A VERSION OF LOMONOSOV'S THEOREM FOR COLLECTIONS OF POSITIVE OPERATORS

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Abstract. It is known that for every Banach space X and every proper WOT-closed subalgebra $\mathcal{A}$ of $L(X)$, if $\mathcal{A}$ contains a compact operator then it is not transitive. That is, there exist non-zero $x \in X$ and $f \in X^*$ such that $\langle f, Tx\rangle = 0$ for all $T \in \mathcal{A}$. In the case of algebras of adjoint operators on a dual Banach space, V. Lomonosov extended this as follows: without having a compact operator in the algebra, one has $\bigl|\langle f, Tx\rangle\bigr| \leqslant \|T_*\|_e$ for all $T \in \mathcal{A}$. In this paper, we prove a similar extension of a result of R. Drnovšek. Namely, we prove that if $\mathcal{C}$ is a collection of positive adjoint operators on a Banach lattice X satisfying certain conditions, then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \|T_*\|_e$ for all $T \in \mathcal{C}$.

In this paper we use techniques which were recently developed for transitive algebras to obtain analogous results for collections of positive operators on Banach lattices. Let us first briefly describe these two branches of the Invariant Subspace research.

Transitive algebras. Suppose that X is a Banach space. A subspace Z of X is said to be invariant under an operator $T \in L(X)$ if $\{0\} \neq Z \neq X$ and $T(Z) \subseteq Z$. The *Invariant Subspace Problem* deals with the question: "Which operators have invariant subspaces?". Lomonosov proved in [7] that an operator which commutes with a compact operator has an invariant subspace. There is also an algebraic version of the problem: which subalgebras of $L(X)$ have no (common) invariant subspaces? Such subalgebras are called ***transitive***. The classical Burnside's theorem asserts that if X is finite-dimensional then $L(X)$ has no proper transitive subalgebras (clearly, $L(X)$ itself is always transitive). Using Lomonosov's technique, Burnside's theorem can be extended to the infinite-dimensional case as follows:

Theorem 1 ([11, Theorem 8.23]). *A proper WOT-closed subalgebra of $L(X)$ containing a compact operator is not transitive.*

A "quantitative" version of the latter theorem was obtained by Lomonosov in [8] for algebras of adjoint operators. Before we state it, we need to introduce some notation. It is easy to see that a subalgebra $\mathcal{A}$ of $L(X)$ has an invariant subspace if and only if

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there exist non-zero $x \in X$ and $f \in X^*$ such that $\langle f, Tx\rangle = 0$ for every $T \in \mathcal{A}$. Now suppose that X is a dual space; that is, $X = Y^*$ for some Banach space Y. If $T \in L(X)$ is a bounded adjoint operator on X then there is a unique operator $S \in L(Y)$ such that $S^* = T$. We will write $S = T_*$; there will be no ambiguity as T_* will always be taken with respect to Y. We will write $\|T\|_e$ for the essential norm of T, i.e., the distance from T to the space of compact operators. Note that in general, for an adjoint operator T, one has $\|T\|_e \leqslant \|T_*\|_e$. See [2] for an example of T such that $\|T\|_e < \|T_*\|_e$.

Theorem 2 ([8])**.** *Let X be a dual Banach space and $\mathcal{A}$ a proper W^*OT-closed subalgebra of $L(X)$ consisting of adjoint operators. Then there exist non-zero $x \in X$ and $f \in X^*$ such that $\bigl|\langle f, Tx\rangle\bigr| \leqslant \|T_*\|_e$ for all $T \in \mathcal{A}$.*

Invariant ideals of collections of positive operators. Suppose now that X is a Banach lattice. Recall that a linear (not necessarily closed) subspace $\mathcal{J} \subseteq X$ is called an ***order ideal*** if it is ***solid***, i.e., $y \in \mathcal{J}$ implies $x \in \mathcal{J}$ whenever $|x| \leqslant |y|$. The following version of Lomonosov's theorem for positive operators was proved by B. de Pagter [10]: *a positive quasinilpotent compact operator on X has a closed invariant order ideal.* There have been many extensions of this result, see, e.g., [1]. In particular, R. Drnovšek [3] showed that a collection of positive operators satisfying certain assumptions has a (common) invariant closed ideal. To state his result precisely, we need to introduce more notations.

As usual, we write X_+, X^*_+, and $L(X)_+$ for the cones of positive elements in X, X^*, and $L(X)$, respectively. Let $\mathcal{C}$ be a collection of positive operators on X. Following [1], we will denote by symbols $\langle\mathcal{C}]$ and $[\mathcal{C}\rangle$ the ***super left*** and the ***super right commutants*** of $\mathcal{C}$, respectively, i.e.,

$$\langle\mathcal{C}] = \bigl\{S \in L(X)_+ \colon ST \leqslant TS \text{ for each } T \in \mathcal{C}\bigr\},$$
$$[\mathcal{C}\rangle = \bigl\{S \in L(X)_+ \colon ST \geqslant TS \text{ for each } T \in \mathcal{C}\bigr\}.$$

If $\mathcal{D}$ is another collection of operators then we write $\mathcal{C}\mathcal{D} = \{TS \colon T \in \mathcal{C}, S \in \mathcal{D}\}$. The symbol $\mathcal{C}^n$ is defined as the product of n copies of $\mathcal{C}$.

An operator T is ***locally quasinilpotent*** at x if $\limsup_n \|T^n x\|^{\frac{1}{n}} = 0$. If U is a subset of X then we write $\|U\| = \sup\bigl\{\|x\| \colon x \in U\bigr\}$. We call a collection $\mathcal{C}$ of operators ***finitely quasinilpotent*** at a vector $x \in X$ if $\limsup_n \|\mathcal{F}^n x\|^{\frac{1}{n}} = 0$ for every finite subcollection $\mathcal{F}$ of $\mathcal{C}$. Clearly, finite quasinilpotence at x implies local quasinilpotence at x of every operator in the collection.

If E is a Banach lattice then an operator $T : E \to E$ is called ***AM-compact*** if the image of every order interval under T is relatively compact. Since order intervals are norm bounded, every compact operator is AM-compact. An operator T is said to ***dominate*** an operator S if $|Sx| \leqslant T|x|$ holds for all $x \in E$.

Theorem 3 ([3])**.** *If $\mathcal{C}$ is a collection of positive operators on a Banach lattice X such that*

(i) *$\mathcal{C}$ is finitely quasinilpotent at some positive non-zero vector, and*
(ii) *some operator in $\mathcal{C}$ dominates a non-zero AM-compact operator,*

then $\mathcal{C}$ and $[\mathcal{C}\rangle$ have a common closed invariant order ideal.

Observe that if a collection $\mathcal{C}$ of positive operators has a (closed nontrivial) invariant ideal then there exist non-zero positive x and f such that $\langle f, Tx\rangle = 0$ for all $T \in \mathcal{C}$. The converse is also true when $\mathcal{C}$ is a semigroup.

The goal of this paper is to "quantize" Theorem 3 in the same manner that Theorem 1 was "quantized" into Theorem 2. Our proofs use ideas from [6] and [9].

In the following lemma, we collect several standard facts that we will use later. See, e.g., [1] for the proofs.

Lemma 4. *Let Z be a vector lattice, $x \in Z_+$. Then for each $y, z \in Z$ one has*

(i) *$|x \wedge y - x \wedge z| \leqslant |y - z|$;*
(ii) *if $|y| \leqslant z$ then $|x - x \wedge z| \leqslant |x - x \wedge y|$;*
(iii) *$|x - x \wedge y| \leqslant |x - y|$.*

From now on, X will be a real Banach lattice. We will also assume that X is a dual Banach space; that is, $X = Y^*$ for some (fixed) Banach space Y. We will start with a version of Theorem 2 for convex collections of positive operators.

Theorem 5. *Let $\mathcal{C}$ be a convex collection of positive adjoint operators on X. If there is $x_0 > 0$ such that every operator from $\mathcal{C}$ is locally quasinilpotent at x_0 then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \|T_*\|_e$ for all $T \in \mathcal{C}$.*

Remark 6. One might try to deduce Theorem 5 from Theorem 2 by considering the W^*OT-closed algebra generated by $\mathcal{C}$. However, the example in [5] shows that there exists an algebra of nilpotent operators on a Hilbert space H which is WOT-dense in $L(H)$.

Proof of Theorem 5. Clearly, we may assume that $\|x_0\| = 1$. Also, without loss of generality, $\mathcal{C}$ is closed under taking positive multiples of its elements, otherwise we replace $\mathcal{C}$ with $\bigl\{\alpha T\colon T\in\mathcal{C}, 0<\alpha\in\mathbb{R}\bigr\}$. Fix $0<\varepsilon<\frac{1}{10}$. Define

$$\begin{aligned}\mathcal{C}_\varepsilon &= \bigl\{T\in\mathcal{C}\colon \|T_*\|_e<\varepsilon\bigr\} \text{ and}\\ H_\varepsilon(x) &= \bigl\{z\in X\colon |z|\leqslant Tx \text{ for some } T\in\mathcal{C}_\varepsilon\bigr\},\quad x\in X_+.\end{aligned}$$

Then $H_\varepsilon(x)$ is convex and solid for all $x\in X_+$.

Suppose that $\overline{H_\varepsilon(x)}\neq X$ for some nonzero $x\in X_+$. Since $H_\varepsilon(x)$ is convex, there is a nonzero $g\in X^*$ such that $g(y)\leqslant 1$ for all $y\in H_\varepsilon(x)$. Consider $h=|g|\in X^*$. Then for any $y\in H_\varepsilon(x)$ we have

$$h(y)\leqslant h\bigl(|y|\bigr)=\sup\bigl\{g(u)\colon -|y|\leqslant u\leqslant |y|\bigr\}\leqslant 1$$

since $H_\varepsilon(x)$ is solid. In particular, $\langle h,Tx\rangle\leqslant 1$ for all $T\in\mathcal{C}_\varepsilon$.

Put $f=\frac{\varepsilon}{2}h$. We claim that $\langle f,Tx\rangle\leqslant\|T_*\|_e$ for each $T\in\mathcal{C}$. Indeed, if T is compact, i.e., $\|T_*\|_e=0$, then $\alpha T\in\mathcal{C}_\varepsilon$ for all $0<\alpha\in\mathbb{R}$. Therefore $\langle h,\alpha Tx\rangle\leqslant 1$ for all $0<\alpha\in\mathbb{R}$, so that $\langle f,Tx\rangle=\langle h,Tx\rangle=0$. If T is not compact then $\frac{\varepsilon T}{2\|T_*\|_e}\in\mathcal{C}_\varepsilon$, whence

$$\langle f,Tx\rangle=\|T_*\|_e\Bigl\langle h,\frac{\varepsilon T}{2\|T_*\|_e}x\Bigr\rangle\leqslant\|T_*\|_e.$$

Suppose now that $\overline{H_\varepsilon(x)}=X$ for all nonzero $x\in X_+$. Then, in particular, for each $x\in X$ there is $y_x\in H_\varepsilon(x)$ such that $\|x_0-y_x\|<\varepsilon$. Fix an operator $T_x\in\mathcal{C}_\varepsilon$ such that $|y_x|\leqslant T_xx$. Then (ii) and (iii) of Lemma 4 yield $\|x_0-x_0\wedge T_xx\|<\varepsilon$.

Let $U_0=\bigl\{x\in X_+\colon \|x-x_0\|\leqslant\frac{1}{2}\bigr\}$. Since $\bigl\|(T_x)_*\bigr\|_e<\varepsilon$, there is an adjoint compact operator $K_x\in\mathcal{K}(X)$ such that $\|K_x-T_x\|<\varepsilon$. As compact adjoint operators are w^*-$\|\cdot\|$ continuous on norm bounded sets, it follows that there is a relative (to U_0) w^*-open neigborhood $W_x\subseteq U_0$ of x such that $\|K_xz-K_xx\|<\varepsilon$ whenever $z\in W_x$. Then for every $y\in W_x$ we have:

$$\begin{aligned}\bigl\|x_0-x_0\wedge T_xy\bigr\| &\leqslant \bigl\|x_0-x_0\wedge T_xx\bigr\|+\bigl\|x_0\wedge T_xx-x_0\wedge K_xx\bigr\|\\ &\qquad+\bigl\|x_0\wedge K_xx-x_0\wedge K_xy\bigr\|+\bigl\|x_0\wedge K_xy-x_0\wedge T_xy\bigr\|\\ &\leqslant \bigl\|x_0-x_0\wedge T_xx\bigr\|+\|T_xx-K_xx\|+\|K_xx-K_xy\|+\|K_xy-T_xy\|\\ &\qquad<\varepsilon+\varepsilon\|x\|+\varepsilon+\varepsilon\|y\|<5\varepsilon<\tfrac{1}{2}.\end{aligned}$$

Together with $T_x\geqslant 0$ this yields $(x_0\wedge T_xy)\in U_0$ for each $y\in W_x$.

Note that U_0 is w^*-compact since U_0 is the intersection of X_+ with a closed ball. Hence, we can find $x_1,\dots,x_n\in U_0$ such that $U_0=\bigcup_{k=1}^n W_{x_k}$. Define $T=T_{x_1}+\dots+T_{x_n}\in\mathcal{C}$. Then by Lemma 4(ii), we have $x_0\wedge Tx\in U_0$ for every $x\in U_0$.

Define a sequence $(y_n) \subseteq U_0$ by $y_0 = x_0$ and $y_{n+1} = x_0 \wedge Ty_n$. Clearly $0 \leqslant y_n$ for all n, and $y_n \leqslant Ty_{n-1} \leqslant \ldots \leqslant T^n y_0$, so that $\|y_n\| \leqslant \|T^n x_0\|$. Thus $y_n \to 0$ as $n \to \infty$ by local quasinilpotence. This is a contradiction by the definition of U_0. $\square$

The next theorem shows that the conclusion of Theorem 5 is also true for some collections of operators which are not necessarily convex. We will, however, use a more restrictive quasinilpotence condition. We will need some additional definitions.

Let $\mathcal{C}$ be a collection of positive operators. Following [1], define

$$\mathcal{D}_{\mathcal{C}} = \Big\{ D \in L(X)_+ \,:\, \exists T_1, \ldots, T_k \in [\mathcal{C}\rangle \text{ and } S_1, \ldots, S_k \in \bigcup_{n=1}^{\infty} \mathcal{C}^n \text{ such that } D \leqslant \sum_{i=1}^{k} T_i S_i \Big\}$$

In other words, $\mathcal{D}_{\mathcal{C}}$ is the smallest additive and multiplicative semigroup which contains the collection $[\mathcal{C}\rangle \cdot \mathcal{C}$ and such that $T \in \mathcal{D}_{\mathcal{C}}$ and $0 \leqslant S \leqslant T$ imply $S \in \mathcal{D}_{\mathcal{C}}$ (see [1]).

Let $\mathcal{C}$ be a collection of positive adjoint operators on X. Define

$$\mathcal{E}_{\mathcal{C}} = \big\{ T \in \mathcal{D}_{\mathcal{C}} \colon T = S^* \text{ for some } S \in L(Y) \big\}.$$

Since adjoint operators are stable under addition and multiplication, $\mathcal{E}_{\mathcal{C}}$ is an additive and multiplicative semigroup. It is also clear that $\mathcal{C} \subseteq \mathcal{E}_{\mathcal{C}}$.

Theorem 7. *Let $\mathcal{C}$ be a collection of positive adjoint operators on X. If $\mathcal{C}$ is finitely quasinilpotent at some $x_0 > 0$ then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx \rangle \leqslant \|T_*\|_e$ for all $T \in \mathcal{E}_{\mathcal{C}}$.*

Proof. Clearly $\mathcal{E}_{\mathcal{C}}$ is convex. Note that the finite quasinilpotence of $\mathcal{C}$ at x_0 implies the finite quasinilpotence of $\mathcal{D}_{\mathcal{C}}$ (and, therefore, of $\mathcal{E}_{\mathcal{C}}$) at x_0 (see, e.g., [1, Lemma 10.43]). Finally, apply Theorem 5 to $\mathcal{E}_{\mathcal{C}}$. $\square$

Now suppose, in addition, that Y is itself a Banach lattice. Then we can improve the conclusion of Theorem 5. For an operator T acting on Y, define

$$\theta(T) = \inf\big\{ \|T - K\| \colon K \text{ is AM-compact} \big\}.$$

Clearly, θ is a seminorm on $L(Y)$ and $\theta(T) = 0$ if and only if T is AM-compact (because the subspace of AM-compact operators in L(Y) is norm closed).

For $\xi \in Y_+$, define a seminorm ρ_ξ on X via $\rho_\xi(x) = |x|(\xi)$.

Lemma 8. *If $\xi \in Y_+$ and $K \in L(Y)$ is AM-compact, then $K^* \colon (B_X, w^*) \to (X, \rho_\xi)$ is continuous.*

Proof. Let $x_\alpha \xrightarrow{w^*} x$, with x_α, $x \in B_X$. Write

$$\rho_\xi\bigl(K^*x_\alpha - K^*x\bigr) = \bigl|K^*x_\alpha - K^*x\bigr|(\xi) = \sup_{-\xi\leqslant\zeta\leqslant\xi}\langle x_\alpha - x, K\zeta\rangle = \sup_{\nu\in A}\langle x_\alpha - x, \nu\rangle,$$

where $A = K\bigl([-\xi,\xi]\bigr)$. By assumption, K is AM-compact, thus A is a $\|\cdot\|$-compact set.

For $\nu \in A$, fix α_ν such that $\bigl|\langle x_\alpha - x, \nu\rangle\bigr| < \frac{\varepsilon}{3}$ whenever $\alpha \geqslant \alpha_\nu$. If $\mu \in Y$ is such that $\|\mu - \nu\| < \frac{\varepsilon}{3}$ then for $\alpha \geqslant \alpha_\nu$ we have

$$\bigl|\langle x_\alpha - x, \mu\rangle\bigr| \leqslant \tfrac{\varepsilon}{3}\|x_\alpha - x\| + \bigl|\langle x_\alpha - x, \nu\rangle\bigr| < \tfrac{2\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

Pick $\nu_1,\dots,\nu_n \in A$ such that $A \subseteq \bigcup_{k=1}^{n} B(\nu_k, \frac{\varepsilon}{3})$. Then for every $\alpha \geqslant \max\bigl\{\alpha_{\nu_1},\dots,\alpha_{\nu_n}\bigr\}$ we must have $\rho_\xi(K^*x_\alpha - K^*x) < \varepsilon$. □

An operator $T \in L(X)$ will be said ***w^*-locally quasinilpotent*** at a pair (x_0,ξ_0), where $x_0 \in X$ and $\xi_0 \in Y$, if $\bigl|T^n x_0(\xi_0)\bigr|^{\frac{1}{n}} \to 0$. Clearly, if T is locally quasinilpotent at x_0 then T is w^*-locally quasinilpotent at (x_0,ξ_0) for every $\xi_0 \in Y$.

Theorem 9. *Suppose that $X = Y^*$ for some Banach lattice Y, and $\mathcal{C}$ is a convex collection of positive adjoint operators on X. Suppose that there exists a pair $(x_0,\xi_0) \in X_+ \times Y_+$ such that $x_0(\xi_0) \neq 0$ and every operator from $\mathcal{C}$ is w^*-locally quasinilpotent at (x_0,ξ_0). Then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \theta(T_*)$ for all $T \in \mathcal{C}$.*

Proof. The proof of the theorem is similar to that of Theorem 5. We may assume that $\|x_0\| = 1$, $\|\xi_0\| = 1$, and $\mathcal{C}$ is closed under taking positive multiples. Put $\rho_{\xi_0}(x) = |x|(\xi_0)$. Evidently $\rho_{\xi_0}(x) \leqslant \|x\|$ for all $x \in X$. It is also clear that $|x| \leqslant |y|$ implies $\rho_{\xi_0}(x) \leqslant \rho_{\xi_0}(y)$.

Fix $0 < \varepsilon < \frac{x_0(\xi_0)}{8}$. Define

$$\begin{aligned} \mathcal{C}_\varepsilon &= \bigl\{T \in \mathcal{C} \colon \theta(T_*) < \varepsilon\bigr\} \text{ and} \\ G_\varepsilon(x) &= \bigl\{z \in X \colon |z| \leqslant Tx \text{ for some } T \in \mathcal{C}_\varepsilon\bigr\}, \quad x \in X_+. \end{aligned}$$

Suppose that $G_\varepsilon(x)$ is not dense in X for some $x \in X_+$. Analogously to the proof of Theorem 5, we find a positive functional $h \in X_+^*$ such that $\langle h, Tx\rangle \leqslant 1$ for all $T \in \mathcal{C}_\varepsilon$. Considering separately the cases $\theta(T_*) = 0$ and $\theta(T_*) \neq 0$, we get the conclusion of the theorem.

Thus, we may assume that $\overline{G_\varepsilon(x)} = X$ for all $x > 0$. Define

$$U_0 = \Bigl\{x \in X_+ \colon x \leqslant x_0 \text{ and } \rho_{\xi_0}(x - x_0) \leqslant \frac{x_0(\xi_0)}{2}\Bigr\}.$$

Clearly, U_0 is w^*-compact.

Let $x \in U_0$ be arbitrary. Since $\overline{G_\varepsilon(x)} = X$, we can find $T_x \in \mathcal{C}_\varepsilon$ such that $\rho_{\xi_0}(x_0 - x_0 \wedge T_x x) \leqslant \|x_0 - x_0 \wedge T_x x\| < \varepsilon$. Fix an operator K_x adjoint to an AM-compact operator such that $\|T_x - K_x\| < \varepsilon$. By Lemma 8, we can find a relative (to U_0) w^*-open neighborhood $V_x \subseteq U_0$ of x such that $\rho_{\xi_0}(K_x x - K_x z) < \varepsilon$ for all $z \in V_x$. Then for an arbitrary $z \in V_x$, we have

$$\begin{aligned}
\rho_{\xi_0}\big(x_0 - x_0 \wedge T_x z\big) &\leqslant \rho_{\xi_0}\big(x_0 - x_0 \wedge T_x x\big) + \rho_{\xi_0}\big(x_0 \wedge T_x x - x_0 \wedge K_x x\big) \\
&\quad + \rho_{\xi_0}\big(x_0 \wedge K_x x - x_0 \wedge K_x z\big) + \rho_{\xi_0}\big(x_0 \wedge K_x z - x_0 \wedge T_x z\big) \\
&\quad < \varepsilon + \|T_x x - K_x x\| + \rho_{\xi_0}(K_x x - K_x z) + \|K_x z - T_x z\| \\
&\qquad < \varepsilon + \|T_x - K_x\| \cdot \|x\| + \varepsilon + \|T_x - K_x\| \cdot \|z\| < 4\varepsilon < \frac{x_0(\xi_0)}{2}.
\end{aligned}$$

Take $x_1, \dots, x_m$ in U_0 such that $\bigcup\limits_{k=1}^{m} V_{x_k} = U_0$. Then $T = T_{x_1} + \dots + T_{x_k} \in \mathcal{C}$ satisfies $\rho_{\xi_0}\big(x_0 - x_0 \wedge Tz\big) \leqslant \frac{x_0(\xi_0)}{2}$ for all $z \in U_0$. Since $x_0 \wedge Tz \leqslant x_0$, we have $x_0 \wedge Tz \in U_0$ for all $z \in U_0$.

Put $z_0 = x_0$ and $z_{n+1} = x_0 \wedge T z_n$. By the w^*-local quasinilpotence of T at (x_0, ξ_0) we have $\rho_{\xi_0}(z_n) \leqslant \rho_{\xi_0}(T^n x_0) = \big|T^n x_0(\xi_0)\big| \to 0$ as $n \to \infty$ which is impossible by the definition of U_0. □

The following result is derived from Theorem 9 in the same way that Theorem 7 was deduced from Theorem 5.

Theorem 10. *Suppose that $X = Y^*$ for some Banach lattice Y, and $\mathcal{C}$ is a collection of positive adjoint operators on X. If $\mathcal{C}$ is finitely quasinilpotent at some $x_0 > 0$ then there exist non-zero $x \in X_+$ and $f \in X_+^*$ such that $\langle f, Tx\rangle \leqslant \theta(T_*)$ for all $T \in \mathcal{E}_{\mathcal{C}}$.*

As every operator on ℓ_p $(1 \leqslant p < \infty)$ is AM-compact, this theorem can be used as an alternative proof of the following (certainly known) result.

Corollary 11. *Every collection of positive operators on ℓ_p, $1 < p < \infty$, which is finitely quasinilpotent at a non-zero positive vector, has a non-trivial closed common invariant ideal.*

Of course, Corollary 11 follows easily from Theorem 3 when $1 \leqslant p < \infty$.

Corollary 12. *Every collection of positive adjoint operators on ℓ_∞ which is finitely quasinilpotent at a non-zero positive vector has a non-trivial closed common invariant ideal.*

The following example shows that the assumptions in Theorems 7 and 10 in general do not guarantee the existence of an invariant subspace.

Example 13. *There is a collection $\mathcal{C}$ of operators which satisfies all the conditions of Theorem 10 and has no common non-trivial invariant subspaces.* Namely, in [4], the authors constructed a multiplicative semigroup $\mathcal{S}_p$ of positive square-zero operators acting on $L_p[0,1]$, $1 \leqslant p < \infty$, having no common non-trivial invariant subspaces. It is not difficult to show that $\mathcal{S}_p$ is in fact finitely quasinilpotent at every positive vector. Hence for $1 < p < \infty$, $\mathcal{C} = \mathcal{S}_p$ satisfies the conditions of Theorem 10.

Remark 14. Even though Theorem 1 is not a special case of Theorem 2, in the case of an algebra of adjoint operators the former can be easily deduced from the latter, see [8, Corollary 1]. Similarly, we will show that in case of adjoint operators, Theorem 3 can be deduced from Theorem 10. Indeed, suppose that $X = Y^*$ for some Banach lattice Y, and $\mathcal{C}$ is a collection of positive adjoint operators which is finitely quasinilpotent at some $x_0 > 0$ and some operator in it dominates a non-zero AM-compact positive[1] adjoint operator K. We will show that there is a non-trivial closed ideal which is invariant under $\mathcal{C}$ and under all adjoint operators in $[\mathcal{C}\rangle$.

Clearly, $K \in \mathcal{E}_{\mathcal{C}}$. Let x and f be as in Theorem 10.

$$\begin{aligned}
\mathcal{J}_1 &= \bigl\{z \in X \,:\, |z| \leqslant T_1KT_2x \text{ for some } T_1, T_2 \in \mathcal{E}_{\mathcal{C}}\bigr\},\\
\mathcal{J}_2 &= \bigl\{z \in X \,:\, T|z| = 0 \text{ for all } T \in \mathcal{E}_{\mathcal{C}}\bigr\}, \text{ and}\\
\mathcal{J}_3 &= \bigl\{z \in X \,:\, |z| \leqslant Tx \text{ for some } T \in \mathcal{E}_{\mathcal{C}}\bigr\}.
\end{aligned}$$

It is easy to see that $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are ideals in X, invariant under $\mathcal{C}$ and under all adjoint operators in $[\mathcal{C}\rangle$. It is left to show that at least one of the three must be non-trivial. Clearly, $\mathcal{J}_2$ is closed and $\mathcal{J}_2 \neq X$. Suppose that $\mathcal{J}_2 = \{0\}$. In particular, $x \notin \mathcal{J}_2$. It follows that $\mathcal{J}_3 \neq \{0\}$. Suppose that $\mathcal{J}_3$ is dense in X. It follows from Theorem 10 that $\mathcal{J}_1 \subseteq \ker f$; hence $\overline{\mathcal{J}_1}$ is proper. Assume that $\mathcal{J}_1 = \{0\}$. Hence, $T_1KT_2x = 0$ for all $T_1, T_2 \in \mathcal{E}_{\mathcal{C}}$. Since $\mathcal{J}_2 = \{0\}$, it follows that K vanishes on $\mathcal{E}_{\mathcal{C}}x$ and, therefore, on $\mathcal{J}_3$. Since $\mathcal{J}_3$ is dense in X it follows that $K = 0$; a contradiction.

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[1]Unlike in Theorem 3, we also require that $K \geqslant 0$ here.

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