MARTINGALES IN BANACH LATTICES, II

HAILEGEBRIEL E. GESSESSE AND VLADIMIR G. TROITSKY

Abstract. This note is a follow-up to [Tro05]. We provide several sufficient conditions for the space M of bounded martingale on a Banach lattice F to be a Banach lattice itself. We also present examples in which M is not a Banach lattice. It is shown that if F is a KB-space and the filtration is dense then F is a projection band in M.

Introduction

This short note is a follow-up to [Tro05], where the second author introduced and studied spaces of bounded martingales on Banach lattices. Let us briefly recall some key definitions from [Tro05]. Throughout this paper, F is a Banach lattice. By a ***filtration*** on F we mean a sequence (E_n) of positive contractive projections such that $E_nE_m = E_{n\wedge m}$. A sequence (x_n) in F is said to be a ***martingale*** (a ***submartingale***) relative to a filtration (E_n) if $E_nx_m = x_n$ ($E_nx_m \geqslant x_n$, respectively) whenever $n \leqslant m$. A (sub)martingale $X = (x_n)$ is ***bounded*** if it has finite ***martingale norm*** given by $\|X\| = \sup_n\|x_n\|$. We write $M = M\big(F, (E_n)\big)$ for the space of all bounded martingale on F relative to filtration (E_n). It is easy to see that M is a Banach space. Also, M can be ordered component-wise, i.e., $(x_n) \leqslant (y_n)$ if $x_n \leqslant y_n$ for every n. It is easy to see that, under this order, M is an ordered Banach space and the norm is monotone, i.e., $0 \leqslant X \leqslant Y$ implies $\|X\| \leqslant \|Y\|$. It was shown in [Tro05] that under certain conditions on F the space M is itself a Banach lattice. In Section 1 of this note, we slightly improve some of these conditions. However, the question whether M is always a Banach lattice was left unanswered in [Tro05]. In Section 2 of this note, we answer this question in the negative by providing examples in which M is not a Banach lattice.

Section 3 is concerned with the case when F is a KB-space. It was shown in [Tro05] that in this case, M is a Banach lattice. It was also claimed in [Tro05] that in this case, F can be identified with a projection band in M. However, the proof of the latter claim in [Tro05] contained a gap. In Section 3 of this note, we present a complete proof of the assertion.

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1. When is M a Banach lattice?

We start by extending Lemmas 5 and 6 of [Tro05] to weakly convergent sequences. The proofs are analogous.

Lemma 1. *Let $X=(x_n)$ and $Y=(y_n)$ be two bounded submartingales.*

(i) *For a fixed n, the sequence $\bigl(E_n(x_m\vee y_m)\bigr)_{m=n}^{\infty}$ is increasing, norm bounded by $\|X\|+\|Y\|$, and bounded below by $x_n\vee y_n$.*

(ii) *If, in addition, this sequence converges weakly to some (z_n) for each n, then $Z=(z_n)$ is a bounded martingale, and it is the least martingale satisfying $X\leqslant Z$ and $Y\leqslant Z$.*

Proof. (i) Let $n\leqslant m$, notice that $E_n(x_m\vee y_m)\geqslant(E_nx_m)\vee(E_ny_m)=x_n\vee y_n$. Furthermore,

$$E_n(x_{m+1}\vee y_{m+1})=E_nE_m(x_{m+1}\vee y_{m+1})\geqslant E_n(E_mx_{m+1}\vee E_my_{m+1})=E_n(x_m\vee y_m).$$

Finally,

$$\bigl\|E_n(x_m\vee y_m)\bigr\|\leqslant\|x_m\vee y_m\|\leqslant\bigl\||x_m|+|y_m|\bigr\|\leqslant\|X\|+\|Y\|.$$

(ii) Suppose that $\operatorname{w-lim}_m E_n(x_m\vee y_m)=z_n$ for each n, and set $Z=(z_n)$. First, observe that Z is a martingale. Indeed, for $k\leqslant n$ we have

$$E_kz_n=E_k\bigl(\operatorname*{w-lim}_{m\to\infty}E_n(x_m\vee y_m)\bigr)=\operatorname*{w-lim}_{m\to\infty}E_kE_n(x_m\vee y_m)=\operatorname*{w-lim}_{m\to\infty}E_k(x_m\vee y_m)=z_k.$$

Furthermore, by properties of weak convergence, we have

$$\|z_n\|\leqslant\liminf_{m\to\infty}\bigl\|E_n(x_m\vee y_m)\bigr\|\leqslant\|X\|+\|Y\|$$

for every n, so that Z is bounded. Since $E_n(x_m\vee y_m)\geqslant x_n\vee y_n$ whenever $m\geqslant n$, we have $z_n\geqslant x_n\vee y_n$ for all n. Thus, $Z\geqslant X$ and $Z\geqslant Y$. On the other hand, suppose that $\widetilde{Z}=(\tilde z_n)$ is a martingale such that $\widetilde{Z}\geqslant X$ and $\widetilde{Z}\geqslant Y$. Then $\tilde z_m\geqslant x_m\vee y_m$ for all m, so that $\tilde z_n=E_n\tilde z_m\geqslant E_n(x_m\vee y_m)$ for all $m\geqslant n$. As $\operatorname{w-lim}_m E_n(x_m\vee y_m)=z_n$, this yields $\tilde z_n\geqslant z_n$, so that $\widetilde{Z}\geqslant Z$. □

Corollary 2. *Suppose that* $\operatorname{w-lim}_m E_n|x_m|$ *exists for each n and for each martingale (x_n) in M. Then M is a Banach lattice with lattice operations given by $(X\vee Y)_n=\operatorname*{w-lim}_{m\to\infty}E_n(x_m\vee y_m)$, $|X|_n=\operatorname*{w-lim}_{m\to\infty}E_n|x_m|$, etc, for any $X,Y\in M$ with $X=(x_n)$ and $Y=(y_n)$.*

Proof. Let $X,Y\in M$, put $Z=X-Y$, then $Z\in M$. Write $X=(x_n)$, $Y=(y_n)$, and $Z=(z_n)$. Then for $n\leqslant m$ we have

$$E_n(x_m\vee y_m)=E_n\Bigl(\frac{x_m+y_m}{2}+\frac{|x_m-y_m|}{2}\Bigr)=\frac{x_n+y_n}{2}+\tfrac12E_n|z_m|,$$

which converges weakly as $m \to +\infty$ by the hypothesis. Thus, by Lemma 1, $X \vee Y$ is a bounded martingale. Hence, M is a vector lattice with lattice operation as in the statement.

It remains to show that $\big\| |X| \big\| = \|X\|$ for every $X \in M$. Let $Z = |X|$. Write $X = (x_n)$ and $Z = (z_n)$. Then $z_n = \text{w-lim}_m E_n|x_m|$ for every n. Let $U = \{f \in F^*_+ \,:\, \|f\| \leqslant 1\}$. Then

$$\|z_n\| = \sup_{f \in U} f(z_n) = \sup_{f \in U} \lim_{m \to \infty} f\big(E_n|x_m|\big). \tag{1}$$

Note that for every $f \in U$ we have $f\big(E_n|x_m|\big) \leqslant \|f\|\|E_n\|\|x_m\| \leqslant \|X\|$, so that $\|Z\| \leqslant \|X\|$. On the other hand, for $n \leqslant m$ we have $|x_n| = |E_n x_m| \leqslant E_n|x_m|$, so that $f\big(E_n|x_m|\big) \geqslant f\big(|x_n|\big)$. It follows from (1) that $\|z_n\| \geqslant \big\| |x_n| \big\| = \|x_n\|$, hence $\|Z\| \geqslant \|X\|$. ☐

It was shown in [Tro05] that if F is a KB-space then M is a Banach lattice. We can now prove the following stronger result.

Corollary 3. *Suppose that every increasing norm bounded sequence in F converges weakly. Then M is a Banach lattice. If $X, Y \in M$ with $X = (x_n)$ and $Y = (y_n)$ then $(X \vee Y)_n = \underset{m \to \infty}{\text{w-lim}}\, E_n(x_m \vee y_m)$.*

Proposition 4. *If E_n is a band projection for every n then M is a Banach lattice with coordinate-wise lattice operations.*

Proof. Let $X = (x_n) \in M$. Then for every $m \geqslant n$ we have $E_n|x_m| = |E_n x_m| = |x_n|$. The conclusion now follows from Corollary 2. ☐

Theorem 5. *If F is order continuous, then the following statments are equivalent.*

(i) *M is a Banach lattice.*
(ii) *For each n, $\big(E_n|x_m|\big)_m$ converges weakly for each $(x_n) \in M$.*
(iii) *For each n, $\big(E_n|x_m|\big)_m$ converges in norm for each $(x_n) \in M$.*

Proof. (i)⇒(ii) Suppose M is a Banach lattice and let $X = (x_n) \in M$. Then $|X|$ exists in M, say, $|X| = (y_n) \in M$. It follows from $|x_m| \leqslant y_m$ that $0 \leqslant E_n|x_m| \leqslant E_n y_m = y_n$ whenever $n \leqslant m$. Thus, the sequence $\big(E_n|x_m|\big)_{m=n}^{\infty}$ is increasing and bounded above. Since F is order continuous, order interval in F are weakly compact, see, e.g. [AB85, Theorem 12.9]. Hence $\big(E_n|x_m|\big)$ has a weakly convergent subsequence. It follows from Lemma 1(i) that $\big(E_n|x_m|\big)$ is increasing, hence the entire sequence converges weakly.

(ii)⇒(iii) By Lemma 1(i), $\big(E_n|x_m|\big)_m$ is increasing, and every increasing weakly convergent sequence is norm convergent (see, e.g., Proposition 1.4.1 of [MN91].

(iii)⇒(i) This is just a special case of Corollary 2. □

2. Examples when M is not a Banach lattice

Example 6. In this example we construct a filtration (E_n) on c_0 such that $M\big(c_0,(E_n)\big)$ is not a Banach lattice.

As usually, an operator $T\in L(c_0)$ can be represented by an infinite matrix where the j-column is Te_j. For $n=0,1,2,\dots$, put

$$E_n=\begin{bmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 1/2 & 1/2 &&& \\ &&& 1/2 & 1/2 &&& \\ &&&&& 1/2 & 1/2 & \\ &&&&& 1/2 & 1/2 & \\ &&&&&&& \ddots \end{bmatrix}$$

with $2n$ ones in the upper-left corner. In other words,

$$E_ne_i=e_i \quad\text{if}\quad i\leqslant 2n,\qquad\text{and}\qquad E_ne_{2k-1}=E_ne_{2k}=\tfrac12(e_{2k-1}+e_{2k})\quad\text{when}\quad n<k.$$

Note that E_0 has no 1's at all. It is easy to see that (E_n) is a dense filtration. Furthermore, for $n=0,1,2,\dots$, put

$$x_n=(\underbrace{-1,1,\dots,-1,1}_{2n},0,\dots).$$

It is easy to see that (x_n) is a bounded martingale relative to (E_n). On the other hand,

$$E_0\lvert x_n\rvert=(\underbrace{1,1,\dots,1,1}_{2n},0,\dots).$$

Clearly, $E_0\lvert x_n\rvert$ diverges, so that M is not a Banach lattice by Theorem 5.

Example 7. This is another example where M is not a Banach lattice. For the definition and properties of vector-valued L_p-spaces we refer the reader to [DU77]. Suppose that X is a Banach space, and let a sequence $(a_n)_{n=0}^\infty$ in X be a tree, that is, $a_n=\frac12(a_{2n+1}+a_{2n+2})$ for every n. Now define a sequence (x_n) in $L_1\big([0,1],X\big)$ via

$$\begin{aligned} x_0 &= a_0\chi_{[0,1]},\\ x_1 &= a_1\chi_{[0,\frac12)}+a_2\chi_{[\frac12,1]},\\ x_2 &= a_3\chi_{[0,\frac14)}+a_4\chi_{[\frac14,\frac24)}+a_5\chi_{[\frac24,\frac34)}+a_6\chi_{[\frac34,1]},\\ &\ \text{etc.}\end{aligned}$$

Then (x_n) is a martingale in the sense of [DU77] relative to the dyadic filtration of $[0,1]$.

Now, suppose that X is a Banach lattice. Then $L_1\big([0,1],X\big)$ also is a Banach lattice. Next, we define a sequence of projections on $L_1\big([0,1],X\big)$ as follows. For $f\in L_1\big([0,1],X\big)$, we put

$$\begin{aligned} E_0f &= \big(\textstyle\int_0^1 f\big)\chi_{[0,1]},\\ E_1f &= 2\Big[\big(\textstyle\int_0^{1/2} f\big)\chi_{[0,\frac12)}+\big(\textstyle\int_{1/2}^1 f\big)\chi_{[\frac12,1]}\Big],\\ E_2f &= 4\Big[\big(\textstyle\int_0^{1/4} f\big)\chi_{[0,\frac14)}+\big(\textstyle\int_{1/4}^{2/4} f\big)\chi_{[\frac14,\frac24)}+\big(\textstyle\int_{2/4}^{3/4} f\big)\chi_{[\frac24,\frac34)}+\big(\textstyle\int_{3/4}^1 f\big)\chi_{[\frac34,1]}\Big],\\ &\ \text{etc.}\end{aligned}$$

It is easy to see that (E_n) is a filtration on $L_1\big([0,1],X\big)$, and (x_n) is a martingale in $L_1\big([0,1],X\big)$ relative to this filtration.

Now put $X=c_0$ and let $F=L_1\big([0,1],c_0\big)$. Let (x_k) be defined as above with

$$a_0=(0,\dots),$$
$$a_1=(1,0,\dots),\qquad a_2=(-1,0,\dots),$$
$$a_3=(1,1,0,\dots),\quad a_4=(1,-1,0\dots),\quad a_5=(-1,1,0,\dots),\quad a_6=(-1,-1,0,\dots),$$

etc. In other words, $x_n=(r_1,\dots,r_n,0,\dots)$, where r_k is the k-th Rademacher function. Then $\|x_n\|=1$ for every n. However,

$$E_0|x_n|=(\underbrace{\mathbb{1},\dots,\mathbb{1}}_{n\text{ times}},0\dots).$$

Again, $E_0|x_n|$ diverges, so that M is not a Banach lattice by Theorem 5.

3. How does F sit in M?

Again, throughout this section we assume that F is a Banach lattice, (E_n) is a filtration on F, and $M=M\big(F,(E_n)\big)$. Moreover, we will assume that (E_n) is ***dense***, i.e., $E_nx\to x$ for every $x\in F$. It was observed in Section 8 of [Tro05] that in this case a bounded martingale (x_n) converges iff it is ***fixed***, i.e., there exists $x\in F$ such that $x_n=E_nx$ for every n. Clearly, in this case we have $x_n\to x$.

Define $\varphi\colon F\to M$ via $\varphi(x)=(E_nx)_{n=1}^\infty$. It is clear that φ is an isometry. We claim that φ is a lattice homomorphism, so that F is lattice isometric to a closed subspace of M.

Indeed, take any $x,y\in F$ and put $x_n=E_nx$ and $y_n=E_ny$ for all n. Since the lattice operations are continuous, we have $x_n\vee y_n\to x\vee y$. Hence, for every n we have

$$\lim_{m\to\infty}E_n(x_m\vee y_m)=E_n\big(\lim_{m\to\infty}x_m\vee y_m\big)=E_n(x\vee y).$$

It follows from Lemma 1 that $\varphi(x) \vee \varphi(y) = \varphi(x \vee y)$. Finally,

$$\varphi(x \wedge y) = -\varphi\big((-x) \vee (-y)\big) = -\big(\varphi(-x) \vee \varphi(-y)\big) = \varphi(x) \wedge \varphi(y).$$

Lemma 8. *Suppose that F is order continuous and M is a Banach lattice. Then M is order complete. If, in addition, (E_n) is dense, then $\varphi(F)$ is an ideal in M.*

Proof. First, show that M is order complete. Suppose that $0 \leqslant X^{(\alpha)} \uparrow\leqslant X$ in M. Put $X = (x_n)$ and $X^{(\alpha)} = (x_n^{(\alpha)})$. Then $0 \leqslant x_n^{(\alpha)} \uparrow\leqslant x_n$ for every n. Since F is order continuous, for every n there exists y_n such that we have $x_n^{(\alpha)} \to y_n$. Put $Y = (y_n)$. It is easy to see that Y is a martingale. It follows from $0 \leqslant Y \leqslant X$ that Y is bounded, hence $Y \in M$.

Now suppose that (E_n) is dense. Put $M_0 = \varphi(F)$. Then F is lattice isometric to M_0. Show that M_0 is an ideal in M. Suppose that $0 \leqslant X \leqslant Y$ for some $X \in M$ and $Y \in M_0$. Put $X = (x_n)$ and $Y = (y_n)$. Then $0 \leqslant x_n \leqslant y_n$ for every n and there exists $y \in F$ such that $y_n = E_n y$ for all n. Fix $\varepsilon > 0$. It follows from $y_n \to y$ that there exists n_0 such that $\|y_n - y\| < \varepsilon$ whenever $n \geqslant n_0$. It follows that

$$|x_n - x_n \wedge y| = |x_n \wedge y_n - x_n \wedge y| \leqslant |y_n - y|,$$

so that $\|x_n - x_n \wedge y\| < \varepsilon$. If follows from $x_n \wedge y \in [0, y]$ that $x_n \in [0, y] + B_\varepsilon$ for all $n \geqslant n_0$. Therefore, (x_n) is almost order bounded. Hence, it converges by Corollary 19 of [Tro05]. Hence, $X \in M_0$. Now suppose that $Y \in M_0$ and $X \in M$ such that $|X| \leqslant Y$. Then $0 \leqslant X^+, X^- \leqslant Y$, so that $X^+, X^- \in M_0$ and, therefore, $X \in M_0$. Thus, M_0 is an ideal. □

Now we are ready to present a new proof of Proposition 16 of [Tro05].

Theorem 9. *If F is a KB-space and (E_n) is dense then $\varphi(F)$ is a projection band in M.*

Proof. By Theorems 7 of [Tro05] and Lemma 8, M is an order complete Banach lattice. Again, denote $M_0 = \varphi(F)$. By Lemma 8, M_0 is an ideal in M. It is left to show that M_0 is a band because every band in an order complete lattice is a projection band by Theorems 3.8 of [AB85].

To show that M_0 is a band, suppose that $0 \leqslant X^{(\alpha)} \uparrow X$ for some net $(X^{(\alpha)})$ in M_0 and some $X \in M$. Put $X = (x_n)$ and $X^{(\alpha)} = (x_n^\alpha)$. Let $X^{(\alpha)} = \varphi(x^{(\alpha)})$ for some $x^{(\alpha)} \in F$. Clearly, $\|x^{(\alpha)}\| = \|X^{(\alpha)}\| \leqslant \|X\|$ for every α, hence the net $(x^{(\alpha)})$ is norm bounded. Since F is a KB-space, this net converges in norm to some $y \in F$, see [AB85, p. 225]. It follows also that $x^{(\alpha)} \uparrow y$ in F. Put $Y = \varphi(y)$, $Y = (y_n)$. For every α we have $x^{(\alpha)} \leqslant y$, so that $X^{(\alpha)} \leqslant Y$, hence $X \leqslant Y$. On the other hand, $x^{(\alpha)} \to y$ implies $\lim_\alpha x_n^{(\alpha)} = y_n$ for every n. Together with $x_n^{(\alpha)} \leqslant x_n$ this implies $y_n \leqslant x_n$, so that $Y \leqslant X$. Thus, $X = Y$, so that $X \in M_0$. □

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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada.

E-mail address: vtroitsky@math.ualberta.ca, hgessesse@math.ualberta.ca