MINIMAL VECTORS IN ARBITRARY BANACH SPACES

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Abstract. We extend the method of minimal vectors to arbitrary Banach spaces. It is proved, by a variant of the method, that certain quasinilpotent operators on arbitrary Banach spaces have hyperinvariant subspaces.

The method of *minimal vectors* was introduced by Ansari and Enflo in [AE98] in order to prove the existence of invariant subspaces for certain classes of operators on a Hilbert space. Pearcy used it in [P] to prove a version of Lomonosov's theorem. Androulakis in [A] adapted the technique to super-reflexive Banach spaces. In [CPS] the method was independently generalized to reflexive Banach spaces. There has been hope that this technique could provide a positive solution to the invariant subspace problem for these spaces. In this note we present a version of the method of minimal vectors (based on [A]) that works for arbitrary Banach spaces. In particular, it applies in the spaces where there are known examples of operators without invariant subspaces, e.g., [Enf76, Enf87, Rea84, Rea85]. This shows that the method of minimal vectors alone cannot solve the invariant subspace problem for "good" spaces.

Suppose that X is a Banach space. For simplicity, we assume that X is a real Banach space, though the results can be adapted to the complex case in a straightforward manner. In the following, $B(x_0,\varepsilon)$ stands for the closed ball of radius ε centered at x_0 while $B^\circ(x_0,\varepsilon)$ stands for the open ball, and $S(x_0,\varepsilon)$ stands for the corresponding sphere.

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Let Q be a bounded operator on X. Since we will be interested in the hyperinvariant subspaces of Q, we can assume, without loss of generality, that Q is one-to-one and has dense range, since otherwise $\ker Q$ or $\overline{\operatorname{Range} Q}$ would be hyperinvariant for Q. By $\{Q\}'$ we denote the commutant of Q.

Fix a point $x_0 \neq 0$ in X and a positive real $\varepsilon < \|x_0\|$. Let $K = Q^{-1}B(x_0,\varepsilon)$. Clearly, K is a convex closed set. Note that $0 \notin K$ and $K \neq \varnothing$ because Q has dense range. Let $d = \inf_K \|z\|$. Then $d > 0$. It is observed in [AE98, A] that if X is reflexive, then there exists $z \in K$ with $\|z\| = d$. Such a vector is called a ***minimal vector*** for x_0, ε and Q. Even without the reflexivity condition, however, one can always find $y \in K$ with $\|y\| \leqslant 2d$; such a y will be referred to as a ***2-minimal vector*** for x_0, ε and Q.

The set $K \cap B(0,d)$ is the set of all minimal vectors; in general, this set may be empty. If z is a minimal vector, since $z \in K = Q^{-1}B(x_0,\varepsilon)$ then $Qz \in B(x_0,\varepsilon)$. Since z is an element of minimal norm in K, then, in fact, $Qz \in S(x_0,\varepsilon)$. Since Q is one-to-one, we have

$$QB(0,d) \cap B(x_0,\varepsilon) = Q\big(B(0,d) \cap K\big) \subseteq S(x_0,\varepsilon).$$

It follows that $QB(0,d)$ and $B^\circ(x_0,\varepsilon)$ are two disjoint convex sets. Since one of them has nonempty interior, they can be separated by a continuous linear functional (see, e.g., [AB99, Theorem 5.5]). That is, there exists a functional f with $\|f\| = 1$ and a positive real c such that $f_{|QB(0,d)} \leqslant c$ and $f_{|B^\circ(x_0,\varepsilon)} \geqslant c$. By continuity, $f_{|B(x_0,\varepsilon)} \geqslant c$. We say that f is a ***minimal functional*** for x_0, ε, and Q.

We claim that $f(x_0) \geqslant \varepsilon$. Indeed, for every x with $\|x\| \leqslant 1$ we have $x_0 - \varepsilon x \in B(x_0,\varepsilon)$. It follows that $f(x_0 - \varepsilon x) \geqslant c$, so that $f(x_0) \geqslant c + \varepsilon f(x)$. Taking sup over all x with $\|x\| \leqslant 1$ we get $f(x_0) \geqslant c + \varepsilon\|f\| \geqslant \varepsilon$.

Observe that the hyperplane $Q^*f = c$ separates K and $B(0,d)$. Indeed, if $z \in B(0,d)$, then $(Q^*f)(z) = f(Qz) \leqslant c$, and if $z \in K$, then $Qz \in B(x_0,\varepsilon)$ so that $(Q^*f)(z) = f(Qz) \geqslant c$. For every z with $\|z\| \leqslant 1$ we have $dz \in B(0,d)$, so that $(Q^*f)(dz) \leqslant c$. It follows that $\big\|Q^*f\big\| \leqslant \frac{c}{d}$. On the other hand, for every $\delta > 0$ there exists $z \in K$ with $\|z\| \leqslant d+\delta$. Then $(Q^*f)(z) \geqslant c \geqslant \frac{c}{d+\delta}\|z\|$, whence $\big\|Q^*f\big\| \geqslant \frac{c}{d+\delta}$. It follows that $\big\|Q^*f\big\| = \frac{c}{d}$. For every $z \in K$ we have $(Q^*f)(z) \geqslant c = d\big\|Q^*f\big\|$. In particular, if y is a 2-minimal vector, then

$$(Q^*f)(y) \geqslant \tfrac{1}{2}\big\|Q^*f\big\|\|y\|. \tag{1}$$

We proceed to the main theorem.

Theorem. *Let Q be a quasinilpotent operator on a Banach space X, and suppose that there exists a closed ball B such that $0 \notin B$ and for every sequence (x_n) in B there is a subsequence (x_{n_i}) and a sequence (K_i) in $\{Q\}'$ such that $\|K_i\| \leqslant 1$ and $(K_i x_{n_i})$ converges in norm to a nonzero vector. Then Q has a hyperinvariant subspace.*

Remark. The hypothesis of the theorem is slightly weaker than the condition $(*)$ in [A], where it is required that for every $\varepsilon \in (0,1)$, there exists x_0 of norm one such that the ball $B(x_0,\varepsilon)$ satisfies the rest of the condition.

Proof. Without loss of generality, Q is one-to-one and has dense range. Let $x_0 \neq 0$ and $\varepsilon \in \big(0,\|x_0\|\big)$ be such that $B = B(x_0,\varepsilon)$. For every $n \geqslant 1$ choose a 2-minimal vector y_n and a minimal functional f_n for x_0, ε, and Q^n.

Since Q is quasinilpotent, there is a subsequence (y_{n_i}) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$. Indeed, otherwise there would exist $\delta > 0$ such that $\frac{\|y_{n-1}\|}{\|y_n\|} > \delta$ for all n, so that $\|y_1\| \geqslant \delta\|y_2\| \geqslant \ldots \geqslant \delta^n\|y_{n+1}\|$. Since $Q^n y_{n+1} \in Q^{-1}B$, we have

$$\big\|Q^n y_{n+1}\big\| \geqslant d \geqslant \frac{\|y_1\|}{2} \geqslant \frac{\delta^n}{2}\|y_{n+1}\|.$$

It follows that $\|Q^n\| \geqslant \delta^n/2$, which contradicts the quasinilpotence of Q.

Since $\|f_{n_i}\| = 1$ for all i, we can assume (by passing to a further subsequence), that (f_{n_i}) weak*-converges to some $g \in X^*$. Since $f_n(x_0) \geqslant \varepsilon$ for all n, it follows that $g(x_0) \geqslant \varepsilon$. In particular, $g \neq 0$.

Consider the sequence $(Q^{n_i-1}y_{n_i-1})_{i=1}^{\infty}$. It is contained in B, so that by passing to yet a further subsequence, if necessary, we find a sequence (K_i) in $\{Q\}'$ such that $\|K_i\| \leqslant 1$ and $K_iQ^{n_i-1}y_{n_i-1}$ converges in norm to some $w \neq 0$. Put

$$Y = \{Q\}'Qw = \bigl\{TQw \mid T \in \{Q\}'\bigr\}.$$

One can easily verify that Y is a linear subspace of X invariant under $\{Q\}'$. Notice that Y is nontrivial because Q is one-to-one and $0 \neq Qw \in Y$. We will show that $Y \subseteq \ker g$, so that $\overline{Y}$ is a proper Q-hyper-invariant subspace.

Take $T \in \{Q\}'$; we will show that $g(TQw) = 0$. It follows from (1) that $\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i}) \neq 0$ for every i, so that $X = \operatorname{span}\{y_{n_i}\} \oplus \ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. Then one can write $TK_iy_{n_i-1} = \alpha_iy_{n_i} + r_i$, where α_i is a scalar and $r_i \in \ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. We claim that $\alpha_i \to 0$. Indeed,

$$\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr) = \alpha_i\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i}), \tag{2}$$

and, combining this with (1), we get

$$\bigl|\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr)\bigr| \geqslant \frac{|\alpha_i|}{2}\bigl\|Q^{*n_i}f_{n_i}\bigr\|\|y_{n_i}\|. \tag{3}$$

On the other hand,

$$\bigl|\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr)\bigr| \leqslant \bigl\|Q^{*n_i}f_{n_i}\bigr\| \cdot \|T\| \cdot \|y_{n_i-1}\|. \tag{4}$$

It follows from (3) and (4) that

$$|\alpha_i| \leqslant 2\|T\|\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0.$$

Then (2) yields that

$$\Bigl|f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr)\Bigr| = \Bigl|\alpha_if_{n_i}\bigl(Q^{n_i}y_{n_i}\bigr)\Bigr| \leqslant |\alpha_i| \cdot \|f_{n_i}\| \cdot \|Q^{n_i}y_{n_i}\| \leqslant |\alpha_i| \cdot 1 \cdot \bigl(\|x_0\| + \varepsilon\bigr) \to 0,$$

so that $f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr) \to 0$. On the other hand, since $T, K_i \in \{Q\}'$ we have

$$Q^{n_i}TK_iy_{n_i-1} = TQK_iQ^{n_i-1}y_{n_i-1} \to TQw$$

in norm, while $f_{n_i} \xrightarrow{w^*} g$, so that $f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr) \to g(TQw)$. Hence, $g(TQw) = 0$. □

Clearly, the argument will work as well for λ-minimal vectors for any $\lambda > 1$.

Suppose that Q is a quasinilpotent operator commuting with a compact operator K. Then Q satisfies the hypothesis of the theorem. Indeed, without loss of generality, $\|K\| = 1$. Fixing $\varepsilon = \frac{1}{3}$, there exists x_0 with $\|x_0\| = 1$ such that $\|Kx_0\| \geqslant \frac{2}{3}$ and $0 \notin KB(x_0, \varepsilon)$. For every sequence (x_n) in $B(x_0, \varepsilon)$, the sequence (Kx_n) has a convergent subsequence (Kx_{n_i}). Take $K_i = K$ for all i; since $0 \notin KB(x_0, \varepsilon)$ then $\lim_i K_i x_{n_i} \neq 0$. It follows from the theorem that *if Q is a quasinilpotent operator on a real or complex Banach space commuting with a nonzero compact operator, then Q has a hyperinvariant subspace.* This fact is not new though: for complex Banach spaces it is a special case of the celebrated Lomonosov's theorem [Lom73], and for real Banach spaces it follows from Theorem 2 of [Hoo81].

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