ and $\mathcal{B}_b(E, F)$. For each $T \in \mathcal{B}_b(E, F)$, there exists $c > 0$ such that, for each $x \in E^+$, there exists $y \in F^+$ with $T(\Delta_x) \subset \Delta_y$ and $\|y\| \leqslant c\|x\|$. The infimum of these constants $c$ is denoted by $\|T\|_b$. Details of this result are given in [20, Proposition 1.26], which is based on [60].

For $T \in \mathcal{B}_r(E, F)$, set

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{B}(E, F)^+,\ |Tz| \leqslant S(|z|) \quad (z \in E)\}\,.$$

Proposition 4.15. *Let $E$ and $F$ be Banach lattices. Then:*

(i) *$\|\cdot\|_b$ is a norm on the space $\mathcal{B}_b(E, F)$ such that*

$$\|T\|_b \geqslant \|T\| \quad (T \in \mathcal{B}_b(E, F))\,,$$

*and $(\mathcal{B}_b(E, F), \|\cdot\|_b)$ is a Banach space;*

(ii) *$\|\cdot\|_r$ is a norm on $\mathcal{B}_r(E, F)$ such that*

$$\|T\|_r \geqslant \|T\|_b \geqslant \|T\| \quad (T \in \mathcal{B}_r(E, F))\,,$$

*and $(\mathcal{B}_r(E, F), \|\cdot\|_r)$ is a Banach space.* ■

In the case where $F = E$, the spaces $(\mathcal{B}_r(E), \|\cdot\|_r)$ and $(\mathcal{B}_b(E), \|\cdot\|_b)$ are unital Banach subalgebras of $\mathcal{B}(E)$.

The following result is proved in [3, pp. 12–13], for example; formula (4.2.4), below, is a *Riesz–Kantorovich formula*.

Proposition 4.16. *Let $E$ and $F$ be Banach lattices, with $F$ Dedekind complete. Then $\mathcal{B}_r(E, F) = \mathcal{B}_b(E, F)$ is a Dedekind complete Banach lattice. Suppose that $T \in \mathcal{B}_r(E, F)$. Then*

$$|T|\,(x) = \sup\{|Tz| : |z| \leqslant x\} \quad (x \in E^+) \tag{4.2.4}$$

*and, further, $\|T\|_r = \|\,|T|\,\|$ and $|Tz| \leqslant |T|\,(|z|)$ $(z \in E)$.* ■

Let $E$ and $F$ be Banach lattices. Often, but not always, the two spaces $\mathcal{B}_r(E, F)$ and $\mathcal{B}_b(E, F)$ are the same; by the above result, this holds when $F$ is Dedekind complete, and, in particular, when $F$ is a dual Banach lattice. In the case where $E$ and $F$ are $AL$-spaces, it follows from [1, Theorem 3.9 and Corollary 3.10] and [3, Theorem 15.3] (where we note that each $AL$-space is a '$KB$-space') that $\mathcal{B}_r(E, F) = \mathcal{B}_b(E, F) = \mathcal{B}(E, F)$ and that

$\|T\|_r = \|T\|$ $(T \in \mathcal{B}(E,F))$. On the other hand, suppose that $p > 1$, that $E = L^p(\Omega)$ for a measure space $\Omega$, and that $E$ is an infinite-dimensional space. Then, by [4], $\mathcal{B}_r(E)$ is not even dense in $(\mathcal{B}(E), \|\cdot\|)$ and $\|\cdot\|_r$ and $\|\cdot\|$ are not equivalent on $\mathcal{B}_r(E)$. Examples with $\mathcal{B}_r(E,F) \subsetneq \mathcal{B}_b(E,F)$ and with $\mathcal{B}_b(E,F) \subsetneq \mathcal{B}(E,F)$ are given in [3, Examples 1.11 and 15.1]. An example given in [60, §2] shows that there may be operators in $\mathcal{B}_b(E,F)$ that are not even in the $\|\cdot\|$-closure of $\mathcal{B}_r(E,F)$, and Example 4.1 of [60] exhibits Banach lattices $E$ and $F$ and a compact, order-bounded operator $V : E \to F$ which is not in the $\|\cdot\|_b$-closure of $\mathcal{B}_r(E,F)$. Suppose that $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F)$. Then the norms $\|\cdot\|_r$ and $\|\cdot\|_b$ are equivalent on $\mathcal{B}_r(E,F)$, but examples in [60] show that the norms are not necessarily equal in this case. For general Banach lattices $E$ and $F$, the two norms $\|\cdot\|_r$ and $\|\cdot\|_b$ are not necessarily equivalent on $\mathcal{B}_r(E,F)$.

More information on regular and order-bounded operators can be found in the fine survey article [61]. In this article, Theorems 2.1 and 2.4, respectively, characterize the lattices $F$ such that $\mathcal{B}_r(E,F) = \mathcal{B}(E,F)$ for every Banach lattice $E$ and lattices $E$ such that $\mathcal{B}_r(E,F) = \mathcal{B}(E,F)$ for every Banach lattice $F$; some extra cases are provided by Example 2.7 and Theorems 2.8 and 2.9 of [61]. Further, conditions for the equality $\mathcal{B}_b(E,F) = \mathcal{B}_r(E,F)$ are given in [61, Section 4].

Let $E$ and $F$ be Banach lattices, and take $T \in \mathcal{B}_r(E,F)$. Then $T \mid G \in \mathcal{B}_r(G,F)$ for each closed sublattice $G$ of $E$.

Definition 4.17. Let $E$ and $F$ be Banach lattices, and take $T \in \mathcal{B}(E,F)$. Then $T$ is *pre-regular* if $T' \in \mathcal{B}(F',E')$ is regular, and then

$$\|T\|_{pr} = \|T'\|_r$$

for each such operator $T$. The space of pre-regular operators from $E$ to $F$ is denoted by $\mathcal{B}_{pr}(E,F)$.

Thus $\mathcal{B}_{pr}(E,F)$ is a linear subspace of $\mathcal{B}(E,F)$,

$$\|T\|_{pr} \geqslant \|T\| \quad (T \in \mathcal{B}_{pr}(E,F)),$$

and $(\mathcal{B}_{pr}(E,F), \|\cdot\|_{pr})$ is a Banach space.

It is clear that $T'$ is regular and that $\|T'\|_r \leqslant \|T\|_r$ for each $T \in \mathcal{B}_r(E,F)$, and so a regular operator is pre-regular. Further, the dual of an order-bounded operator is order-bounded [3, Theorem 5.8], and so an order-bounded operator is pre-regular by Proposition 4.16. Thus we have

$$\mathcal{B}(E,F)^+ \subset \mathcal{B}_r(E,F) \subset \mathcal{B}_b(E,F) \subset \mathcal{B}_{pr}(E,F) \subset \mathcal{B}(E,F).$$

The following example shows that $\mathcal{B}_b(E,F)$ can be a proper subset of $\mathcal{B}_{pr}(E,F)$.

Example 4.18. In [3, Example 15.1], it is shown that the map

$$T : f \mapsto (f(1/n) - f(0) : n \in \mathbb{N}), \quad C(\mathbb{I}) \to c_0,$$

is a bounded linear operator that is not order-bounded, and hence not regular. However the dual $T'$ of $T$ is an operator $T' : \ell^1 \to C(\mathbb{I})'$ between two $AL$-spaces, and so $T'$ is regular, and hence $T$ is pre-regular. ∎

PROPOSITION 4.19. *Let $E$ and $F$ be Banach lattices, and suppose that $F$ is Dedekind complete and has the Fatou property. Take $T \in \mathcal{B}_r(E,F)$. Then $T' \in \mathcal{B}_r(F',E')$ and $\|T'\|_r = \|T\|_r$.*

*Proof.* We shall show that $\|T\|_r \leqslant \|T''\|_r$. Since $\|T''\|_r \leqslant \|T'\|_r \leqslant \|T\|_r$, this implies the result.

Fix $\varepsilon > 0$. By (4.2.1), there exists $x \in E^+$ with $\|x\| = 1$ and

$$\| \, |T| \, \| \leqslant \| \, |T| \, (x) \| + \varepsilon \, .$$

Set $S = \{|Tz| : |z| \leqslant x\}$, a subset of $F^+$, so that, by equation (4.2.4), $\sup S = |T| \, (x)$. The family $\mathcal{F}$ of finite subsets of $S$, when ordered by inclusion, is a directed set. For each $\alpha \in \mathcal{F}$, set $y_\alpha = \sup \alpha$, so that $(y_\alpha : \alpha \in \mathcal{F})$ is an increasing net in $F^+$ such that $\sup\{y_\alpha : \alpha \in \mathcal{F}\} = |T| \, (x)$. Since $F$ has the Fatou property,

$$\| \, |T| \, (x) \, \| = \sup\{\|y_\alpha\| : \alpha \in \mathcal{F}\} \, .$$

Note that $\widehat{x} = \kappa_E(x)$ belongs to $(E'')^+$. Now set $\widetilde{S} = \{|T''\zeta| : |\zeta| \leqslant \widehat{x}\}$, a subset of $(F'')^+$, and let $\widetilde{\mathcal{F}}$ be the family of finite subsets of $\widetilde{S}$; suppose that the elements $\widetilde{y_\beta}$ are defined in an analogous way to the elements $y_\alpha$, now with respect to $\widetilde{\mathcal{F}}$. Since $F''$ has the Fatou property,

$$\| \, |T''| \, (\widehat{x}) \, \| = \sup\{\|\widetilde{y_\beta}\| : \beta \in \widetilde{\mathcal{F}}\} \, .$$

Since $\{|T''\zeta| : |\zeta| \leqslant \widehat{x}\} \supset \{|T''\widehat{z}| : |z| \leqslant x\}$ and the embedding of $F$ into $F''$ is a lattice homomorphism, so that $\{\widehat{y_\alpha} : \alpha \in \mathcal{F}\}$ is a subset of $\{\widetilde{y_\beta} : \beta \in \widetilde{\mathcal{F}}\}$, it follows that $\| \, |T''| \, (\widehat{x}) \, \| \geqslant \| \, |T| \, (x) \, \|$. Thus

$$\| \, |T''| \, \| \geqslant \| \, |T''| \, (\widehat{x}) \, \| \geqslant \| \, |T| \, \| - \varepsilon \, .$$

This holds true for each $\varepsilon > 0$, and so $\|T\|_r \leqslant \|T''\|_r$, as required. ∎

COROLLARY 4.20. *Let $E$ and $F$ be Banach lattices, and suppose that $F$ is a dual Banach lattice. Take $T \in \mathcal{B}_r(E,F)$. Then $T' \in \mathcal{B}_r(F',E')$ and $\|T'\|_r = \|T\|_r = \| \, |T| \, \|$.* ∎

THEOREM 4.21. *Let $E$ and $F$ be Banach lattices, and take $T \in \mathcal{B}(E,F)$. Then the following are equivalent:*

(a) *$T' : F' \to E'$ is regular, so that $T$ is pre-regular;*

(b) *$T'' : E'' \to F''$ is regular;*

(c) *$\kappa_F \circ T : E \to F''$ is regular.*

*In this case, the three operators have the same regular norm.*

*Proof.* Certainly (a) $\Rightarrow$ (b) and $\|T''\|_r \leqslant \|T'\|_r$. Since $\kappa_F \circ T = T'' \circ \kappa_E : E \to F''$, we see that (b) $\Rightarrow$ (c) and $\|\kappa_F \circ T\|_r \leqslant \|T''\|_r$

Finally, suppose that (c) holds. Then $(\kappa_F \circ T)' : F''' \to E'$ is regular. Now we have $T' = (\kappa_F \circ T)' \circ \kappa_{F'} : F' \to E'$, and so $T'$ is regular, giving (a); further, we have $\|T'\|_r \leqslant \|(\kappa_F \circ T)'\|_r$. By Corollary 4.20, $\|\kappa_F \circ T\|_r = \|(\kappa_F \circ T)'\|_r$, and hence $\|T'\|_r \leqslant \|\kappa_F \circ T\|_r$.

The result follows. ∎

**4.3. Multi-norms based on Banach lattices.** We now define the canonical lattice $p$–multi-norm based on a Banach lattice.

Definition 4.22. Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p \leqslant \infty$. For each $n \in \mathbb{N}$, set

$$\|\boldsymbol{x}\|_n^{L,p} = \left\| \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)\,. \tag{4.3.1}$$

The corresponding definition to (4.3.1) in the special case where $p = \infty$ is

$$\|\boldsymbol{x}\|_n^{L} = \|\, |x_1| \vee \cdots \vee |x_n|\, \| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)\,;$$

the above definition in the special case where $p = 1$ is

$$\|\boldsymbol{x}\|_n^{DL} = \|\, |x_1| + \cdots + |x_n|\, \| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n)\,.$$

Then $(\|\cdot\|_n^L)$ and $(\|\cdot\|_n^{DL})$ are the *lattice multi-norm* and the *dual lattice multi-norm*, respectively, based on $E$, as defined in [20, Definition 4.41].

Let $E$ be a Banach lattice. Then the Banach space $(E^n, \|\cdot\|_n^{L,p})$ is the space that is sometimes denoted by $E(\ell_n^p)$, slightly modifying the notation of [43, p. 46], and we shall do this at some later points. See also [45, p. 8].

The space $(E^n, \|\cdot\|_n^{L,p})$ is itself a Banach lattice with respect to the coordinatewise operations.

Theorem 4.23. *Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p \leqslant \infty$. Then the sequence $(\|\cdot\|_n^{L,p})$ based on $E$ is a strong $p$–multi-norm.*

*Proof.* As in [20, Theorem 4.42], it is immediately checked that $(\|\cdot\|_n^L)$ is an $\infty$–multi-norm. By Theorem 2.25, each $\infty$–multi-norm is a strong $\infty$–multi-norm, and so the result holds in the case where $p = \infty$.

Now suppose that $1 \leqslant p < \infty$, and set $q = p'$. By Proposition 2.23, we know that a strong $p$–multi-norm is a $p$–multi-norm, and so it suffices to verify the condition in Definition 1.37.

Take $m, n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$, and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$ with $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$. Thus, for each positive linear functional $\lambda$ on $E$, each $\mu \in E'$ with $|\mu| \leqslant \lambda$, and each $(\alpha_1, \dots, \alpha_n) \in B_{\ell_n^q}$, we have

$$\begin{aligned}\left| \left\langle \sum_{j=1}^n \alpha_j y_j,\, \mu \right\rangle \right| &\leqslant \sum_{j=1}^n |\alpha_j|\, |\langle y_j, \mu\rangle| \leqslant \left( \sum_{j=1}^n |\alpha_j|^q \right)^{1/q} \left( \sum_{j=1}^n |\langle y_j, \mu\rangle|^p \right)^{1/p} \\ &\leqslant \left( \sum_{i=1}^m |\langle x_i,\, \mu\rangle|^p \right)^{1/p} \leqslant \left( \sum_{i=1}^m \langle |x_i|\,,\, \lambda\rangle^p \right)^{1/p} \quad \text{by} \quad (4.1.12) \\ &\leqslant \left\langle \left( \sum_{i=1}^m |x_i|^p \right)^{1/p},\, \lambda \right\rangle \quad \text{by} \quad (4.2.3)\,,\end{aligned}$$

and so it follows from (4.1.12) that

$$\left\langle \left|\sum_{j=1}^{n} \alpha_j y_j\right|, \lambda \right\rangle \leqslant \left\langle \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p}, \lambda \right\rangle .$$

This holds for each positive linear functional $\lambda$, and so

$$\left|\sum_{j=1}^{n} \alpha_j y_j\right| \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \quad \text{in} \quad E$$

for each $(\alpha_1, \dots, \alpha_n) \in B_{\ell_n^q}$. It now follows from (4.1.6) or (4.1.7) that

$$\left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p},$$

and hence $\|\boldsymbol{y}\|_n^{L,p} \leqslant \|\boldsymbol{x}\|_m^{L,p}$, giving the result. ∎

Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p < \infty$. It follows from Theorem 2.11 that

$$\mu_{p,n}(\boldsymbol{x}) \leqslant \|\boldsymbol{x}\|_n^{L,p} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N}),$$

A short calculation shows that we have equality in the case where $E = C(K)$ for a compact space $K$.

Definition 4.24. Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p \leqslant \infty$. Then the sequence $(\|\cdot\|_n^{L,p})$ defined in (4.3.1) is the *canonical lattice $p$–multi-norm* based on $E$.

Example 4.25. Take $p$ with $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$. We give a specific example of a space $E(\ell_n^p) = (E^n, \|\cdot\|_n^{L,p})$.

Indeed, we take $r$ with $1 \leqslant r < \infty$, and consider the Banach lattice $E = \ell^r$. The space $E(\ell_n^p)$ consists of $n$-tuples $\boldsymbol{x} = (x_1, \dots, x_n)$, where $x_i = (x_{i,j} : j \in \mathbb{N}) \in \ell^r$ for $i \in \mathbb{N}_n$, and the norm of such an element is

$$\|\boldsymbol{x}\|_n^{L,p} = \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{n} |x_{i,j}|^p\right)^{r/p}\right)^{1/r}. \tag{4.3.2}$$

Now consider the Banach space $F = \ell_n^p$. For $1 \leqslant r < \infty$, the space $\ell^r(F)$ consists of sequences $\boldsymbol{y} = (y_j : j \in \mathbb{N})$, where $y_j = (y_{j,i} : i \in \mathbb{N}_n) \in \ell_n^p$ for $j \in \mathbb{N}$, and the norm of such an element is

$$\|\boldsymbol{y}\|_{\ell^r(F)} = \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{n} |y_{j,i}|^p\right)^{r/p}\right)^{1/r}. \tag{4.3.3}$$

Thus $E(\ell_n^p)$ with $E = \ell^r$ is isometrically isomorphic to $\ell^r(F)$ with $F = \ell_n^p$. ∎

Theorem 4.26. *Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p < \infty$. Then the canonical lattice $p$–multi-norm based on $E$ is $p$–convex if and only if $E$ is $p$–convex as a Banach lattice.*

*Proof.* Suppose first that $E$ is $p$–convex as a Banach lattice, and suppose that $m, n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_m) \in E^m$, and $\boldsymbol{y} = (y_1, \dots, y_n) \in E^n$. Set $\|\cdot\|_n = \|\cdot\|_n^{L,p}$ $(n \in \mathbb{N})$ and

$$u = \left(\sum_{i=1}^m |x_i|^p\right)^{1/p} \quad \text{and} \quad v = \left(\sum_{j=1}^n |y_j|^p\right)^{1/p}$$

in $E$. It follows that

$$\|(\boldsymbol{x}, \boldsymbol{y})\|_{m+n} = \left\|(u^p + v^p)^{1/p}\right\| \leqslant (\|u\|^p + \|v\|^p)^{1/p} = (\|\boldsymbol{x}\|_m^p + \|\boldsymbol{y}\|_n^p)^{1/p} .$$

Hence the $p$–multi-norm $(\|\cdot\|_n^{L,p})$ is $p$–convex.

Conversely, suppose that the $p$–multi-norm $(\|\cdot\|_n^{L,p})$ is $p$–convex, and take $x, y \in E$. Then

$$\left\|(|x|^p + |y|^p)^{1/p}\right\| = \|(x, y)\|_2^{L,p} \leqslant (\|x\|^p + \|y\|^p)^{1/p} ,$$

and so $E$ is a $p$–convex Banach lattice. ∎

COROLLARY 4.27. *Take $p$ with $1 \leqslant p < \infty$, and suppose that $E$ is a $p$–convex Banach lattice. Then*

$$\|\boldsymbol{x}\|_n^{L,p} \leqslant \|\boldsymbol{x}\|_{\ell_n^p(E)} \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N}) . \ ∎$$

It is shown in [20, §4.3.1] that the two sequences $(\|\cdot\|_n^L : n \in \mathbb{N})$ and $(\|\cdot\|_n^{DL} : n \in \mathbb{N})$ are multi-norms and dual multi-norms, respectively, and that the duals of the lattice multi-norm and the dual lattice multi-norm based on $E$ are the dual lattice multi-norm and the lattice multi-norm, respectively, based on $E'$. We now generalize these facts; the proof is similar to one on pages 47 and 48 of [43] that shows (for the case of real Banach lattices) that the dual space of $E(\ell_n^p)$ is $E'(\ell_n^{p'})$.

THEOREM 4.28. *Let $E$ be a Banach lattice, and take $p$ with $1 \leqslant p \leqslant \infty$. Then the dual of the canonical lattice $p$–multi-norm based on $E$ is the canonical lattice $p'$-multi-norm based on $E'$.*

*Proof.* The cases where $p = 1$ and $p = \infty$ have already been covered, and so we may suppose that $1 < p < \infty$. Set $q = p'$. For $n \in \mathbb{N}$, we write $\|\cdot\|_n'$ for the dual of the norm $\|\cdot\|_n^{L,p}$, so that $\|\cdot\|_n'$ is defined on the space $(E')^n$.

Take $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$. For $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$|\langle \boldsymbol{x},\, \boldsymbol{\lambda}\rangle| \leqslant \sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle| \leqslant \left\langle \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\rangle \leqslant \left\| \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\| \|\boldsymbol{x}\|_n^{L,p}$$

by Proposition 4.8(ii), and so

$$\|\boldsymbol{\lambda}\|_n' \leqslant \left\| \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\| = \|\boldsymbol{\lambda}\|_n^{L,q} .$$

For the reverse inequality, take $x \in E^+$, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ such that $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \leqslant x$. Then $\|\boldsymbol{x}\|_n^{L,p} \leqslant \|x\|$. For each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$, we have

$$\left|\sum_{i=1}^n \langle x_i,\, \lambda_i\rangle\right| \leqslant \|\boldsymbol{\lambda}\|_n' \,\|x\| \ ,$$

and so it follows from Proposition 4.9 that

$$\left\langle x,\, \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\rangle \leqslant \|\boldsymbol{\lambda}\|_n' \,\|x\| \ .$$

Hence, by (4.1.3), we have

$$\|\boldsymbol{\lambda}\|_n^{L,q} = \left\| \left(\sum_{i=1}^n |\lambda_i|^q\right)^{1/q} \right\| \leqslant \|\boldsymbol{\lambda}\|_n' \ .$$

This concludes the proof. ∎

Take $p$ with $1 \leqslant p \leqslant \infty$. We now consider the canonical lattice $p$–multi-norms associated with sublattices and quotients of a Banach lattice.

First, let $F$ be a closed sublattice of a Banach lattice $E$, and consider the canonical lattice $p$–multi-norm based on $E$. Then $F$ is a Banach lattice, and the $p$–multi-norm induced on the family $\{F^n : n \in \mathbb{N}\}$ is exactly the canonical lattice $p$–multi-norm based on $F$.

Next suppose that $F$ is a closed order-ideal in $E$, so that $E/F$ is again a Banach lattice; we again write $Q_F : E \to E/F$ for the quotient map, so that $Q_F$ is a lattice homomorphism. Then there are a quotient power-norm, temporarily called $(\|\cdot\|_{n,\mathrm{quot}})$, and a canonical lattice $p$–multi-norm, temporarily called $(\|\cdot\|_{n,\mathrm{can}})$, based on $E/F$. We *claim* that these two $p$–multi-norms coincide.

Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E$. Then

$$Q_F\left(\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}\right) = Q_F\left(\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right) \quad (y_1, \dots, y_n \in F)\,,$$

and so $\|\boldsymbol{x} + F^n\|_{n,\mathrm{can}} \leqslant \|\boldsymbol{x} + F^n\|_{n,\mathrm{quot}}$.

To prove that, conversely, we have $\|\boldsymbol{x} + F^n\|_{n,\mathrm{quot}} \leqslant \|\boldsymbol{x} + F^n\|_{n,\mathrm{can}}$, it suffices to show that, for each $n \in \mathbb{N}$, each $x_1, \dots, x_n \in E$, and each $y \in F$, there exist $y_1, \dots, y_n \in F$ such that

$$\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} \leqslant \left|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} - y\right| \,, \tag{4.3.4}$$

and we shall do this. Set

$$u = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \ .$$

Without loss of generality, we may suppose that $0 \leqslant y \leqslant u$, for otherwise, replacing $y$ by $(\Re y)^+ \wedge u$ will reduce the right-hand side of (4.3.4). It suffices to prove that, for each such $y$, there exist $y_1, \dots, y_n \in E$ such that (4.3.4) holds and such that $|y_i| \leqslant y \ \ (i \in \mathbb{N}_n)$, for the latter condition guarantees that $y_1, \dots, y_n \in F$. We can work in the order ideal $I_u$, which we can identify with $C(K)$ for a compact space $K$, and so it suffices to establish the inequality (4.3.4) in the special case in which $E = C(K)$.

For $i \in \mathbb{N}_n$, define $y_i$ such that

$$y_i(t) = (|x_i(t)| \wedge y(t))\,\frac{x_i(t)}{|x_i(t)|} \quad \text{when} \quad t \in K \ \text{ and } \ x_i(t) \neq 0$$

and $y_i(t) = 0$ when $t \in K$ and $x_i(t) = 0$. Then we see that $y_1, \dots, y_n \in C(K)$ and also that $|x_i - y_i| = |x_i| - |y_i|\ \ (i \in \mathbb{N}_n)$. By replacing each $x_i$ by $|x_i|$, we may suppose that $x_i \geqslant 0\ \ (i \in \mathbb{N}_n)$ in (4.3.4). Hence $y_1, \dots, y_n \in C(K)^+$ and $y_i = x_i \wedge y$ for each $i \in \mathbb{N}_n$, and so we see that it suffices to prove that

$$\left(\sum_{i=1}^{n} (x_i - x_i \wedge y)^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} - y \tag{4.3.5}$$

whenever $x_1, \dots, x_n \in C(K)^+$ and $y \in C(K)^+$ with $y \leqslant u$. Since the order in $C(K, \mathbb{R})$ is pointwise, it suffices to prove equation (4.3.5) in the case where $x_1, \dots, x_n, y \in \mathbb{R}^+$. Set $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y, y, \dots, y) \in \mathbb{R}^n$; without loss of generality, we may suppose that $\|\boldsymbol{x}\|_{\ell_n^p} = 1$, in which case $0 \leqslant y \leqslant 1$. Thus we need to show that $\|(\boldsymbol{x} - \boldsymbol{y})^+\|_{\ell_n^p} \leqslant 1 - y$.

We may suppose that $x_1, \dots, x_k \geqslant y$ and that $x_{k+1}, \dots, x_n \leqslant y$ for some $k \in \mathbb{N}_n$. Take $\alpha_1, \dots, \alpha_k \geqslant 0$ such that

$$\|(\boldsymbol{x} - \boldsymbol{y})^+\|_{\ell_n^p} = \sum_{i=1}^{k} (x_i - y)\alpha_i \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i^q = 1\,,$$

where $q = p'$. Then $\sum_{i=1}^{k} \alpha_i \geqslant 1$, and so

$$\|(\boldsymbol{x} - \boldsymbol{y})^+\|_{\ell_n^p} \leqslant \sum_{i=1}^{k} x_i\alpha_i - y \leqslant \|\boldsymbol{x}\|_{\ell_n^p}\,\|(\alpha_1, \dots, \alpha_k)\|_{\ell_k^q} - y = 1 - y\,,$$

as required. Thus we have proved the following theorem.

THEOREM 4.29. *Let $E$ be a Banach lattice, and suppose that $F$ is a closed order-ideal in $E$. Take $p$ with $1 \leqslant p \leqslant \infty$. Then the quotient power-norm induced on $E/F$ by the canonical lattice $p$–multi-norm on $E$ is the canonical lattice $p$–multi-norm on $E/F$.* ∎

**4.4. Interpolation between Banach lattices.** We consider interpolation between complex Banach lattices. In particular we wish to note first that in certain circumstances, a particular interpolation space between two Banach lattices is itself a Banach lattice. This topic has been previously considered; the seminal work is [11], and some works have shown the result for Banach lattices of particular types. The result is also stated without proof by Raynaud and Tradacete in [53, p. 96]. However we have not found exactly the result that we seek, and so we provide details here; we are grateful to Michael Cwikel for some valuable comments, based on [14].

The initial definition and results apply to both real and complex Banach lattices.

Definition 4.30. Let $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ be Banach lattices such that $\{E_0, E_1\}$ is a compatible couple of Banach spaces with an ambient space $H$ that is a Banach lattice. Suppose, further, that, for $i = 0, 1$, each $E_i$ is an order-ideal (not necessarily closed) in $H$. Then $\{E_0, E_1\}$ is a *compatible couple of Banach lattices*.

Later, we shall use the following remark. Let $E_0$ and $E_1$ be compatible couple of Banach lattices, and take $x \in E_0 + E_1$ such that $0 \leqslant x \leqslant y_0 + y_1$, where $y_0 \in E_0^+$ and $y_1 \in E_1^+$. Then

$$\|x\|_{E_0+E_1} \leqslant \|y_0\|_0 + \|y_1\|_1 .$$

Indeed, by the Riesz decomposition property, Proposition 4.1, there exist $x_0, x_1 \in H^+$ such that $x_0 \leqslant y_0$, $x_1 \leqslant y_1$, and $x = x_0 + x_1$. Since $E_0$ and $E_1$ are ideals in $H$, we see that $x_0 \in E_0^+$ and $x_1 \in E_1^+$. Thus $\|x\|_{E_0+E_1} \leqslant \|x_0\|_0 + \|x_1\|_1 \leqslant \|y_0\|_0 + \|y_1\|_1$, as required.

Theorem 4.31. *Let $\{E_0, E_1\}$ be a compatible couple of Banach lattices. Then*

$$(E_0 \cap E_1, \|\cdot\|_{E_0 \cap E_1}) \quad \text{and} \quad (E_0 + E_1, \|\cdot\|_{E_0+E_1})$$

*are Banach lattices that are sublattices of the ambient space.*

*Proof.* We know that $E_0 \cap E_1$ and $E_0 + E_1$ are Banach spaces, and they are sublattices of the ambient space.

It is clear that $E_0 \cap E_1$ is a Banach lattice; we shall show that $E_0 + E_1$ (with the norm $\|\cdot\| = \|\cdot\|_{E_0+E_1}$) is a Banach lattice.

We first *claim* the following: Take $x, y \in E_0 + E_1$ with $0 \leqslant x \leqslant y$. Then $\|x\| \leqslant \|y\|$. Indeed, fix $\varepsilon > 0$. Then there exist $y_0 \in E_0$ and $y_1 \in E_1$ such that $y = y_0 + y_1$ and

$$\|y_0\|_0 + \|y_1\|_1 \leqslant \|y\| + \varepsilon .$$

We may suppose that $y_0 \in (E_0)_{\mathbb{R}}$ and $y_1 \in (E_1)_{\mathbb{R}}$. We have $x \leqslant y \leqslant y_0^+ + y_1^+$ and

$$\|y_0^+\|_0 + \|y_1^+\|_1 \leqslant \|y\| + \varepsilon .$$

By the remark, $\|x\| \leqslant \|y_0^+\|_0 + \|y_1^+\|_1 \leqslant \|y\| + \varepsilon$. This holds true for each $\varepsilon > 0$, and so the first claim is proved.

Second, we *claim* the following: For each $z \in E_0 + E_1$, we have $\|\,|z|\,\| = \|z\|$.

Indeed, take $z \in E_0 + E_1$ and fix $\varepsilon > 0$. Then there exist $z_0 \in E_0$ and $z_1 \in E_1$ such that $z = z_0 + z_1$ and

$$\|z_0\|_0 + \|z_1\|_1 \leqslant \|z\| + \varepsilon .$$

Then $|z| \leqslant |z_0| + |z_1|$. By the remark, $\|\,|z|\,\| \leqslant \|\,|z_0|\,\|_0 + \|\,|z_1|\,\|_1 = \|z_0\|_0 + \|z_1\|_1$, and so $\|\,|z|\,\| \leqslant \|z\| + \varepsilon$. Hence $\|\,|z|\,\| \leqslant \|z\|$.

For the reverse inequality, again fix $\varepsilon > 0$. There exist $z_0 \in E_0$ and $z_1 \in E_1$ such that $|z| = z_0 + z_1$ and

$$\|z_0\|_0 + \|z_1\|_1 \leqslant \|\,|z|\,\| + \varepsilon .$$

Since $|z| \leqslant |z_0| + |z_1|$, there exist $x_0 \in E_0^+$ and $x_1 \in E_1^+$ such that $x_0 \leqslant |z_0|$, $x_1 \leqslant |z_1|$, and also $|z| = x_0 + x_1$. Take $e \in H^+$ such that $z$, $x_0$, and $x_1$ belong to $I_e$. Then $I_e$ is lattice isomorphic to $C(K)$ for a compact space $K$. By working in $C(K)$, we see that there exist $w_0$ and $w_1$ in $I_e$ such that

$$w_0(t) = x_0(t) \cdot \arg z(t) , \quad w_1(t) = x_1(t) \cdot \arg z(t) \quad (t \in K) .$$

Then $|w_0| = x_0$ and $|w_1| = x_1$, so that $w_0 \in E_0$ and $w_1 \in E_1$. Further, we see that $w_0 + w_1 = |z| \cdot \arg z = z$ in $C(K)$, and hence $z = w_0 + w_1$ in $I_e$. It follows that

$$\|z\| \leqslant \|w_0\|_0 + \|w_1\|_1 = \|x_0\|_0 + \|x_1\|_1 \leqslant \|z_0\|_0 + \|z_1\|_1 \leqslant \| \, |z| \, \| + \varepsilon \, .$$

Thus $\|z\| \leqslant \| \, |z| \, \|$. The second claim follows.

Finally, suppose that $z, w \in E_0 + E_1$ with $|z| \leqslant |w|$. Then $\| \, |z| \, \| = \|z\|$ and $\| \, |w| \, \| = \|w\|$ by the second claim, and $\| \, |z| \, \| \leqslant \| \, |w| \, \|$ by the first claim, and so $\|z\| \leqslant \|w\|$. This shows that $(E_0 + E_1, \| \cdot \|_{E_0+E_1})$ is indeed a Banach lattice. ∎

We also note the following. Suppose that $E_0$ and $E_1$ are complex Banach lattices that are the complexifications of $F_0$ and $F_1$, respectively. Then $E_0 \cap E_1$ and $E_0 + E_1$ are the complexifications of $F_0 \cap F_1$ and $F_0 + F_1$, respectively.

Let $H$ be a Banach lattice. Take $x_0, x_1 \in H^+$ and $\theta \in (0, 1)$. Then the element $x_0^{1-\theta} x_1^{\theta}$ is defined in $H^+$; here we identify $x_0^{1-\theta} x_1^{\theta}$ with $|x_0|^{1-\theta} |x_1|^{\theta}$, which is defined by the Youdine–Krivine calculus, as in [43]. By [43, Proposition 1.d.2(i)], we have

$$\left\| x_0^{1-\theta} x_1^{\theta} \right\| \leqslant \|x_0\|^{1-\theta} \|x_1\|^{\theta} \, . \tag{4.4.1}$$

Recall from inequality (4.1.9) (with $p_0 = p_1 = 1$) that

$$\sum_{i=1}^{n} y_i^{1-\theta} z_i^{\theta} \leqslant \left( \sum_{i=1}^{n} y_i \right)^{1-\theta} \left( \sum_{i=1}^{n} z_i \right)^{\theta} \tag{4.4.2}$$

for each $n \in \mathbb{N}$ and $y_1, \dots, y_n, z_1, \dots, z_n \in H^+$.

Definition 4.32. Let $\{E_0, E_1\}$ be a compatible couple of Banach lattices, and take $\theta$ with $0 < \theta < 1$. Then the *Calderón–Lozanovskii space*, denoted by $E_0^{1-\theta} E_1^{\theta}$, is the set of all $x \in E_0 + E_1$ such that $|x| \leqslant x_0^{1-\theta} x_1^{\theta}$ for some $x_0 \in E_0^+$ and $x_1 \in E_1^+$. For $x \in E_0^{1-\theta} E_1^{\theta}$, set

$$\|x\|_L = \inf \left\{ c : |x| \leqslant c x_0^{1-\theta} x_1^{\theta}, \ x_i \in B_{E_i}^+ \ (i = 0, 1) \right\} . \tag{4.4.3}$$

We see that

$$\|x\|_L = \inf \left\{ \|y_0\|_0^{1-\theta} \|y_1\|_1^{\theta} : |x| \leqslant y_0^{1-\theta} y_1^{\theta}, \ y_i \in E_i^+ \ (i = 0, 1) \right\} \quad (x \in E_0^{1-\theta} E_1^{\theta}) \, . \tag{4.4.4}$$

The following result is implicit in [53, §4], but no explicit proof was given in that source.

Proposition 4.33. *Let $\{E_0, E_1\}$ be a compatible couple of Banach lattices, and take $\theta$ with $0 < \theta < 1$. Then the Calderón–Lozanovskii space $(E_0^{1-\theta} E_1^{\theta}, \| \cdot \|_L)$ is a Banach lattice and also an intermediate space. Further, the closure of $E_0 \cap E_1$ in $E_0^{1-\theta} E_1^{\theta}$ is a Banach lattice.*

*Proof.* The ambient space for $\{E_0, E_1\}$ is $H$, say.

Set $L = E_0^{1-\theta} E_1^{\theta}$. Clearly $\alpha x \in L$ and $\|\alpha x\|_L = |\alpha| \, \|x\|_L$ whenever $\alpha \in \mathbb{F}$ and $x \in L$. Now take $x_1, x_2 \in L$. We *claim* that $x_1 + x_2 \in L$ and that $\|x_1 + x_2\|_L \leqslant \|x_1\|_L + \|x_2\|_L$, and hence that $\| \cdot \|_L$ is a semi-norm.

To see that $x_1 + x_2 \in L$, it suffices to show that

$$|x_1 + x_2| \leqslant (y_1 + y_2)^{1-\theta} (z_1 + z_2)^{\theta} \tag{4.4.5}$$

whenever $y_j \in E_0^+$, $z_j \in E_1^+$ and $|x_j| \leqslant y_j^{1-\theta} z_j^\theta$ in $H$ for $j = 1, 2$. Since we know that $|x_1 + x_2| \leqslant |x_1| + |x_2|$, inequality (4.4.5) follows from (4.4.2). Hence $x_1 + x_2 \in L$.

We now *claim* that

$$\|x_1 + x_2\|_L \leqslant \|x_1\|_L + \|x_2\|_L .$$

Let $j \in \{1, 2\}$. Given $c_j > \|x_j\|_L$, choose $v_j \in B_{E_0}^+$ and $w_j \in B_{E_1}^+$ with $|x_j| \leqslant c_j v_j^{1-\theta} w_j^\theta$, and set $y_j = c_j v_j \in E_0^+$ and $z_j = c_j w_j \in E_1^+$. Then $|x_j| \leqslant y_j^{1-\theta} z_j^\theta$, so that

$$|x_1 + x_2| \leqslant (y_1 + y_2)^{1-\theta} (z_1 + z_2)^\theta$$

by the inequality (4.4.5). Using equation (4.4.4), we see that

$$\begin{aligned} \|x_1 + x_2\|_L &\leqslant \|y_1 + y_2\|_0^{1-\theta} \|z_1 + z_2\|_1^\theta \\ &\leqslant \left(c_1 \|v_1\|_0 + c_2 \|v_2\|_0\right)^{1-\theta} \left(c_1 \|w_1\|_1 + c_2 \|w_2\|_1\right)^\theta \\ &\leqslant (c_1 + c_2)^{1-\theta} (c_1 + c_2)^\theta = c_1 + c_2. \end{aligned}$$

Since $c_1 > \|x_1\|_L$ and $c_2 > \|x_2\|_L$ were arbitrary, the claim follows.

We have shown that $(L, \|\cdot\|_L)$ is a semi-normed space.

We see easily that the inclusion map of $E_0 \cap E_1$ in $L$ is contractive. To see that the inclusion map of $L$ into $E_0 + E_1$ is a contraction, take $x \in L$ with $\|x\|_L < 1$. Then there exist $x_0 \in B_{E_0}^+$ and $x_1 \in B_{E_1}^+$ with $|x| \leqslant x_0^{1-\theta} x_1^\theta$. But $x_0^{1-\theta} x_1^\theta \leqslant (1-\theta) x_0 + \theta x_1$ (for $x_0, x_1 \in \mathbb{R}^+$, this is [28, Proposition 4.1.3]), and so

$$\|x\|_{E_0 + E_1} = \| \, |x| \, \|_{E_0 + E_1} \leqslant (1 - \theta) \|x_0\|_0 + \theta \|x_1\|_1 \leqslant 1 .$$

It follows that $\|x\|_{E_0 + E_1} \leqslant \|x\|_L \quad (x \in L)$, and so the inclusion is indeed a contraction. In particular, this shows that $x = 0$ when $\|x\|_L = 0$, and so $\|\cdot\|_L$ is a norm on $L$. Hence $(L, \|\cdot\|_L)$ is an intermediate space.

We now *claim* that $(L, \|\cdot\|_L)$ is a Banach space. For this, it suffices to show that $\sum_{j=1}^\infty x_j$ converges in $L$ whenever $(x_j)$ is a sequence in $L$ with $\|x_j\|_L < 2^{-j} \quad (j \in \mathbb{N})$; take $(x_j)$ to be such a sequence.

For each $j \in \mathbb{N}$, there exist $y_{j,0} \in E_0^+$ and $y_{j,1} \in E_1^+$ with $\|y_{j,0}\|_0 = \|y_{j,1}\|_1 < 2^{-j}$ and $|x_j| \leqslant y_{j,0}^{1-\theta} y_{j,1}^\theta \quad (j \in \mathbb{N})$. The two series $\sum_{j=1}^\infty y_{j,0}$ and $\sum_{j=1}^\infty y_{j,1}$ converge, say to $y_0 \in E_0^+$ and $y_1 \in E_1^+$, respectively. Set

$$u_k = \sum_{j=1}^k x_j \quad (k \in \mathbb{N}) .$$

The sequence $(u_k)$ converges in $(E_0 + E_1, \|\cdot\|_{E_0 + E_1})$, say to $x$, and so $(|u_k|)$ converges to $|x|$ in the same space. For each $k \in \mathbb{N}$, we have

$$|u_k| \leqslant \sum_{j=1}^k y_{j,0}^{1-\theta} y_{j,1}^\theta \leqslant \left( \sum_{j=1}^k y_{j,0} \right)^{1-\theta} \left( \sum_{j=1}^k y_{j,1} \right)^\theta$$

by inequality (4.4.2), and so $|u_k| \leqslant y_0^{1-\theta} y_1^\theta$. Since this holds for each $k \in \mathbb{N}$, it follows that $|x| \leqslant y_0^{1-\theta} y_1^\theta$, and this implies that $x \in L$ with $\|x\|_L \leqslant \|y_0\|_0^{1-\theta} \|y_1\|_1^\theta$ by (4.4.4).

Again take $k \in \mathbb{N}$. Then

$$|x - u_k| \leqslant \left(y_0 - \sum_{j=1}^{k} y_{j,0}\right)^{1-\theta} \left(y_1 - \sum_{j=1}^{k} y_{j,1}\right)^{\theta},$$

and so

$$\|x - u_k\|_L \leqslant \left\|y_0 - \sum_{j=1}^{k} y_{j,0}\right\|_0^{1-\theta} \left\|y_1 - \sum_{j=1}^{k} y_{j,1}\right\|^{\theta} \leqslant \frac{1}{2^k},$$

again using inequality (4.4.4). It follows that $(u_k)$ converges to $x$ in $(L, \|\cdot\|_L)$. We have shown that $(L, \|\cdot\|)$ is a Banach space.

It is clear that $(L, \|\cdot\|)$ is a Banach lattice, and that the closure of $E_0 \cap E_1$ in $E_0^{1-\theta} E_1^{\theta}$ is also a Banach lattice. ∎

We remark that, in the case where $E_0$ and $E_1$ are the complexifications of real Banach lattices $F_0$ and $F_1$, respectively, the Calderón–Lozanovskii space $E_0^{1-\theta} E_1^{\theta}$ is the complexification of the space $F_0^{1-\theta} F_1^{\theta}$.

Now suppose that $\{E_0, E_1\}$ is a compatible couple of *complex* Banach lattices, and take $\theta$ with $0 < \theta < 1$. Then, as in §1.10, we can define the intermediate Banach space $((E_0, E_1)_\theta, \|\cdot\|_{[\theta]})$. The following key result of Raynaud and Tradacete is [53, Theorem 9].

THEOREM 4.34. *Let $\{E_0, E_1\}$ be a compatible couple of complex Banach lattices, and take $\theta$ with $0 < \theta < 1$. Then the intermediate space $((E_0, E_1)_\theta, \|\cdot\|_{[\theta]})$ is the closure in the Calderón–Lozanovskii space $(E_0^{1-\theta} E_1^{\theta}, \|\cdot\|_L)$ of the space $E_0 \cap E_1$. Further, $\|x\|_{[\theta]} = \|x\|_L$ for each $x \in E_0 \cap E_1$.* ∎

COROLLARY 4.35. *Let $\{E_0, E_1\}$ be a compatible couple of complex Banach lattices, and take $\theta$ with $0 < \theta < 1$. Then the intermediate space $((E_0, E_1)_\theta, \|\cdot\|_{[\theta]})$ is a Banach lattice.* ∎

THEOREM 4.36. *Let $E$ be a complex Banach lattice. Take $\theta$ with $0 < \theta < 1$, take $n \in \mathbb{N}$, and take $p_0, p_1$ with $1 \leqslant p_0, p_1 < \infty$. Then the interpolation space*

$$((E^n, \|\cdot\|_n^{L,p_0}), (E^n, \|\cdot\|_n^{L,p_1})_\theta$$

*is isometrically isomorphic to the Banach lattice $(E^n, \|\cdot\|_n^{L,p})$, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

*Proof.* We shall use Theorem 4.34. We may suppose that $p_0 \neq p_1$, for the result is trivial when $p_0 = p_1$.

Set $F_i = (E^n, \|\cdot\|_n^{L,p_i})$ for $i = 0$ and $i = 1$. The space $\ell_n^\infty(E)$ plays the rôle of an ambient Banach lattice for the Banach lattices $F_0$ and $F_1$, where we note that the natural injections of $F_0$ and $F_1$ in $\ell_n^\infty(E)$ are continuous lattice homomorphisms and that $F_0$ and $F_1$ are order-ideals in $\ell_n^\infty(E)$.

We denote the Calderón–Lozanovskii space $F_0^{1-\theta} F_1^{\theta}$ specified in Definition 4.32 by $(L, \|\cdot\|_L)$.

The only non-trivial fact that we must show is that

$$\|\boldsymbol{x}\|_L = \left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_E \qquad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n), \tag{4.4.6}$$

and we shall now do this. Fix $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$; without loss of generality, we may suppose that $x_1, \dots, x_n \in E^+$.

As a preliminary, we set $\alpha_i = p/p_i$ and $\beta_i = \alpha_i - 1$ for $i = 0, 1$, so that $\beta_i \neq 0$. We note that $(1-\theta)\alpha_0 + \theta\alpha_1 = 1$ and $(1-\theta)\beta_0 + \theta\beta_1 = 0$.

Consider the functions

$$F_{j,i} : (t_1, \dots, t_n) \mapsto |t_j|^{\alpha_i} \left( \sum_{k=1}^{n} |t_k|^p \right)^{-\beta_i/p}, \quad \mathbb{R}^n \to \mathbb{R},$$

defined for $j \in \mathbb{N}_n$ and $i = 0, 1$, where $F_{j,i}(0, \dots, 0) = 0$. It is clear that each function $F_{j,i}$ is continuous and positively homogeneous, and so operates on $E_{\mathbb{R}}$ by the Youdine–Krivine calculus. We note that

$$F_{j,0}(\boldsymbol{t})^{1-\theta} F_{j,1}(\boldsymbol{t})^{\theta} = |t_j| \quad (\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n,\ j \in \mathbb{N}_n). \tag{4.4.7}$$

Take $\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$, and set $t_{j,i} = F_{j,i}(\boldsymbol{t})$ for $j \in \mathbb{N}_n$ and $i = 0, 1$. Then

$$\left( \sum_{j=1}^{n} |t_{j,i}|^{p_i} \right)^{1/p_i} = \left( \sum_{j=1}^{n} |t_j|^{\alpha_i p_i} \right)^{1/p_i} \left( \sum_{k=1}^{n} |t_k|^p \right)^{-\beta_i/p} = \left( \sum_{k=1}^{n} |t_k|^p \right)^{1/p_i - \beta_i/p}.$$

Also $1/p_i - \beta_i/p = 1/p$, and hence

$$\left( \sum_{j=1}^{n} |t_{j,i}|^{p_i} \right)^{1/p_i} = \left( \sum_{k=1}^{n} |t_k|^p \right)^{1/p} \quad (i = 0, 1). \tag{4.4.8}$$

For $j \in \mathbb{N}_n$ and $i = 0, 1$, set $x_{j,i} = F_{j,i}(x_1, \dots, x_n) \in E^+$. It follows from equation (4.4.7) that

$$x_j = x_{j,0}^{1-\theta} x_{j,1}^{\theta} \quad (j \in \mathbb{N}_n).$$

Set $\boldsymbol{x}_i = (x_{1,i}, \dots, x_{n,i}) \in (E^+)^n$ for $i = 0, 1$. Then $\boldsymbol{x} = \boldsymbol{x}_0^{1-\theta} \boldsymbol{x}_1^{\theta}$, and so

$$\|\boldsymbol{x}\|_L \leqslant \left( \|\boldsymbol{x}_0\|_n^{L,p_0} \right)^{1-\theta} \left( \|\boldsymbol{x}_1\|_n^{L,p_1} \right)^{\theta} = \left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_E$$

by equation (4.4.8).

For the reverse inequality, again take $\boldsymbol{x} = (x_1, \dots, x_n) \in (E^+)^n$, and suppose that $\boldsymbol{x}_0, \boldsymbol{x}_1 \in (E^+)^n$ satisfy $\boldsymbol{x} \leqslant \boldsymbol{x}_0^{1-\theta} \boldsymbol{x}_1^{\theta}$, say $\boldsymbol{x}_0 = (x_{1,0}, \dots, x_{n,0})$ and $\boldsymbol{x}_1 = (x_{1,1}, \dots, x_{n,1})$. Since the lattice operations in $\ell_n^\infty(E)$ are defined coordinatewise, we have

$$x_j \leqslant x_{j,0}^{1-\theta} x_{j,1}^{\theta} \quad (j \in \mathbb{N}_n).$$

It follows from inequality (4.1.9) that

$$\left( \sum_{j=1}^{n} x_j^p \right)^{1/p} \leqslant \left( \sum_{j=1}^{n} x_{j,0}^{p_0} \right)^{(1-\theta)/p_0} \left( \sum_{j=1}^{n} x_{j,1}^{p_1} \right)^{\theta/p_1},$$

and so, by inequality (4.4.1),

$$\left\|\left(\sum_{k=1}^{n} x_k^p\right)^{1/p}\right\|_E \leqslant \left\|\left(\sum_{j=1}^{n} x_{j,0}^{p_0}\right)^{1/p_0}\right\|^{1-\theta} \left\|\left(\sum_{j=1}^{n} x_{j,1}^{p_1}\right)^{1/p_1}\right\|^{\theta}$$
$$= \left(\|\boldsymbol{x}_0\|_n^{L,p_0}\right)^{1-\theta} \left(\|\boldsymbol{x}_1\|_n^{L,p_1}\right)^{\theta}.$$

Taking the infimum over all such choices of $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, we conclude that

$$\left\|\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}\right\|_E \leqslant \|\boldsymbol{x}\|_L .$$

We have established equation (4.4.6), and hence the theorem follows. ■

We believe that a similar result holds when we start with a compatible couple $\{E_0, E_1\}$ of complex Banach lattices, rather than one fixed Banach lattice, but we do not have a proof of such a general result; certain special cases are listed by Calderón in [11].

**4.5. Regular and multi-bounded operators.** Let $E$ and $F$ be Banach lattices, take $p$ with $1 \leqslant p \leqslant \infty$, and consider the canonical lattice $p$–multi-norms based on $E$ and $F$. As before, the norm of a $p$–multi-bounded operator $T \in \mathcal{M}_p(E,F)$ is denoted by $\|T\|_{p-\mathrm{mb}}$. To be specific, we have $T \in \mathcal{M}_p(E,F)$ if and only if there exists a constant $C > 0$ with

$$\left\|\left(\sum_{i=1}^{n} |Tx_i|^p\right)^{1/p}\right\| \leqslant C\left\|\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\right\| \quad (x_1,\dots,x_n \in E,\ n \in \mathbb{N}), \tag{4.5.1}$$

and then $\|T\|_{p-\mathrm{mb}}$ is the infimum of the constants $C$.

The space of multi-bounded operators between two Banach lattices $E$ and $F$, each equipped with the lattice multi-norm $(\|\cdot\|_n^L)$, is discussed and often identified in [20, §6.4]. First, we note that each order-bounded operator $T$ from $E$ to $F$ is $\infty$–multi-bounded and that $\|T\|_{\infty-\mathrm{mb}} \leqslant \|T\|_b$ [20, Theorem 6.31], so that

$$\mathcal{B}_r(E,F) \subset \mathcal{B}_b(E,F) \subset \mathcal{M}_\infty(E,F) \subset \mathcal{B}(E,F),$$

and all the inclusions are contractions. There is a comprehensive statement of some conditions for equality in the above inclusions in [20, Theorem 6.33]; here we state just one result.

Proposition 4.37. *Let $E$ and $F$ be Banach lattices, considered with their Banach lattice multi-norms. Suppose that $F$ is monotonically bounded and Dedekind complete. Then $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F) = \mathcal{M}_\infty(E,F)$.* ■

Corollary 4.38. *Let $E$ and $F$ be Banach lattices. Then*

$$\mathcal{B}_r(E,F') = \mathcal{B}_b(E,F') = \mathcal{M}_\infty(E,F').$$

*Proof.* For a Banach lattice $F$, the dual Banach lattice $F'$ is monotonically bounded and Dedekind complete. ■

Let $E$ and $F$ be Banach lattices. As mentioned above, the 'opérateurs réguliers' of [45, Définition 3.2] are exactly the operators in our class $\mathcal{M}_\infty(E,F)$; this class is denoted by $\mathcal{B}_r(E,F)$ in [45, Définition 3.2]. Note that these 'opérateurs réguliers' are not always the same as the usual 'regular operators'. The 'opérateurs $\ell^1$–réguliers' of [45] are our 1–multi-bounded operators. It is shown in [45, Lemme 1.1] that, in our notation, $\mathcal{M}_\infty(E,F) = \mathcal{M}_1(E,F)$; this will also be a consequence of our Theorem 4.40, to be given below. Our '$p$–multi-bounded operators' correspond to the 'opérateurs $p$–réguliers' of [45, Remarque, p. 21].

Take $p$ with $1 \leqslant p \leqslant \infty$. It follows from equation (4.2.2) that each positive operator in $\mathcal{B}(E,F)$ is $p$–multi-bounded, with $\|T\|_{p-\mathrm{mb}} \leqslant \|T\|$, and so each regular operator is $p$–multi-bounded. In fact, the following stronger statement is true.

THEOREM 4.39. *Let $E$ and $F$ be Banach lattices, and suppose that $T \in \mathcal{B}(E,F)$ is pre-regular. Take $p$ with $1 \leqslant p \leqslant \infty$. Then $T$ is $p$–multi-bounded with*

$$\|T\|_{p-\mathrm{mb}} \leqslant \|T'\|_r \,.$$

*Proof.* We write $\kappa T$ for $\kappa_F \circ T : E \to F''$; by Theorem 4.21, (a) $\Rightarrow$ (c), $\kappa T$ is regular.

Take $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$. Then

$$\left\| \left( \sum_{i=1}^n |Tx_i|^p \right)^{1/p} \right\|_F = \left\| \left( \sum_{i=1}^n |(\kappa T)x_i|^p \right)^{1/p} \right\|_{F''} \leqslant \left\| \left( \sum_{i=1}^n (|\kappa T|\,|x_i|)^p \right)^{1/p} \right\|_{F''} .$$

By equation (4.2.2),

$$\left\| \left( \sum_{i=1}^n (|\kappa T|\,|x_i|)^p \right)^{1/p} \right\|_{F''} \leqslant \|\,|\kappa T|\,\| \left\| \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_E ,$$

and so

$$\left\| \left( \sum_{i=1}^n |Tx_i|^p \right)^{1/p} \right\|_F \leqslant \|\,|\kappa T|\,\| \left\| \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_E .$$

In terms of the canonical lattice $p$–multi-norms, this says that

$$\|(Tx_1, \dots, Tx_n)\|_n^{L,p} \leqslant \|\kappa T\|_r \, \|(x_1, \dots, x_n)\|_n^{L,p} .$$

By Theorem 4.21, $\|\kappa T\|_r = \|T'\|_r$, and so the result follows. ∎

Thus, in the above setting, we have

$$\mathcal{B}(E,F)^+ \subset \mathcal{B}_r(E,F) \subset \mathcal{B}_b(E,F) \subset \mathcal{B}_{pr}(E,F) \subset \mathcal{M}_p(E,F) \subset \mathcal{B}(E,F)$$

for each $p$ with $1 \leqslant p \leqslant \infty$.

We shall now show that, in the case where $p = 1$ or $p = \infty$, the converse of Theorem 4.39 holds, in the sense that each $p$–multi-bounded operator is pre-regular, and, further, that $\|T\|_{p-\mathrm{mb}} = \|T'\|_r$ for such operators $T$. However Example 4.44 will show that there are 2–multi-bounded operators on certain Banach lattices that are not pre-regular and that there are pre-regular operators $T$ such that $\|T\|_{2-\mathrm{mb}} \neq \|T'\|_r$.

THEOREM 4.40. *Let $E$ and $F$ be Banach lattices, and suppose that $T \in \mathcal{B}(E,F)$. Then the following conditions on $T$ are equivalent:*

(a) *$T$ is $\infty$–multi-bounded;*

(b) *$T$ is $1$–multi-bounded;*

(c) *$T$ is pre-regular.*

*Further, in this case, $\|T\|_{\infty-\mathrm{mb}} = \|T\|_{1-\mathrm{mb}} = \|T'\|_r$.*

*Proof.* Suppose that $T$ satisfies (c). Then, by Theorem 4.39, $T$ satisfies (a) and (b), and $\|T\|_{\infty-\mathrm{mb}} \leqslant \|T'\|_r$ and $\|T\|_{1-\mathrm{mb}} \leqslant \|T'\|_r$.

Suppose that $T$ satisfies (a). Then $\kappa T$ is $\infty$–multi-bounded, again writing $\kappa T$ for $\kappa_F \circ T : E \to F''$, and so, by Corollary 4.38, $\kappa T$ is regular. By the implication (c) $\Rightarrow$ (a) of Theorem 4.21, $T$ is pre-regular, and so $T$ satisfies (c).

Suppose that $T$ satisfies (b). Then, by Proposition 3.4, $T' : F' \to E'$ is $\infty$–multi-bounded, and so, by Corollary 4.38, $T'$ is regular. Hence $T$ satisfies (c).

Thus (a), (b), and (c) are equivalent.

To establish the equality of the three norms in the case where (a), (b), and (c) are satisfied, fix $x \in E^+$, and set

$$A = \left\{ \bigvee_{i=1}^{n} |Tx_i| : x_1, \dots, x_n \in E^+ \cap \Delta_x,\, n \in \mathbb{N} \right\} .$$

Then we can regard $A$ as an increasing net in both $F^+$ and $(F'')^+$; also

$$\|a\| \leqslant \|T\|_{\infty-\mathrm{mb}} \|x\| \quad (a \in A)\,,$$

and so $A$ has a supremum, say $\Lambda$, in $F''$ with

$$\|\Lambda\| = \sup\{\|a\| : a \in A\} \leqslant \|T\|_{\infty-\mathrm{mb}} \|x\| \,.$$

It follows that

$$|\kappa T|\,(x) = \sup\{|\kappa T(z)| : |z| \leqslant x\} \leqslant \Lambda\,,$$

and so

$$\|\,|\kappa T|\,(x)\| \leqslant \|\Lambda\| \leqslant \|T\|_{\infty-\mathrm{mb}} \|x\| \,,$$

whence $\|\,|\kappa T|\| \leqslant \|T\|_{\infty-\mathrm{mb}}$. By Theorem 4.21, $\|T'\|_r = \|\kappa T\|_r$, and so $\|T'\|_r \leqslant \|T\|_{\infty-\mathrm{mb}}$. Finally, we have

$$\|T\|_{1-\mathrm{mb}} = \|T'\|_{\infty-\mathrm{mb}} = \|T''\|_r = \|T'\|_r \,,$$

again by Theorem 4.21. Thus $\|T\|_{\infty-\mathrm{mb}} = \|T\|_{1-\mathrm{mb}} = \|T'\|_r$. ∎

COROLLARY 4.41. *Let $E$ and $F$ be Banach lattices. Then $\mathcal{M}_\infty(E,F) = \mathcal{B}(E,F)$ if and only if $T' \in \mathcal{B}_r(F',E')$ for each $T \in \mathcal{B}(E,F)$.* ∎

In [45], Banach lattices $E$ such that $\mathcal{M}_\infty(E) = \mathcal{B}(E)$ are said to be *homogènes*; by [45, Corollaire 4.2], they are characterized as being the lattices that are lattice isomorphic to either $AL$- or $AM$-spaces. (Here we are using [43, Theorem 1.b.12] and [46, 2.1.12] to see that the definitions of $AL$- and $AM$-spaces in [45] coincide with the Banach lattices that are lattice isomorphic to $AL$- or $AM$-spaces, in our terminology.)

THEOREM 4.42. *Let $E$ and $F$ be Banach lattices, and suppose that $T \in \mathcal{B}(E,F)$. Then $T$ is 2–multi-bounded, and $\|T\|_{2-\mathrm{mb}} \leqslant K_G \|T\|$.*

*Proof.* This follows from Krivine's theorem, Theorem 4.11. ∎

We summarize the above results in the following theorem; it follows from Theorems 4.39, 4.40, and 4.42, and from a remark on page 84.

THEOREM 4.43. *Let $E$ and $F$ be Banach lattices, and take $p$ with $1 < p < \infty$. Then*

$$\mathcal{B}_b(E,F) \subset \mathcal{B}_{pr}(E,F) = \mathcal{M}_1(E,F) = \mathcal{M}_\infty(E,F) \subset \mathcal{M}_p(E,F) \subset \mathcal{M}_2(E,F) = \mathcal{B}(E,F)\,.$$

*In the case where $E$ and $F$ are AL-spaces and $1 \leqslant p \leqslant \infty$, we have*

$$\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F) = \mathcal{M}_p(E,F) = \mathcal{B}(E,F)\,. \ ∎$$

EXAMPLE 4.44. We *claim* that there is reflexive Banach lattice $E$ with $\mathcal{B}_{pr}(E) \subsetneq \mathcal{B}(E)$. Indeed, take $E = \ell^p$, where $1 < p < \infty$, and assume towards a contradiction that each 2–multi-bounded operator in $\mathcal{B}(E)$ is pre-regular. Then each dual operator in $\mathcal{B}(E')$ is regular, and so $\mathcal{B}_r(E') = \mathcal{B}(E')$. But, as noted above, it is shown in [4] that $\mathcal{B}_r(E')$ is not even dense in $\mathcal{B}(E')$. Since $\|\cdot\|_{2-\mathrm{mb}}$ is equivalent to $\|\cdot\|$, it also follows from [4] that $\|\cdot\|_r$ is not equivalent to $\|\cdot\|_{2-\mathrm{mb}}$ on $\mathcal{B}_r(E)$. ∎

THEOREM 4.45. *Let $E$ and $F$ be Banach lattices, and take $p_1, p_2 \in \mathbb{R}$ such that either $1 < p_1 < p_2 < 2$ or $2 < p_2 < p_1 < \infty$. Then*

$$\mathcal{M}_1(E,F) = \mathcal{M}_\infty(E,F) \subset \mathcal{M}_{p_1}(E,F) \subset \mathcal{M}_{p_2}(E,F) \subset \mathcal{M}_2(E,F) = \mathcal{B}(E,F)\,. \quad (4.5.2)$$

*Proof.* We suppose that $1 < p_1 < p_2 < 2$.

Take $T \in \mathcal{M}_{p_1}(E,F)$, say with $\|T\|_{p_1-\mathrm{mb}} \leqslant 1$. Then also $T \in \mathcal{B}(E,F) = \mathcal{M}_2(E,F)$, with $\|T\|_{2-\mathrm{mb}} \leqslant K_G$.

First, suppose that $E$ and $F$ are complex Banach lattices, and take $n \in \mathbb{N}$. By Theorem 4.36, the spaces $E(\ell_n^{p_2})$ and $F(\ell_n^{p_2})$ are isometrically isomorphic to $(E(\ell_n^{p_1}), E(\ell_n^2))_\theta$ and $(F(\ell_n^{p_1}), F(\ell_n^2))_\theta$, respectively, for a suitable choice of $\theta \in (0,1)$. Further, $T^{(n)}$ is a linear map from $E(\ell_n^{p_1}) + E(\ell_n^2)$ to $F(\ell_n^{p_1}) + F(\ell_n^2)$ such that $T^{(n)} : E(\ell_n^{p_1}) \to F(\ell_n^{p_1})$ is bounded with norm at most 1 and $T^{(n)} : E(\ell_n^2) \to F(\ell_n^2)$ is bounded with norm at most $K_G$. By Theorem 1.46, $T^{(n)}$ is a bounded linear map from $E(\ell_n^{p_2})$ to $F(\ell_n^{p_2})$ with norm at most $K_G^\theta$, a bound independent of $n$. It follows that $T \in \mathcal{M}_{p_2}(E,F)$, and so equation (4.5.2) holds.

Next, suppose that $E$ and $F$ are real Banach lattices, and again take $n \in \mathbb{N}$. For an arbitrary $p$ with $1 \leqslant p \leqslant \infty$, we again write $\ell_n^p(\mathbb{R})$ and $\ell_n^p(\mathbb{C})$ for the appropriate spaces taken over real and complex scalars, respectively. It is easy to see that the complexification $E(\ell_n^p(\mathbb{R})) \oplus \mathrm{i}E(\ell_n^p(\mathbb{R}))$ of $E(\ell_n^p(\mathbb{R}))$ may be identified with $(E \oplus \mathrm{i}E)(\ell_n^p(\mathbb{C}))$, and that this identification is isometric. Using this identification, we may also identify the $n^{\mathrm{th}}$ amplification $(T_\mathbb{C})^{(n)}$ of the complexification $T_\mathbb{C}$ with the complexification of $T^{(n)}$, namely with

$$(T^{(n)})_\mathbb{C} \colon E(\ell_n^p(\mathbb{R})) \oplus \mathrm{i}E(\ell_n^p(\mathbb{R})) \to F(\ell_n^p(\mathbb{R})) \oplus \mathrm{i}F(\ell_n^p(\mathbb{R}))\,.$$

In particular, the two operators have the same norms, and so

$$\left\|(T_{\mathbb{C}})^{(n)}\colon (E\oplus \mathrm{i}E)(\ell_n^{p_1}(\mathbb{C}))\to (F\oplus \mathrm{i}F)(\ell_n^{p_1}(\mathbb{C}))\right\|$$

is equal to

$$\left\|(T^{(n)})_{\mathbb{C}}\colon E(\ell_n^{p_1}(\mathbb{R}))\oplus \mathrm{i}E(\ell_n^{p_1}(\mathbb{R}))\to F(\ell_n^{p_1}(\mathbb{R}))\oplus \mathrm{i}F(\ell_n^{p_1}(\mathbb{R}))\right\|.$$

The latter norm is bounded by $2\left\|T^{(n)}\colon E(\ell_n^{p_1}(\mathbb{R}))\to F(\ell_n^{p_1}(\mathbb{R}))\right\|\leqslant 2$; this is because $\|T_{\mathbb{C}}\|\leqslant 2\,\|T\|$ and $\|T\|_{p_1-\mathrm{mb}}\leqslant 1$. It follows from the first part of the proof that

$$\begin{aligned}&\left\|T^{(n)}\colon E(\ell_n^{p_2}(\mathbb{R}))\to F(\ell_n^{p_2}(\mathbb{R}))\right\|\\ &\quad\leqslant \left\|(T^{(n)})_{\mathbb{C}}\colon E(\ell_n^{p_2}(\mathbb{R}))\oplus \mathrm{i}E(\ell_n^{p_2}(\mathbb{R}))\to F(\ell_n^{p_2}(\mathbb{R}))\oplus \mathrm{i}F(\ell_n^{p_2}(\mathbb{R}))\right\|\\ &\quad= \left\|(T_{\mathbb{C}})^{(n)}\colon (E\oplus \mathrm{i}E)(\ell_n^{p_2}(\mathbb{C}))\to (F\oplus \mathrm{i}F)(\ell_n^{p_2}(\mathbb{C}))\right\|\leqslant 2K_G^{\theta},\end{aligned}$$

and hence $\|T\|_{p_2-\mathrm{mb}}\leqslant 2K_G^{\theta}$. Thus the result follows in this real case.

The case where $2<p_2<p_1<\infty$ is similar. ∎

The following example leads to the determination of $\mathcal{M}_p(E,F)$ in some cases.

EXAMPLE 4.46. Let $E$ and $F$ be Banach lattices, and take $p$ with $1<p<\infty$ and $n\in\mathbb{N}$. As before the space $E^n$ with the canonical lattice $p$–multi-norm $\|\cdot\|_n^{L,p}$ is denoted by $E(\ell_n^p)$ and the space $E^n$ with the $p$–sum power-norm is denoted by $\ell_n^p(E)$. (We recall that the $p$–sum power-norm is always a power-norm, and that it is a $p$–multi-norm for certain Banach spaces $E$.) Thus we may consider the space of $p$–multi-bounded operators from $E$ to $F$ with respect to these power-norms.

Specifically consider two operators $S\in\mathcal{B}(E,F)$ with $S^{(n)}:\ell_n^p(E)\to F(\ell_n^p)$ and $T\in\mathcal{B}(E,F)$ with $T^{(n)}:E(\ell_n^p)\to\ell_n^p(F)$. By the definitions given in [43, Definition 1.d.3]: $S$ is $p$–multi-bounded if and only if $S$ is a *$p$–convex* operator, and the $p$–multi-bounded norm of $S$ is $M^{(p)}(S)$; $T$ is $p$–multi-bounded if and only if $T$ is a *$p$–concave* operator, and the $p$–multi-bounded norm of $T$ is $M_{(p)}(T)$. The Banach lattice $E$ is $p$–convex or $p$–concave if the identity operator on $E$ is $p$–convex or $p$–concave, respectively. Thus the canonical lattice $p$–multi-norm and the $p$–sum power-norm based on $E$ are equivalent if and only if $E$ is $p$–convex and $p$–concave; this holds for the spaces $L^p(\Omega)$ for a measure space $\Omega$.

By Theorem 4.28, $(E(\ell_n^p))'=E'(\ell_n^{p'})$, and so it follows from our Proposition 3.4 that an operator $T$ between two Banach lattices is $p$–convex if and only if $T'$ is $p'$–concave and that $T$ is $p$–concave if and only if $T'$ is $p'$–convex, as in [43, Proposition 1.d.4]. ∎

PROPOSITION 4.47. *Let $\Omega$ and $\Sigma$ be measure spaces, and suppose that $1\leqslant r\leqslant p\leqslant s\leqslant\infty$. Then*

$$\mathcal{M}_p(L^r(\Omega),L^s(\Sigma))=\mathcal{B}(L^r(\Omega),L^s(\Sigma))\,, \tag{4.5.3}$$

*with equality of norms.*

*Proof.* Set $E = L^r(\Omega)$ and $F = L^s(\Sigma)$, and take $T \in \mathcal{B}(E, F)$. The Banach lattice $F$ is $p$–convex with constant 1, and so

$$\left\| \left( \sum_{i=1}^n |Tf_i|^p \right)^{1/p} \right\| \leqslant \left( \sum_{i=1}^n \|Tf_i\|^p \right)^{1/p} \qquad (f_1, \dots, f_n \in E,\, n \in \mathbb{N})\,.$$

The Banach lattice $E$ is $p$–concave with constant 1, and so

$$\left( \sum_{i=1}^n \|f_i\|^p \right)^{1/p} \leqslant \left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\| \qquad (f_1, \dots, f_n \in E,\, n \in \mathbb{N})\,.$$

It follows that $T \in \mathcal{M}_p(E, F)$ with $\|T\|_{p-\mathrm{mb}} \leqslant \|T\|$. Since the inequality $\|T\| \leqslant \|T\|_{p-\mathrm{mb}}$ always holds, we obtain equality of norms in (4.5.3). ∎

Corollary 4.48. *Let $\Omega$ and $\Sigma$ be measure spaces, and take $r, s$ with $1 \leqslant r \leqslant s \leqslant 2$ or $2 \leqslant r \leqslant s \leqslant \infty$. Then*

$$\mathcal{M}_p(L^r(\Omega), L^s(\Sigma)) = \mathcal{B}(L^r(\Omega), L^s(\Sigma)) \tag{4.5.4}$$

*for each $p \in [r, 2]$ or each $p \in [2, s]$, respectively.*

*Proof.* First suppose that $1 \leqslant r \leqslant s \leqslant 2$ and that $p = r$. Then equation (4.5.4) holds by Proposition 4.47. Thus (4.5.4) holds for each $p \in [r, 2]$ by Theorem 4.45. The case where $2 \leqslant r \leqslant s \leqslant \infty$ is similar. ∎

The following result essentially contains a converse to Corollary 4.48 in a special case.

Proposition 4.49. *Take $r$ with $1 < r < \infty$.*

(i) *Suppose that $1 \leqslant p < 2$. Then $\mathcal{M}_p(\ell^r) = \mathcal{B}(\ell^r)$ if and only if $1 < r \leqslant p$.*

(ii) *Suppose that $2 < p \leqslant \infty$. Then $\mathcal{M}_p(\ell^r) = \mathcal{B}(\ell^r)$ if and only if $r \geqslant p$.*

*Proof.* The facts that $\mathcal{M}_p(\ell^r) = \mathcal{B}(\ell^r)$ for $p \in [r, 2]$, and hence for $r \in (1, p]$, when $1 \leqslant p < 2$, and for $p \in [2, r]$, and hence for $r \geqslant p$, when $2 < p \leqslant \infty$ are special cases of Corollary 4.48. We must show that these are the only cases for which $\mathcal{M}_p(\ell^r) = \mathcal{B}(\ell^r)$.

In the case where $p = 1$, it follows from Theorem 4.40 that $\mathcal{M}_1(\ell^r) = \mathcal{B}_{pr}(\ell^r)$ for each $r \in (1, \infty]$. Further, $\mathcal{B}_{pr}(\ell^r) = \mathcal{B}_r(\ell^r)$ for each $r \in (1, \infty)$, and we have noted that $\mathcal{B}_r(\ell^r)$ is not even dense in $\mathcal{B}(\ell^r)$. Thus $\mathcal{M}_1(\ell^r) \neq \mathcal{B}(\ell^r)$.

Now suppose that $1 < p < \infty$, that $1 < r < \infty$, and that $\mathcal{M}_p(\ell^r) = \mathcal{B}(\ell^r)$. Thus there exists $C \geqslant 1$ such that

$$\|T\|_{p-\mathrm{mb}} \leqslant C\, \|T\| \quad (T \in \mathcal{B}(\ell^r))\,. \tag{4.5.5}$$

Take $m, n \in \mathbb{N}$. As in Example 4.25, we see that there is an isometric isomorphism from $((\ell^r_m)^n, \|\cdot\|^{L,p}_n)$ onto $(\ell^r_m(\ell^p_n), \|\cdot\|_{\ell^r_m(\ell^p_n)})$ formed by 'taking transposes'. Now take $T \in \mathcal{B}(\ell^r_m)$ and regard $T$ as an $m \times m$ matrix $(T_{j,k})$ and as an element of $\mathcal{B}(\ell^r)$, and take

$\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell^{\,r}_m)^{\,n}$, where $x_i = (x_{i,k} : k \in \mathbb{N}_m)$ for $i \in \mathbb{N}_n$. Then

$$\left\| T^{(n)}\boldsymbol{x} \right\|_n^{L,p} = \left\| \left( \left( \sum_{k=1}^m T_{j,k} x_{i,k} : i \in \mathbb{N}_n \right) : j \in \mathbb{N}_m \right) \right\|_{\ell^{\,r}_m(\ell^{\,p}_n)}$$
$$= \left( \sum_{j=1}^m \left( \sum_{i=1}^n \left| \sum_{k=1}^m T_{j,k} x_{i,k} \right|^{\,p} \right)^{r/p} \right)^{1/r} .$$

On the other hand,

$$\|\boldsymbol{x}\|_n^{L,p} = \left( \sum_{j=1}^m \left( \sum_{i=1}^n |x_{i,j}|^{\,p} \right)^{r/p} \right)^{1/r} .$$

Thus equation (4.5.5) implies that

$$\|T : \ell^{\,r}_m(E) \to \ell^{\,r}_m(E)\| \leqslant C \, \|T : \ell^{\,r}_m \to \ell^{\,r}_m\|$$

for each $T \in \mathcal{B}(\ell^{\,r}_m)$ and $m \in \mathbb{N}$, where $E = \ell^{\,p}_n$. It follows from Theorem 1.43, (d) $\Rightarrow$ (a), (with $r$ replacing $p$ in the notation) that $\ell^{\,p}_n$ is $C$–isomorphic to an $r$–space for each $n \in \mathbb{N}$. By the final claim of Corollary 1.44, $r \in [p, 2]$ when $1 < p < 2$ and $r \in [2, p]$ when $2 < p < \infty$, as required. ∎

Corollary 4.50. *Take $p_1, p_2$ such that $1 \leqslant p_1, p_2 \leqslant \infty$. Then the inclusion*

$$\mathcal{M}_{p_1}(E) \subset \mathcal{M}_{p_2}(E)$$

*holds for every Banach lattice $E$ in each of the following three cases:*

(i) $p_1 \in \{1, \infty\}$*;*

(ii) $1 \leqslant p_1 \leqslant p_2 \leqslant 2$*;*

(iii) $2 \leqslant p_2 \leqslant p_1 \leqslant \infty$*.*

*For all other pairs $\{p_1, p_2\}$, there is a Banach lattice $E$ such that $\mathcal{M}_{p_1}(E) \not\subset \mathcal{M}_{p_2}(E)$.*

*Proof.* The proof of the inclusions $\mathcal{M}_{p_1}(E) \subset \mathcal{M}_{p_2}(E)$ in the specified cases follows from Theorem 4.45. To show that the inclusion fails in all other cases, take $E$ to be the Banach lattice $\ell^{\,p_1}$. ∎

## 5. Representation theorems

We now seek canonical representation theorems for certain $p$–multi-normed spaces.

**5.1. Representations as subspaces of lattices.** Let $E$ be a Banach space. The memoir [45] contains a representation theorem for spaces $c_0 \otimes E$ satisfying property (P), which was defined on page 51, and hence gives a representation theorem for multi-normed spaces, in terms of closed subspaces of Banach lattices, or as 'sous-espaces de treillis'; the theorem is [45, Théorème 2.1], where the result and proof are attributed to Pisier. The theorem is also stated as [20, Theorem 4.56]. We now give a simpler and shorter version of this proof in the language of multi-norms; further, we shall generalize the result to apply to certain $p$–multi-norms.

After the relevant part of this memoir was completed, we discovered that a different proof of Pisier's representation theorem was given by Casazza and Nielsen in [12, Theorem 1.7]; this proof uses ultraproducts and is also different from our proof. Further, a proof of our Theorem 5.5 (in a different language) is contained in the thesis [44] of McClaran; again, the proof is different from ours. We are grateful to Professor W. B. Johnson for discussing this thesis with us.

We commence by setting the scene for the results.

Let $(E, \|\cdot\|)$ be a normed space. We write $K$ for the closed unit ball $B_{E'}$ of $E'$, so that $K$ is a compact space with respect to the relative weak* topology $\sigma(E', E)$, and the space $(C(K), \|\cdot\|_\infty)$ is a Banach lattice. As before, to every element $x \in E$ one can associate the element $\widehat{x}$ in $E''$ defined by $\widehat{x}(\lambda) = \langle x, \lambda \rangle$ $(\lambda \in E')$; with a slight abuse of notation, we also denote the restriction of $\widehat{x}$ to $K$ by $\widehat{x}$, so that we are considering $\widehat{x}$ as an element of $C(K)$. The map

$$x \mapsto \widehat{x}\,, \quad (E, \|\cdot\|) \to (C(K), \|\cdot\|_\infty)\,,$$

is a linear isometry. Throughout this section $V$ denotes the order-ideal in $C(K)$ generated by (the image of) $E$. Thus, for each $f \in C(K)$, we have $f \in V$ if and only if $|f| \leqslant \bigvee_{i=1}^n |\widehat{x}_i|$ holds in $C(K)$ for some $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$.

Let $E$ be a normed space, and fix $p$ with $1 \leqslant p \leqslant \infty$. We shall be especially interested in functions on $K$ of the form

$$f_{\boldsymbol{x}} := \left( \sum_{i=1}^n |\widehat{x}_i|^p \right)^{1/p} \quad \text{for} \quad \boldsymbol{x} = (x_1, \dots, x_n) \in E^n\,,$$

where $n \in \mathbb{N}$; here we interpret $f_{\boldsymbol{x}}$ as $\max\{|\widehat{x}_1|, \dots, |\widehat{x}_n|\}$ in the case where $p = \infty$.

Since the lattice operations in $C(K)$ are defined pointwise, we have

$$f_{\boldsymbol{x}}(\lambda) = \left( \sum_{i=1}^{n} |\langle x_i,\, \lambda\rangle|^p \right)^{1/p} \qquad (\lambda \in K)\,, \tag{5.1.1}$$

and so $f_{\boldsymbol{x}} \in C(K)^+$. We note that $f_{\boldsymbol{x}}$ depends on $p$, although this is not shown explicitly in the notation. Take $\alpha \in \mathbb{F}$, $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$. Then $f_{\alpha\boldsymbol{x}} = |\alpha|\, f_{\boldsymbol{x}}$ and

$$f_{(\boldsymbol{x},\boldsymbol{y})} = \left(f_{\boldsymbol{x}}^p + f_{\boldsymbol{y}}^p\right)^{1/p} \leqslant f_{\boldsymbol{x}} + f_{\boldsymbol{y}}\,. \tag{5.1.2}$$

Further, $f_{\boldsymbol{y}} \leqslant f_{\boldsymbol{x}}$ in $(C(K)^+, \leqslant)$ if and only if $\boldsymbol{y} \preccurlyeq_p \boldsymbol{x}$ (in the notation of Definition 1.37), and so, in the particular case that $(E^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space, $\|\boldsymbol{y}\|_n \leqslant \|\boldsymbol{x}\|_m$ whenever $f_{\boldsymbol{y}} \leqslant f_{\boldsymbol{x}}$ in $(C(K)^+, \leqslant)$. (Indeed, this fact motivated us to formulate the definition of a strong $p$–multi-norm.)

Take $n \in \mathbb{N}$. There are constants $C_1$ and $C_2$ (depending on $n$) such that

$$C_1 \sum_{i=1}^{n} |f_i| \leqslant \left( \sum_{i=1}^{n} |f_i|^p \right)^{1/p} \leqslant C_2 \bigvee_{i=1}^{n} |f_i|$$

for $f_1, \dots, f_n \in C(K)$, and so $f \in V$ if and only if $|f| \leqslant f_{\boldsymbol{x}}$ for some $n \in \mathbb{N}$ and $\boldsymbol{x} \in E^n$.

Definition 5.1. Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. For each $f \in V$, set

$$\rho_p(f) = \inf\left\{\|\boldsymbol{x}\|_n : |f| \leqslant f_{\boldsymbol{x}} \ \text{ for some } \boldsymbol{x} \in E^n \text{ and } n \in \mathbb{N}\right\}\,. \tag{5.1.3}$$

Thus $\rho_p(f) \in \mathbb{R}^+$ for each $f \in V$. The first lemma is immediate.

Lemma 5.2. *Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Then:*

(i) $\rho_p(\alpha f) = |\alpha|\, \rho_p(f)$ $(\alpha \in \mathbb{F},\ f \in V)$;

(ii) $\rho_p(|f|) = \rho_p(f)$ $(f \in V)$;

(iii) $\rho_p(f) \leqslant \rho_p(g)$ *whenever* $f, g \in V$ *with* $|f| \leqslant |g|$ *in* $C(K)^+$;

(iv) $\rho_p(f_{\boldsymbol{x}}) \leqslant \|\boldsymbol{x}\|_m$ *whenever* $m \in \mathbb{N}$ *and* $\boldsymbol{x} \in E^m$. ∎

Lemma 5.3. *Let $(E^n, \|\cdot\|_n)$ be a strong $p$–multi-normed space, where $1 \leqslant p \leqslant \infty$. Then*

$$\rho_p(f_{\boldsymbol{x}}) = \|\boldsymbol{x}\|_m \ \ (\boldsymbol{x} \in E^m,\ m \in \mathbb{N})\,.$$

*Proof.* Take $m \in \mathbb{N}$ and $\boldsymbol{x} \in E^m$. By Lemma 5.2(iv), $\rho_p(f_{\boldsymbol{x}}) \leqslant \|\boldsymbol{x}\|_m$. Now suppose that $\boldsymbol{y} \in E^n$, where $n \in \mathbb{N}$, and that $|f_{\boldsymbol{x}}| \leqslant f_{\boldsymbol{y}}$. Then $\|\boldsymbol{y}\|_n \geqslant \|\boldsymbol{x}\|_m$, and so $\rho_p(f_{\boldsymbol{x}}) = \|\boldsymbol{x}\|_m$. ∎

Suppose that $(E^n, \|\cdot\|_n)$ is a power-normed space and that $1 \leqslant p \leqslant \infty$. In addition, assume that $\rho_p$ is subadditive, so that

$$\rho_p(f+g) \leqslant \rho_p(f) + \rho_p(g) \quad (f, g \in V)\,. \tag{5.1.4}$$

Then $\rho_p$ is a lattice semi-norm on $V$, and so $\ker \rho_p = \{f \in V : \rho_p(f) = 0\}$ is an order-ideal in $V$ and $V/\ker \rho_p$ is a normed lattice with respect to the norm induced by $\rho_p$. Let $X$ be the completion of this normed space, so that $X$ is a Banach lattice, and define

$$J : x \mapsto \widehat{x} + \ker \rho\,, \quad E \to X\,.$$

Then $J$ is a linear map and $(\sum_{i=1}^{n} |Jx_i|^p)^{1/p} = f_{\boldsymbol{x}} + \ker \rho$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

As in Definition 4.22, we write $(\|\cdot\|_n^{L,p})$ for the canonical lattice $p$–multi-norm based on the Banach lattice $X$, and we suppose throughout that this is the $p$–multi-norm that is based on $X$.

LEMMA 5.4. *Let $(E^n, \|\cdot\|_n)$ be a power-normed space, and take $p$ with $1 \leqslant p \leqslant \infty$. Further, assume that $\rho_p$ is subadditive. Then $J : E \to X$ is a multi-contraction. In the case where $(E^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space, $J : E \to X$ is a multi-isometry.*

*Proof.* Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$. Then

$$\left\| J^{(n)}\boldsymbol{x} \right\|_n^{L,p} = \left\| \left( \sum_{i=1}^{n} |Jx_i|^p \right)^{1/p} \right\| = \|f_{\boldsymbol{x}} + \ker \rho\| = \rho_p(f_{\boldsymbol{x}}) \leqslant \|\boldsymbol{x}\|_n \tag{5.1.5}$$

by Lemma 5.2(iv). Thus $J : E \to X$ is a multi-contraction. In the case where $(E^n, \|\cdot\|_n)$ is a strong $p$–multi-normed space, $\rho_p(f_{\boldsymbol{x}}) = \|\boldsymbol{x}\|_n$ by Lemma 5.3, and so equation (5.1.5) shows that $J : E \to X$ is a multi-isometry. ∎

Clearly the point to be resolved before we can claim a satisfactory representation theorem is when the above map $\rho_p$ is subadditive. We shall first show that this is certainly the case when we are considering multi-norms themselves, so recovering the theorem of Pisier.

THEOREM 5.5. *Let $(E^n, \|\cdot\|_n)$ be a multi-Banach space. Then there are a Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^L)$.*

*Proof.* The multi-norm $(\|\cdot\|_n)$ is a strong multi-norm by Theorem 2.25. As we remarked, it suffices to show that the function $\rho = \rho_\infty$ defined above (in the case where $p = \infty$) is subadditive.

Take $f, g \in V$, and fix $\varepsilon > 0$. Then we can find $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$ such that

$$|f| \leqslant f_{\boldsymbol{x}}, \quad |g| \leqslant f_{\boldsymbol{y}}, \quad \|\boldsymbol{x}\|_m \leqslant \rho(f) + \varepsilon, \quad \text{and} \quad \|\boldsymbol{y}\|_n \leqslant \rho(g) + \varepsilon\,.$$

Set $G = \ell_m^\infty \oplus_1 \ell_n^\infty$, so that

$$\|(u,v)\| = \|u\|_{\ell_m^\infty} + \|v\|_{\ell_n^\infty} \quad (u \in \ell_m^\infty\,, v \in \ell_n^\infty)\,.$$

By Proposition 1.9, there exist $k \in \mathbb{N}$ and a linear embedding $T : G \to \ell_k^\infty$ such that

$$\|(u,v)\| \leqslant \|T(u,v)\| \leqslant (1+\varepsilon)\,\|(u,v)\| \quad (u \in \ell_m^\infty\,, v \in \ell_n^\infty)\,. \tag{5.1.6}$$

Clearly there are linear mappings $A : \ell_m^\infty \to \ell_k^\infty$ and $B : \ell_n^\infty \to \ell_k^\infty$ for which

$$\max\{\|A\|\,, \|B\|\} \leqslant \|T\| \leqslant 1 + \varepsilon$$

such that each element $T(u,v)$ can be written in the form $Au + Bv$.

We can regard $T$ as a matrix, and hence as a linear map from $E^m \times E^n$ to $E^k$. Similarly, we can regard $A$ and $B$ as linear maps from $E^m$ and $E^n$, respectively, to $E^k$.

Define $\boldsymbol{z} = T(\boldsymbol{x}, \boldsymbol{y}) \in E^k$, say $\boldsymbol{z} = (z_1, \dots, z_k)$, and take $\lambda \in E'$. Then it follows from (1.4.3) that

$$\langle \boldsymbol{z},\, \lambda \rangle = \lambda^{(k)}(T(\boldsymbol{x}, \boldsymbol{y})) = T(\lambda^{(m+n)}(\boldsymbol{x}, \boldsymbol{y})) = T(\lambda^{(m)}(\boldsymbol{x}), \lambda^{(n)}(\boldsymbol{y}))\,.$$

Combining this with (5.1.6), we obtain

$$\begin{aligned} f_{\boldsymbol{z}}(\lambda) &= \max_{i=1,\dots,k} |\langle z_i, \lambda\rangle| = \|\langle \boldsymbol{z}, \lambda\rangle\|_{\ell_k^\infty} = \left\|T(\lambda^{(m)}(\boldsymbol{x}), \lambda^{(n)}(\boldsymbol{y}))\right\|_{\ell_k^\infty} \\ &\geqslant \left\|(\lambda^{(m)}(\boldsymbol{x}), \lambda^{(n)}(\boldsymbol{y}))\right\|_G = \left\|\lambda^{(m)}(\boldsymbol{x})\right\|_{\ell_m^\infty} + \left\|\lambda^{(n)}(\boldsymbol{y})\right\|_{\ell_n^\infty} = f_{\boldsymbol{x}}(\lambda) + f_{\boldsymbol{y}}(\lambda)\,, \end{aligned}$$

and hence $f_{\boldsymbol{z}} \geqslant f_{\boldsymbol{x}} + f_{\boldsymbol{y}} \geqslant |f| + |g| \geqslant |f+g|$. This shows that

$$\begin{aligned} \rho(f+g) &\leqslant \|\boldsymbol{z}\|_k = \|A\boldsymbol{x} + B\boldsymbol{y}\|_k \leqslant \|A\boldsymbol{x}\|_k + \|B\boldsymbol{y}\|_k \\ &\leqslant (1+\varepsilon)(\|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n) \leqslant (1+\varepsilon)(\rho(f) + \rho(g) + 2\varepsilon)\,. \end{aligned}$$

The above inequality holds true for each $\varepsilon > 0$, and so $\rho(f+g) \leqslant \rho(f) + \rho(g)$, which shows that $\rho$ is indeed subadditive.

This completes the proof of the theorem. ■

We next consider the representation of 1–multi-norms, i.e., of dual multi-norms. As we saw in Example 2.33, there are 1–multi-norms that are not strong 1–multi-norms, and so we must impose this condition on the 1–multi-norm. Indeed, since the canonical lattice 1–multi-norm $(\|\cdot\|_n^{DL}) = (\|\cdot\|_1^{L,1})$ of the following result is strong (by Theorem 4.23), the hypothesis that the 1–multi-norm based on $E$ be strong is clearly necessary for the following theorem to hold.

THEOREM 5.6. *Let $(E^n, \|\cdot\|_n)$ be a strong 1–multi-Banach space. Then there are a Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^{DL})$.*

*Proof.* Again it suffices to show that the function $\rho = \rho_1$ defined above (in the case where $p = 1$) is subadditive.

Take $f, g \in V$, and fix $\varepsilon > 0$. Then we can find $m, n \in \mathbb{N}$ , $\boldsymbol{x} \in E^m$ and $\boldsymbol{y} \in E^n$ such that

$$|f| \leqslant f_{\boldsymbol{x}}, \quad |g| \leqslant f_{\boldsymbol{y}}, \quad \|\boldsymbol{x}\|_m \leqslant \rho(f) + \varepsilon, \quad \text{and} \quad \|\boldsymbol{y}\|_n \leqslant \rho(g) + \varepsilon\,.$$

Then $|f+g| \leqslant f_{\boldsymbol{x}} + f_{\boldsymbol{y}} = f_{(\boldsymbol{x},\boldsymbol{y})}$, and so

$$\rho(f+g) \leqslant \|(\boldsymbol{x}, \boldsymbol{y})\|_{m+n} \leqslant \|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n \leqslant \rho(f) + \rho(g) + 2\varepsilon\,.$$

This holds true for each $\varepsilon > 0$, and so $\rho(f+g) \leqslant \rho(f) + \rho(g)$, as required. ■

We now seek a result that is applicable in the case where $1 < p < \infty$. In the following theorem, we impose the extra condition that the $p$–multi-norm be strong, which is certainly a necessary condition, and that the $p$–multi-norm be $p$–convex; for each $p$, this latter condition is necessary if we require that the Banach lattice be $p$–convex, for then the corresponding canonical $p$–multi-norm is $p$–convex by Theorem 4.26, and so the initial $p$–multi-norm must be $p$–convex, where we note that $p$–convexity passes to closed subspaces. In Example 5.9, we shall exhibit a strong 2–multi-Banach space (that is not 2–convex) which is not multi-isometric to $(Y^n, \|\cdot\|_n^{L,2})$ for any closed subspace $Y$ of a Banach lattice.

THEOREM 5.7. *Take $p$ with $1 < p < \infty$, and let $(E^n, \|\cdot\|_n)$ be a strong $p$–multi-Banach space that is $p$–convex. Then there are a $p$–convex Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^{L,p})$.*

*Proof.* To establish the existence of $X$ and $Y$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^{L,p})$, it suffices to show that $\rho_p$ defined above is subadditive. Set $q = p'$.

Again take $f, g \in V$, and fix $\varepsilon > 0$. Then we can find $m, n \in \mathbb{N}$, $\boldsymbol{x} \in E^m$, and $\boldsymbol{y} \in E^n$ such that

$$|f| \leqslant f_{\boldsymbol{x}}, \quad |g| \leqslant f_{\boldsymbol{y}}, \quad \|\boldsymbol{x}\|_m \leqslant \rho_p(f) + \varepsilon, \quad \text{and} \quad \|\boldsymbol{y}\|_n \leqslant \rho_p(g) + \varepsilon\,.$$

We may suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-zero. We set

$$\alpha = \left(\frac{\|\boldsymbol{x}\|_m}{\|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n}\right)^{1/q}, \quad \beta = \left(\frac{\|\boldsymbol{y}\|_n}{\|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n}\right)^{1/q},$$

so that $\alpha^q + \beta^q = 1$. By Hölder's inequality applied pointwise in $C(K)$, we have

$$|f| + |g| \leqslant \left(\frac{|f|^p}{\alpha^p} + \frac{|g|^p}{\beta^p}\right)^{1/p}.$$

Set $\boldsymbol{z} = (\boldsymbol{x}/\alpha, \boldsymbol{y}/\beta) \in E^{m+n}$, say $\boldsymbol{z} = (z_1, \dots, z_{m+n})$. Then

$$\left(\frac{|f|^p}{\alpha^p} + \frac{|g|^p}{\beta^p}\right)^{1/p} \leqslant \left(\sum_{i=1}^{m+n} |\widehat{z}_i|^p\right)^{1/p} = f_{\boldsymbol{z}}\,,$$

and so $\rho_p(f+g) \leqslant \rho_p(f_{\boldsymbol{z}}) = \|\boldsymbol{z}\|_{m+n}$. Since the multi-norm $(\|\cdot\|_n)$ is $p$–convex, we have

$$\|\boldsymbol{z}\|_{m+n} \leqslant \left(\frac{\|\boldsymbol{x}\|_m^p}{\alpha^p} + \frac{\|\boldsymbol{y}\|_n^p}{\beta^p}\right)^{1/p},$$

and the expression on the right-hand side is just $\|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n$. Therefore

$$\rho_p(f+g) \leqslant \|\boldsymbol{x}\|_m + \|\boldsymbol{y}\|_n \leqslant \rho_p(f) + \rho_p(g) + 2\varepsilon\,.$$

This holds true for each $\varepsilon > 0$, and so $\rho_p$ is indeed subadditive.

We must also show that the Banach lattice $X$ is $p$–convex. For this, take $f, g \in V$, as above. Then

$$\begin{aligned}\rho_p\left((|f|^p + |g|^p)^{1/p}\right) &\leqslant \|(\boldsymbol{x}, \boldsymbol{y})\|_{m+n} \leqslant (\|\boldsymbol{x}\|_m^p + \|\boldsymbol{y}\|_n^p)^{1/p} \\ &\leqslant ((\rho_p(f) + \varepsilon)^p + (\rho_p(g) + \varepsilon)^p)^{1/p}\,.\end{aligned}$$

This also holds true for each $\varepsilon > 0$, and so

$$\rho_p\left((|f|^p + |g|^p)^{1/p}\right) \leqslant (\rho_p(f)^p + \rho_p(g)^p)^{1/p}\,.$$

This implies that the Banach lattice $X$ is $p$–convex. ∎

Recall that a sequential norm is a 2–multi-norm that is 2–convex.

COROLLARY 5.8. *Let $E$ be a Banach space, and let $(\|\cdot\|_n)$ be a sequential norm based on $E$. Then there are a 2-convex Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^{L,2})$.*

*Proof.* By Theorem 2.25, every 2-multi-norm is a strong 2-multi-norm. ∎

EXAMPLE 5.9. First, for each $p$ with $1 < p < \infty$, we shall construct an example of a $p$–multi-normed space based on a Banach space $E$ that is not multi-isomorphic to any closed subspace of a Banach lattice with the canonical $p$–multi-norm.

Let $(E, \|\cdot\|_E)$ be a Banach space, and consider the dual weak $p$–summing norm $(\nu_{p,n})$ based on $E$, as in Example 2.7(iv); we recall from Theorem 2.11 that $(\nu_{p,n})$ is the maximum $p$–multi-norm based on $E$ and that, for $n \in \mathbb{N}$, $\nu_{p,n}$ corresponds to the projective tensor norm $\|\cdot\|_{\pi,n}$ on $\ell^p_n \widehat{\otimes} E$. Suppose that $X$ is a Banach lattice equipped with the canonical lattice $p$–multi-norm $(\|\cdot\|_n^{L,p})$ and that $T : E \to X$ is an embedding onto a closed subspace $Y$ of $X$; we may suppose that $\|T\| = 1$. Define $M_n = \left\|(T^{-1})^{(n)}\right\|$ $(n \in \mathbb{N})$.

In fact we take $E = \ell^q(\mathbb{R})$, where $q = p'$. We write $(\delta_n)$ for the standard basis in $\ell^p(\mathbb{R})$, as before, and now write $(\delta'_n)$ for the standard basis in $E$. Fix $n \in \mathbb{N}$, and set $\boldsymbol{e} = (\delta'_1, \dots, \delta'_n) \in E^n$. Then, using equation (1.5.10), we have

$$\nu_{p,n}(\boldsymbol{e}) = \left\|\sum_{i=1}^n \delta_i \otimes \delta'_i\right\|_{\pi,n} .$$

Consider $\boldsymbol{\lambda} = (\delta_1, \dots, \delta_n) \in (E')^n = (\ell^p)^n$. By equation (1.5.3), $\mu_{q,n}(\boldsymbol{\lambda}) = 1$, and so $n = \langle \boldsymbol{e}, \boldsymbol{\lambda}\rangle \leqslant \nu_{p,n}(\boldsymbol{e})$. Thus $\nu_{p,n}(\boldsymbol{e}) = n$.

Take $n \in \mathbb{N}$, and set $x_i = T\delta'_i$ $(i \in \mathbb{N}_n)$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in Y^n$, so that $T^{(n)}\boldsymbol{e} = \boldsymbol{x}$. Then

$$n = \nu_{p,n}(\boldsymbol{e}) \leqslant M_n \|\boldsymbol{x}\|_n^{L,p} = M_n \left\|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right\| . \tag{5.1.7}$$

By Proposition 4.10,

$$\frac{1}{\sqrt{2}}\left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \leqslant \frac{1}{2^n}\sum\left|\sum_{i=1}^n \varepsilon_i x_i\right| ,$$

where the outer sum on the right-hand side is taken over all choices of $\varepsilon_i = \pm 1$ for $i \in \mathbb{N}_n$. We have

$$\left\|\sum_{i=1}^n \varepsilon_i x_i\right\| \leqslant \left\|\sum_{i=1}^n \varepsilon_i \delta'_i\right\|_E = n^{1/q} ,$$

and so

$$\frac{1}{\sqrt{2}}\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| \leqslant \left\|\frac{1}{2^n}\sum\left|\sum_{i=1}^n \varepsilon_i x_i\right|\right\| \leqslant \frac{1}{2^n}\sum\left\|\sum_{i=1}^n \varepsilon_i \delta'_i\right\| = n^{1/q} .$$

Suppose that $p \geqslant 2$. Then

$$\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} ,$$

and hence, using (5.1.7), we see that

$$\frac{n}{\sqrt{2}M_n} \leqslant \frac{1}{\sqrt{2}}\left\|\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\right\| \leqslant n^{1/q} ,$$

and so $M_n \geqslant (1/\sqrt{2})n^{1/p}$ $(n \in \mathbb{N})$.

Suppose that $1 \leqslant p \leqslant 2$. Then

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leqslant n^{1/p-1/2} \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2},$$

and now

$$\frac{n}{\sqrt{2}M_n} \leqslant \frac{1}{\sqrt{2}} \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \right\| \leqslant n^{1/p-1/2+1/q} = n^{1/2},$$

and so $M_n \geqslant (n/2)^{1/2}$ $(n \in \mathbb{N})$.

In each case, $M_n \to \infty$ as $n \to \infty$, and so there is no embedding of $E$ onto a closed subspace of a Banach lattice such that the inverse is multi-bounded.

In the case where $p = 2$, the multi-norm $(\nu_{2,n})$ is a strong 2–multi-norm. This shows that the convexity condition in Corollary 5.8 is not redundant. ■

**5.2. Representations as quotients of lattices.** We now give a related representation theorem for dual multi-normed spaces and certain other $p$–multi-normed spaces. We state two theorems, but we shall give one combined proof.

THEOREM 5.10. *Let $(E^n, \|\cdot\|_n)$ be a 1–multi-Banach space. Then there are a Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to the space $((X/Y)^n, |||\cdot|||_n)$, where $(|||\cdot|||_n)$ is the 1–multi-norm based on $X/Y$ that is the quotient of the canonical lattice 1–multi-norm $(\|\cdot\|_n^{L,1}) = (\|\cdot\|_n^{DL})$ based on $X$.*

The above theorem is related to [44, Theorem 4.18].

THEOREM 5.11. *Take $p$ with $1 < p \leqslant \infty$, and let $(E^n, \|\cdot\|_n)$ be a $p$–multi-Banach space. Suppose that $(E^n, \|\cdot\|_n)$ is $p$–concave and that, for each finite-dimensional subspace $F$ of $E$, equipped with the $p$–multi-norm inherited from $(E^n, \|\cdot\|_n)$, the dual $p'$–multi-norm based on $F'$ is a strong $p'$–multi-norm. Then there are a Banach lattice $X$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to*

$$((X/Y)^n, |||\cdot|||_n),$$

*where $(|||\cdot|||_n)$ is the $p$–multi-norm based on $X/Y$ that is the quotient of the canonical $p$–multi-norm $(\|\cdot\|_n^{L,p})$ based on $X$.*

Before giving the proof, we make a preliminary remark.

The hypothesis that arises in Theorem 5.11 implies that the dual $p'$–multi-norm based on $E'$ is a strong $p'$–multi-norm. Indeed, set $q = p'$, take $m, n \in \mathbb{N}$, and suppose that $\boldsymbol{\lambda} \in (E')^m$ and $\boldsymbol{\mu} \in (E')^n$ satisfy

$$\|\langle x, \boldsymbol{\lambda}\rangle\|_{\ell_m^q} \leqslant \|\langle x, \boldsymbol{\mu}\rangle\|_{\ell_n^q} \quad (x \in E). \tag{5.2.1}$$

For each $\varepsilon > 0$, there is a unit vector $\boldsymbol{y} = (y_1, \dots, y_m)$ in $E^m$ with $|\langle \boldsymbol{y}, \boldsymbol{\lambda}\rangle| \geqslant \|\boldsymbol{\lambda}\|_m - \varepsilon$. Set $F = \operatorname{lin}\{y_1, \dots, y_m\}$, a finite-dimensional subspace of $E$. For each $x \in F$, inequality (5.2.1) holds, and so, by the hypothesis in Theorem 5.11, we have $\|\boldsymbol{\lambda} \mid F^m\|_m \leqslant \|\boldsymbol{\mu} \mid F^n\|_n$. Hence

$$\|\boldsymbol{\mu}\|_n \geqslant \|\boldsymbol{\mu} \mid F^n\|_n \geqslant \|\boldsymbol{\lambda} \mid F^m\|_m \geqslant |\langle \boldsymbol{y}, \boldsymbol{\lambda}\rangle| \geqslant \|\boldsymbol{\lambda}\|_m - \varepsilon.$$

This holds true for each $\varepsilon > 0$, and so $\|\boldsymbol{\mu}\|_n \geqslant \|\boldsymbol{\lambda}\|_m$. Thus the $p'$–multi-norm based on $E'$ is strong. Unfortunately, the converse to this statement does not hold in general; we shall show this in Example 5.13, below.

*Proof of Theorems 5.10 and 5.11.* Set $q = p'$ (with $p = 1$ and $q = \infty$ in the case of Theorem 5.10), and set

$$\mathcal{I} = \bigcup_{n\in\mathbb{N}} \{\boldsymbol{x} \in E^n : \|\boldsymbol{x}\|_n = 1\}\,.$$

For each $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{I}$, set $E_{\boldsymbol{x}} = \mathrm{lin}\,\{x_1, \dots, x_n\}$, so that $(E_{\boldsymbol{x}}, \|\cdot\|)$ is a finite-dimensional, and hence closed, subspace of $E$. As such, $E_{\boldsymbol{x}}$ inherits a $p$–multi-norm from $(E^n, \|\cdot\|_n)$; we equip $E'_{\boldsymbol{x}}$ with the dual $q$–multi-norm. (By assumption when $p > 1$, or by Theorem 2.25 when $p = 1$, this $q$–multi-norm is strong.) Then there is a multi-isometry $S_{\boldsymbol{x}}$ of $E'_{\boldsymbol{x}}$ into some Banach lattice $Y_{\boldsymbol{x}}$, equipped with its canonical lattice $q$–multi-norm. Indeed, this is immediate from Theorem 5.5 for $q = \infty$, from Theorem 5.6 for $q = 1$ (taking into account the preliminary remark), and from Theorem 5.7 and Proposition 2.41 for $q$ with $1 < q < \infty$.

Being finite-dimensional, the space $E_{\boldsymbol{x}}$ is reflexive, so that we may consider $S'_{\boldsymbol{x}}$ as an operator from $Y'_{\boldsymbol{x}}$ onto $E_{\boldsymbol{x}}$; the relevant power-norm based on $Y'_{\boldsymbol{x}}$ is the dual $p$–multi-norm which agrees with the canonical $p$–multi-norm based on the Banach lattice $Y'_{\boldsymbol{x}}$ by Theorem 4.28. Since $S^{(n)}_{\boldsymbol{x}}$ is an isometry for each $n \in \mathbb{N}$, equation (1.3.20) and Proposition 1.4(ii) imply that $(S^{(n)}_{\boldsymbol{x}})' = (S'_{\boldsymbol{x}})^{(n)}$ is an exact quotient operator, and so this operator maps the closed unit ball of $(Y'_{\boldsymbol{x}})^n$ onto the closed unit ball of $E^n_{\boldsymbol{x}}$.

Define $X$ to be the $\ell^1$-sum of the family $\{Y'_{\boldsymbol{x}} : \boldsymbol{x} \in \mathcal{I}\}$, so that $X$ is the space of functions $f : \mathcal{I} \to \bigcup_{\boldsymbol{x}\in\mathcal{I}} Y'_{\boldsymbol{x}}$ such that

$$f(\boldsymbol{x}) \in Y'_{\boldsymbol{x}} \quad (\boldsymbol{x} \in \mathcal{I}) \quad \text{and} \quad \sum_{\boldsymbol{x}\in\mathcal{I}} \|f(\boldsymbol{x})\| < \infty\,.$$

Then $X$ is a Banach lattice with respect to the pointwise-defined vector lattice operations; we equip $\{X^n : n \in \mathbb{N}\}$ with its canonical $p$–multi-norm $(\|\cdot\|_n^{L,p})$.

We shall now show that, for each $n \in \mathbb{N}$, the $n^{\text{th}}$ amplification $T^{(n)}$ of the linear mapping $T : X \to E$ that is defined by the formula

$$Tf = \sum_{\boldsymbol{x}\in\mathcal{I}} S'_{\boldsymbol{x}}(f(\boldsymbol{x})) \quad (f \in X)$$

maps the closed unit ball of $X^n$ onto the closed unit ball of $E^n$. This will clearly imply that $T^{(n)}$ maps the open unit ball of $X^n$ onto the open unit ball of $E^n$, and hence complete the proof by Proposition 3.7.

Let $n \in \mathbb{N}$. On the one hand, the following calculation for $\boldsymbol{f} = (f_1, \dots, f_n) \in X^n$

shows that $T^{(n)}$ maps the closed unit ball of $X^n$ *into* the closed unit ball of $(E^n, \|\cdot\|_n)$:

$$\begin{aligned}\left\|T^{(n)}\boldsymbol{f}\right\|_n &= \left\|\left(\sum_{\boldsymbol{x}\in\mathcal{I}} S'_{\boldsymbol{x}}\left(f_j(\boldsymbol{x})\right)\right)_{j=1}^{n}\right\|_n = \left\|\sum_{\boldsymbol{x}\in\mathcal{I}}(S'_{\boldsymbol{x}})^{(n)}\left((f_j(\boldsymbol{x}))_{j=1}^n\right)\right\|_n \\ &\leqslant \sum_{\boldsymbol{x}\in\mathcal{I}}\left\|(S'_{\boldsymbol{x}})^{(n)}\left((f_j(\boldsymbol{x}))_{j=1}^n\right)\right\|_n \leqslant \sum_{\boldsymbol{x}\in\mathcal{I}}\left\|(f_j(\boldsymbol{x}))_{j=1}^n\right\|_n^{L,p} \\ &= \sum_{\boldsymbol{x}\in\mathcal{I}}\left\|\left(\sum_{j=1}^n |f_j(\boldsymbol{x})|^p\right)^{1/p}\right\| = \left\|\left(\sum_{j=1}^n |f_j|^p\right)^{1/p}\right\| = \|\boldsymbol{f}\|_n^{L,p}\,.\end{aligned}$$

On the other hand, let $\boldsymbol{x}\in\mathcal{I}$, say $\boldsymbol{x}\in E^n$ for $n\in\mathbb{N}$. Then $\boldsymbol{x}\in E^n_{\boldsymbol{x}}$, so that $\boldsymbol{x} = (S'_{\boldsymbol{x}})^{(n)}(\boldsymbol{\lambda})$ for some unit vector $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n)\in(Y'_{\boldsymbol{x}})^n$. Moreover, since $\boldsymbol{x}\in\mathcal{I}$, we can define $\boldsymbol{f} = (f_1,\dots,f_n)\in X^n$ by setting $f_j(\boldsymbol{y}) = \lambda_j$ if $\boldsymbol{y}=\boldsymbol{x}$ and $f_j(\boldsymbol{y}) = 0$ otherwise, for $j\in\mathbb{N}_n$. Then we have

$$\|\boldsymbol{f}\|_n^{L,p} = \left\|\left(\sum_{j=1}^n |f_j|^p\right)^{1/p}\right\| = \sum_{\boldsymbol{y}\in\mathcal{I}}\left\|\left(\sum_{j=1}^n |f_j(\boldsymbol{y})|^p\right)^{1/p}\right\| = \left\|\left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}\right\| = \|\boldsymbol{\lambda}\|_n^{L,p}$$

and $T^{(n)}\boldsymbol{f} = (S'_{\boldsymbol{x}}\lambda_j)_{j=1}^n = \boldsymbol{x}$, and so $T^{(n)}$ maps the closed unit ball of $X^n$ *onto* the closed unit ball of $(E^n, \|\cdot\|_n)$.

As indicated, this completes the proof of Theorems 5.10 and 5.11. ∎

Example 5.12. This example shows that the quotient of a canonical lattice 1–multi-norm is not necessarily a strong 1–multi-norm. (We have seen a similar example of a strong $p$–multi-norm with a quotient that is not a strong $p$–multi-norm in the case where $1 < p < \infty$ and $p \neq 2$ in Example 2.39.)

Indeed, let $(E^n, \|\cdot\|_n)$ be a 1–multi-Banach space. Then, by Theorem 5.10, there are a Banach lattice $X$ with the canonical lattice 1–multi-norm $(\|\cdot\|_n^{DL})$ and a closed subspace $Y$ of $X$ such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $((X/Y)^n, |||\cdot|||_n)$. Now $(\|\cdot\|_n^{DL})$ is a strong 1–multi-norm by Theorem 4.23, but the quotient $((X/Y)^n, |||\cdot|||_n)$ is not necessarily a strong 1–multi-norm; this would imply that every 1–multi-norm is strong, and this is not true by Example 2.33. ∎

Example 5.13. Take $p$ with $1 < p \leqslant \infty$, let $(E^n, \|\cdot\|_n)$ be a $p$–concave $p$–multi-Banach space, and suppose that the dual $p'$–multi-norm based on $E'$ is a strong $p'$–multi-norm. As we remarked before the proof of Theorems 5.10 and 5.11, above, it is not in general true that this implies that the hypotheses of Theorem 5.11 are satisfied. To substantiate this remark, we shall now show that, for certain values of $p$, there exists a $p$–concave $p$–multi-norm based on a Banach space $E$ such that: (i) the dual $p'$–multi-norm based on $E'$ is strong; (ii) $E$ has a finite-dimensional subspace $F$ such that the dual $p'$–multi-norm based on $F'$ is not strong.

Indeed, take $p$ with $1 < p < 2$, set $q = p'$, and let $E$ be the Banach space $L^p(\mathbb{I})$. We consider the $p$–sum power-norm based on $E$. By Theorem 2.28, this is a strong $p$–multi-norm, and it is $p$–concave; the dual multi-norm based on $E' = L^q(\mathbb{I})$ is the $q$–sum power-norm based on $E'$, and this is also a strong $q$–multi-norm by Theorem 2.28.

Now take $r$ with $p < r < 2$, and set $s = r'$. Then it follows from Proposition 1.22 that $\ell^r$ embeds isometrically into $E$, and so, for each $n \in \mathbb{N}$, the space $E$ has a subspace $F_n$ that is isometrically isomorphic to $\ell_n^r$. We have $F_n' \cong \ell_n^s$ $(n \in \mathbb{N})$.

For $n \in \mathbb{N}$, consider the $p$–sum power-norm based on $F_n$ and the dual $q$–multi-norm, which is the $q$–sum power-norm based on $F_n' = \ell_n^s$. Since $2 < s < q$, it follows from Corollary 2.29(iii) that there exists $n \in \mathbb{N}$ such that the $q$–sum power-norm based on $F_n'$ is not strong.

Although $E$ does not satisfy the hypotheses of Theorem 5.11, it obviously does satisfy the conclusions of the theorem. ∎

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# Index of Terms

# Index of Symbols