INFINITELY FINE PARTITIONS OF MEASURE SPACES

VLADIMIR G. TROITSKY

1. Infinitely fine partitions

Throughout this paper $(\mathcal{X}, \mathcal{A})$ always stands for a measurable space, i.e., $\mathcal{X}$ is an arbitrary set and $\mathcal{A}$ is an algebra of its subsets.

By ${}^*\mathcal{X}$ we denote the nonstandard extension of $\mathcal{X}$. We assume that the nonstandard model is κ-saturated where κ is the cardinality of $\mathcal{A}$.

1.1**. Definition.** A measurable partition P of ${}^*\mathcal{X}$ is said to be an ***infinitely fine partition*** (abbreviated ***ifp***) if ${}^*A = \bigcup\{p \in P \mid p \subseteq {}^*A\}$ for every measurable set A, or, equivalently, if P is finer than every standard partition.

It follows from the saturation principle that every measurable space (and even a Boolean algebra) has a hyperfinite ifp. From now on the symbol P stands for a fixed hyperfinite ifp of the measurable space $(\mathcal{X}, \mathcal{A})$. Infinitely fine partitions were originally introduced in [Loe72]. Some properties of ifp's were investigated in [Tro93].

Notice that if $\mathcal{A}$ contains all the singletons, then every standard singleton belongs to P.

1.2**. Definition.** We say that $p_1, p_2 \in P$ are ***equivalent*** (denoted $p_1 \sim p_2$) if $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is again an ifp.

1.3**. Theorem.** *An element of P is not equivalent to any other element of P if and only if it is standard.*

Proof. Obviously every standard elements of P is not equivalent to another element of P. Suppose that $p \in P$ is non-standard. Consider

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the collection $q = \{A \in \mathcal{A} \mid p \subseteq {}^*A\}$. Let $\mathcal{S}$ be a finite subset of q and put $S = \bigcap \mathcal{S}$, then $S \in \mathcal{A}$. Evidently $p \subseteq {}^*S$, but since p is nonstandard, it follows that $p \neq {}^*S$. Therefore, there exists $p' \in P$ such that $p' \subset {}^*S$ and $p' \neq p$. Since $\mathcal{S}$ was chosen as an arbitrary finite sub-collection of q, by the saturation principle there exists an element $p_1 \in P$ distinct from p contained in the extension of every set belonging to q, so that $p \sim p_1$. ☐

It was shown in [Loe72] that a standard bounded measurable function is approximately constant on every element of an ifp.

1.4. **Lemma.** *Let $p_1, p_2 \in P$, then $p_1 \sim p_2$ if and only if every standard bounded measurable function $f\colon X \to \mathbb{R}$ takes approximately equal values on p_1 and p_2.*

Proof. The implication from left to right follows from the fact that $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is an ifp. The converse can be easily obtained by considering characteristic functions of standard measurable sets. ☐

2. Nonatomicity

In this section we present some interpretations of nonatomicity in terms of infinitely fine partitions.

Suppose now that $(\mathcal{X}, \mathcal{A})$ is endowed with a standard finitely additive measure μ. We introduce the following notations:

$$\begin{array}{rclcrcl} P_+ &=& \{p \in P \mid {}^*\mu(p) > 0\} & \qquad & P_0 &=& \{p \in P \mid {}^*\mu(p) = 0\} \\ \mathcal{A}_+ &=& \{A \in \mathcal{A} \mid \mu(A) > 0\} & \qquad & \mathcal{A}_0 &=& \{A \in \mathcal{A} \mid \mu(A) = 0\} \end{array}$$

The elements of P_+ will be referred to as the ***essential*** elements of P. Notice that every points of an essential element of P is random (recall that a point $x \in {}^*\mathcal{X}$ is said to be ***random*** if $x \notin {}^*N$ for each $N \in \mathcal{A}_0$).

2.1. **Definition.** Let p be an essential element of P. We say that p is ***essentially joinable*** if p is equivalent to another essential element of P. We call p ***essentially divisible*** if it can be written as a union of two disjoint sets of positive measure.

Recall that a set $F \in \mathcal{A}_+$ is said to be an ***atom*** of μ if for every measurable set $E \subseteq F$ we have either $\mu(E) = 0$ or $\mu(E) = \mu(F)$. A measure is said to be ***nonatomic*** if it has no atoms. A measure is said to be ***strongly continuous*** if for every $\varepsilon > 0$ there exists a finite measurable partition of $\mathcal{X}$ into sets of measure less than ε. A measure μ is said to be ***strongly nonatomic*** if for every set $F \in \mathcal{A}_+$ and every $0 < c < \mu(F)$ there exists a measurable set $E \subset F$ such that $\mu(E) = c$.

It was shown in [BRBR83] that strong nonatomicity implies strong continuity, strong continuity implies nonatomicity, and all the three properties are equivalent for sigma-additive measures.

2.2**. Theorem.** *The following statements are equivalent:*

(1) *every essential element of P is essentially divisible;*
(2) *every essential element of P is essentially joinable;*
(3) *μ is nonatomic.*

Proof. To prove the implication (1)⇒(2), consider an essential element $p \in P$. By the assumption, it is essentially divisible. Let $q = \{A \in \mathcal{A} \mid p \subset {}^*A\}$, and consider a finite sub-collection $\mathcal{S}$ of q. Let $S = \bigcap \mathcal{S}$. Evidently, S is a standard set and $p \subset {}^*S$. Essential divisibility of p implies that there exists $p_1 \in {}^*A_+$ such that $0 < {}^*\mu(p_1) < {}^*\mu(p) \leqslant \mu(S)$. By the transfer principle there exists a standard set $B \subset S$ such that the both B and $S \setminus B$ are of positive measure. Assume without loss of generality that $p_1 \subset {}^*B$. Obviously there is an essential element $p_2 \subset {}^*S \setminus {}^*B \subset {}^*S$. Since $\mathcal{S}$ was chosen arbitrarily, it follows by saturation that there exists $p' \in P_+$ distinct from p, such that $p' \subseteq {}^*A$ for each $A \in q$, and it follows that $p' \sim p$.

To prove (2)⇒(3) take an arbitrary set $A \in \mathcal{A}_+$. There exists $p \in P_+$ such that $p \subseteq {}^*A$. Since p is essentially joinable, then there is $p' \in P_+$ such that $p' \subset {}^*A \setminus p$ and it follows that ${}^*\mu(p) < \mu(A)$. By using the transfer principle we obtain the conclusion.

The implication (3)⇒(1) is straightforward. □

2.3. **Theorem.** *A measure is strongly continuous if and only if the measure of every element of P is infinitesimal.*

Proof. The proof is elementary. $\square$

2.4. **Theorem.** *A measure μ is strongly nonatomic if and only if* $(\forall p \in P_+)(\forall \lambda \in {}^*[0,1])(\exists p' \in {}^*\mathcal{A}_+)(p' \subset p \;\&\; \frac{{}^*\mu(p')}{{}^*\mu(p)} = \lambda)$.

Proof. The implication from left to right is obvious. To prove the converse implication, fix $F \in \mathcal{A}_+$ and $c \in (0, \mu(F))$. Consider an internal measurable subset E of ${}^*F$ such that

(1) $E = \bigcup\{p \in P \mid p \subset E\}$;
(2) ${}^*\mu(E) \leqslant c$;
(3) $p \nsubseteq E$ implies ${}^*\mu(E \cup p) > c$ for each $p \in P_+$.

Such a set exists because ${}^*F$ is a union of a hyperfinite number of elements of P. We claim that ${}^*\mu(E) = c$. Indeed, if ${}^*\mu(E) < c$, then we could take any $p \subseteq {}^*F \setminus E$ and split it into the disjoint union of p_1 and p_2 so that ${}^*\mu(E \cup p_1) = c$, which would contradict to the assumptions on E. The transfer principle completes the proof. $\square$

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3. Representation of L_∞

In the following section we extend a results obtained by P. Loeb in [Loe72]. For this subsection, let μ be a standard sigma-additive measure, P a hyperfinite ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$, and L_∞ stands for $L_\infty(\mathcal{X}, \mathcal{A}, \mu)$. In [Loe72] Loeb introduced a map $T_0\colon L_\infty \to \mathbb{R}^P$ by the following rule. For every $p \in P_+$ pick a point $c_p \in p$. For $f \in L_\infty$ define

$$(T_0(f))_p = \begin{cases} {}^\circ f(c_p) & \text{if } p \in P_+; \\ 0 & \text{if } p \in P_0. \end{cases}$$

Loeb showed in [Loe72] that the variation of a standard bounded measurable function on each element of an ifp is infinitesimal, so that T_0 does not depend on the choise of c_p's. Loeb proved that a vector

$\mathbf{v} \in \mathbb{R}^P$ equals $T_0(f)$ for some $f \in L_\infty$ if and only if the following three conditions hold:

(1) $\mathbf{v}_p = 0$ for all $p \in P_0$;
(2) $\max\limits_{p \in P} |\mathbf{v}_p|$ is nearstandard;
(3) for each p in P_+ and for each $\varepsilon \in \mathbb{R}_+$ there exists a set $A \in \mathcal{A}$ such that $p \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| < \varepsilon$ for every essential $p' \subseteq {}^*A$.

3.1. **Lemma.** *Condition* (3) *above is equivalent to the following statement:*

(3') $p_1 \sim p_2$ *implies* $\mathbf{v}_{p_1} \approx \mathbf{v}_{p_2}$ *for any* $p_1, p_2 \in P_+$.

Proof. Suppose that $p_1 \sim p_2$ for $p_1, p_2 \in P_+$. Then it follows from (3) that $|\mathbf{v}_{p_1} - \mathbf{v}_{p_2}| < \varepsilon$ for every $\varepsilon \in \mathbb{R}_+$, hence $\mathbf{v}_{p_1} \approx \mathbf{v}_{p_2}$, so that (3) implies (3'). To show the reverse implication, assume that there exists $p \in P_+$ and $\varepsilon \in \mathbb{R}_+$ such that for each $A \in \mathcal{A}$ satisfying $p \subseteq {}^*A$ one can find $p' \in P_+$ such that $p' \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| > \varepsilon$. Let $q = \{A \in \mathcal{A} \mid p \subseteq {}^*A\}$ and consider a finite sub-collection $\mathcal{S}$ of q. Let $S = \bigcap \mathcal{S}$. By the assumption, there exists an essential $p' \subseteq {}^*S$ such that $|\mathbf{v}_p - \mathbf{v}_{p'}| \geqslant \varepsilon$. By the saturation principle there exists an element $p' \in P_+$, such that $p' \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| \geqslant \varepsilon$, but this contradicts to (3'). $\square$

4. Monads of ifp

Again, let P be an ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$. For $p \in P$ we denote by $[p]$ the equivalence class of p, i.e., $[p] = \{p' \in P \mid p' \sim p\}$. The union of the elements of $[p]$ will be denoted by m_p and referred to as a ***monad*** of P. Since "$\sim$" is an equivalence relation, $\mathfrak{M} = \{m_p\}_{p \in P}$ is an partition of ${}^*\mathcal{X}$. It is easy to see that $\mathfrak{M}$ is exactly the partition of $\mathcal{X}$ generated by all standard measurable sets. It implies, in particular, that the monads are independent of a particular ifp.

Let Q be the Stone space of the algebra $\mathcal{A}$. Recall that the points of Q are the ultra-filters on $\mathcal{A}$. It is well known that there exists a canonical boolean algebra isomorphism ι between $\mathcal{A}$ and the algebra $\operatorname{Clop} Q$ of all clopen subsets of Q given by $\iota(A) = \{q \in Q \mid A \in q\}$.

Notice that there is a one-to-one correspondences between the monads of P, the ultra-filters on $\mathcal{A}$, the zero-one measures on $\mathcal{A}$, and the points of Q. Consider a monad m of P. The corresponding ultra-filter q_m is $\{A \in \mathcal{A} \mid m \subseteq {}^*A\}$, i.e., the elements of q_m are exactly the standard measurable sets containing the monad m. On the other hand, since $m = \bigcup [p]$ where $[p]$ is exactly the equivalence class of those elements of P which are contained in each A in q_m, we have $m = \bigcap q_m$.

Any monad m corresponds to some zero-one measure on $\mathcal{A}$ defined by

$$\delta_m(A) = \begin{cases} 1 & \text{if } m \subseteq {}^*A, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $q_m = \{A \in \mathcal{A} \mid \delta_m(A) = 1\}$. Finally, for any monad m we can consider the ultra-filter q_m as a point of the Stone space Q of $\mathcal{A}$. Notice that the set $\bigcup\{\iota(p) \mid p \in P, p \subseteq m\}$ is the usual topological monad of q_m in Q. Notice also, that $\{q_m \mid m \subset {}^*A\} = \iota(A)$ for any A in $\mathcal{A}$.

It follows from [Loe72, Proposition 4.2] that there is a one-to-one correspondence between the monads of P and the multiplicative linear functionals on $L_\infty(X)$. Namely, if m is a monad of P, then the map $f \mapsto {}^{\circ}({}^*f)_{|m}$ is the multiplicative linear functional corresponding to m.

Consider a monad m of P. Since q_m is a standard ultra-filter on $\mathcal{A}$, it follows that ${}^*q_m$ contains exactly one element of P. This element will be referred as the ***central*** element of m, we will denote it by p_m. Obviously $p_m \subseteq m$. We mentioned before that $\{\iota(p) \mid p \in P, p \subseteq m\}$ form a partition of the topological monad of the (standard) point q_m in Q. Then p_m can be characterized as the element of P for which $q_m \in {}^*\iota(p_m)$. Finally, central elements can also be characterized in terms of zero-one measures. Namely, $p = p_m$ if and only if ${}^*\delta_m(p) = 1$. Denote by P_c the collection of all central elements of P.

It will be shown that properties of measures are essentially determined by their values at the central elements. For the rest of this section we suppose that μ is a standard finitely additive measure on $\mathcal{A}$. We assume for simplicity that μ is finite.

4.1. **Theorem.** *If p is a central element with ${}^*\mu(p)$ standard, then*

(1) ${}^*\mu(p') = 0$ *for any noncentral p' equivalent to p;*
(2) *There exists a set $A \in \mathcal{A}$ such that $p \subseteq {}^*A$ and $\mu(A) = {}^*\mu(p)$.*

Proof. It follows from $p \in P_c$ that $p = p_m$ for some monad m of P. Since $p \in {}^*q_m$ and ${}^*\mu(p)$ is standard, then by the transfer principle we can find a standard set $A \in q_m$ such that $\mu(A) = {}^*\mu(p)$. This proves (2). Finally, (1) follows immediately from (2). □

4.2. **Corollary.** *Suppose that p is a central element of measure zero. If $p' \sim p$ then p' is of measure zero.*

4.3. **Theorem.** *If p is a central element of P, then for every standard $\varepsilon > 0$ there exists a standard set $A \in \mathcal{A}$ such that $m_p \subseteq {}^*A$ and $\mu(A) \leqslant {}^*\mu(p_m) + \varepsilon$.*

Proof. Let $\lambda = {}^*\mu(p) + \frac{\varepsilon}{2}$, then ${}^*\mu(p) < {}^\circ\lambda$. Since $p \in {}^*q_m$ then, by the transfer principle, there exists a standard set $A \in q_m$ such that $\mu(A) < {}^\circ\lambda < {}^*\mu(p_m) + \varepsilon$. □

4.4. **Corollary.** *The measure of every noncentral element is infinitesimal. Moreover, the union of any collection of non-central elements of m has infinitesimal measure. Furthermore, if m is a monad of P and $D \subset m$ for some $D \in {}^*\mathcal{A}$ such that $q_m \notin {}^*\imath(D)$, then ${}^*\mu(D) \approx 0$.*

4.5. **Remark.** There is another important consequence of Theorem 4.3. We have noticed before that the monads do not depend on a particular ifp, but only on the algebra. Now it follows immediately from Theorem 4.3 that if m is a monad, then ${}^\circ\big({}^*\mu(p_m)\big)$ does not depend on a particular ifp either.

4.6. **Lemma.** *If p is a central element of P then ${}^*\mu(p) \geqslant {}^\circ\big({}^*\mu(p)\big)$.*

Proof. For every standard $A \in q_{m_p}$ we have $p \subset A$, and, therefore, ${}^\circ\big({}^*\mu(p)\big) \leqslant \mu(A)$. By the transfer principle it follows that ${}^\circ\big({}^*\mu(p)\big) \leqslant {}^*\mu(p)$. $\square$

5. Sobczik-Hammer Decomposition Theorem

The technique of ifp gives us an opportunity to present a simple proof of the Sobczik-Hammer Decomposition Theorem.

5.1. **Theorem** (Sobczik-Hammer Decomposition Theorem). *Let μ be a finite finitely additive measure on a measurable space $(\mathcal{X}, \mathcal{A})$. Then there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $\mathcal{A}$, a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers, and a strongly continuous measure $\bar{\mu}$ on $\mathcal{A}$, such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\mu = \bar{\mu} + \sum_{n=1}^{\infty} a_n\delta_n$. Further, this decomposition is unique up to the order of terms.*

Proof. Let P be a hyperfinite ifp for $(\mathcal{X}, \mathcal{A})$. Let p_1 be an element of P of maximum measure, put $a_1 = {}^\circ\big(({}^*\mu(p_1)\big)$. For δ_1 we take the zero-one measure corresponding to m_{p_1}. If μ is strongly continuous, then $a_1 = 0$ by Theorem 2.3. Otherwise, it follows from Corollary 4.4 that p_1 is central. Then Lemma 4.6 implies that $a_1 \leqslant {}^*\mu(p_1)$. It follows that $\mu_1 = \mu - a_1\delta_1$ is a standard measure which is nonnegative on each element of P, hence μ_1 is nonnegative. We proceed in the similar fashion defining $\mu_{n+1} = \mu_n - a_n\delta_n$, etc. This process may stop after a finite number of steps if $a_n = 0$ for some n. In this case we get $\mu = \sum_{i=1}^{n} a_i\delta_i + \mu_n$ where μ_n is a strongly continuous measure, so that the conclusion of the theorem is satisfied. Otherwise, we and obtain a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of standard non-negative reals, a sequence $(\delta_n)_{n\in\mathbb{N}}$ of standard distinct zero-one measures, and a sequence $(\mu_n)_{n\in\mathbb{N}}$ of standard non-negative measures, such that $\mu_n = \mu - \sum_{i=1}^{n} a_i\delta_i$ for every $n \in \mathbb{N}$. In particular, $\mu(\mathcal{X}) \geqslant \sum_{i=1}^{n} a_i\delta_i(\mathcal{X}) = \sum_{i=1}^{n} a_i$ for every natural $n \in \mathbb{N}$; it follows that $\sum_{i=1}^{\infty} a_i < \infty$.

It can be easily verified that $\bar{\mu}$ defined by $\bar{\mu} = \mu - \sum_{n=1}^{\infty} a_n\delta_n$ is a standard nonnegative measure, and $\bar{\mu} \leqslant \mu_n$ for all $n \in \mathbb{N}$. Assume that $\bar{\mu}$ is not strongly continuous, then there exists $p \in P$ such that $\operatorname{st}({}^*\bar{\mu}(p)) > 0$. Since $\lim_n a_n = 0$, we can find $n \in \mathbb{N}$ such that $\operatorname{st}({}^*\bar{\mu}(p)) > a_n \geqslant \operatorname{st}({}^*\mu_{n-1}(p))$; but this contradicts to the fact that $\bar{\mu} \leqslant \mu_n$ for each $n \in \mathbb{N}$.

We see that the numbers a_n and measures δ_n are completely determined by the values of μ on the central elements of P up to the order. Thus, the constructed decomposition is unique. □

The concept of an infinitely fine partition also gives us an opportunity to prove the Sobczik-Hammer Decomposition Theorem for vector measures. Let Y be a standard Banach space, and let $F\colon \mathcal{A} \to Y$ be a standard Y-valued measure on $(\mathcal{X}, \mathcal{A})$. Recall that the variation of F is a real-valued measure given by $|F|(A) = \sup_\pi \sum_{B\in\pi} \|F(B)\|$, where sup is taken over all finite measurable partitions π of A. A vector measure is said to be nonatomic (strongly continuous, strongly nonatomic) if the same is true for its variation.

To prove a vector analogue of Sobczik-Hammer Theorem we need the following lemma. As usually, we assume that P is a hyperfinite ifp for $(\mathcal{X}, \mathcal{A})$.

5.2. **Lemma.** *If F is a standard vector measure of bounded variation on $(\mathcal{X}, \mathcal{A})$; then $|{}^*F|(p) \approx \|{}^*F(p)\|$ for each $p \in P$.*

Proof. Clearly $|F|(A) \geqslant \|F(A)\|$ for any measurable A. If $p \in P \setminus P_c$ then $|{}^*F|(p) \approx 0$ by Corollary 4.4, so that $|{}^*F|(p) \approx \|{}^*F(p)\|$ holds trivially.

Suppose that $p \in P_c$. Fix an infinitesimal $\varepsilon > 0$ and consider a hyperfinite measurable partition π of p such that $|{}^*F|(p) \leqslant \sum_{p'\in\pi} \|{}^*F(p')\| + \varepsilon$. Then $P \setminus \{p\} \cup \pi$ is again a hyperfinite ifp. Let p_π be the central element of this new ifp, corresponding to the monad of p. By Corollary 4.4 $|{}^*F|(p \setminus p_\pi) \approx 0$, so that

$$\begin{aligned}\big\|{}^*F(p_\pi)\big\| \leqslant |{}^*F|(p) \leqslant \sum_{p'\in\pi} \big\|{}^*F(p')\big\| + \varepsilon \leqslant \sum_{p'\in\pi, p'\neq p_\pi} |{}^*F|(p') + \big\|{}^*F(p_\pi)\big\| + \varepsilon \\ = |{}^*F|(p \setminus p_\pi) + \big\|{}^*F(p_\pi)\big\| + \varepsilon \approx \big\|{}^*F(p_\pi)\big\|.\end{aligned}$$

It follows that $\|{}^*F(p_\pi)\| \approx |{}^*F|(p)$. On the other hand, it follows from ${}^*F(p) = {}^*F(p_\pi) + {}^*F(p \setminus p_\pi)$ and $\|{}^*F(p \setminus p_\pi)\| \leqslant |{}^*F|(p \setminus p_\pi) \approx 0$ that $\|{}^*F(p)\| \approx \|{}^*F(p_\pi)\|$, so that $\|{}^*F(p)\| \approx |{}^*F|(p)$. □

Now we can proceed with the vector-valued version of the Sobczik-Hammer Theorem. A similar result can be found in [KM89], but the proof presented there is based on completely different concepts.

5.3. **Theorem** (Sobczik-Hammer Decomposition Theorem for vector measure). *Let F be a standard Y-valued measure of bounded variation on $(\mathcal{X}, \mathcal{A})$ such that the range of F is relatively compact. Then there exists a strongly additive vector measure $\bar{F}$, a sequence $(x_n)_{n\in\mathbb{N}} \subset E$, and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $(\mathcal{X}, \mathcal{A})$, such that $F = \bar{F} + \sum_{n=1}^\infty x_n\delta_n$. Further, this decomposition is unique up to the order of terms.*

Proof. The proof is analogous to the proof of Theorem 5.1. Consider a hyperfinite ifp P for $(\mathcal{X}, \mathcal{A})$. Let p_1 be an element of P of maximum value of $|{}^*F|$. Since F has compact range we are guaranteed that ${}^*F(p_1)$ is nearstandard. Let $x_1 = \mathrm{st}({}^*F(p_1))$, let δ_1 be the zero-one measure corresponding to $[p]$, and let $F_1 = F - x_1\delta_1$. It follows that $|{}^*F_1|(p) \approx \|{}^*F_1(p_1)\| = \|{}^*F(p_1) - \mathrm{st}(F(p_1))\| \approx 0$. Further, follows from the transfer principle, that $|F_1| \leqslant |F|$.

Iterating this process we obtain standard sequences $(F_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, and $(\delta_n)_{n\in\mathbb{N}}$ such that $F_n = F - \sum_{i=1}^n x_i\delta_i$, $|F_{n+1}| \leqslant |F_n| \leqslant |F|$, and $\|x_{n+1}\| \leqslant \|x_n\|$ for each $n \in \mathbb{N}$. Let $\bar{F} = F - \sum_{i=1}^\infty x_i\delta_1$, then an argument similar to the one in the proof of Theorem 5.1 shows that $\bar{F}$ is strongly continuous. □

6. Horn-Tarsky Theorem

6.1. **Theorem.** *Let $\mathcal{A}$ be an algebra of subsets of a set $\mathcal{X}$, $\mathcal{C}$ a subalgebra of $\mathcal{A}$, and μ a finitely additive measure on $\mathcal{C}$. Then μ can be extended to a finitely additive measure on $\mathcal{A}$.*

Proof. Let $P_{\mathcal{A}}$ and $P_{\mathcal{C}}$ be hyperfinite ifp's for $\mathcal{A}$ and $\mathcal{C}$ respectively. Without loss of generality we can assume $P_{\mathcal{A}}$ is a refinement of $P_{\mathcal{C}}$. Take $p \in P_{\mathcal{C}}$ and let $p_1, p_2, \ldots, p_N$ be a hyperfinite partition of p into elements of $P_{\mathcal{A}}$. Next, we assign a weight $w(p_i) \in {}^*\mathbb{R}_+$ to each p_1 as $1 \leqslant i \leqslant N$ so that $\sum_{i=1}^{N} w(p_i) = {}^*\mu(p)$.

We apply this procedure to every $p \in P_{\mathcal{C}}$. Now each element of $P_{\mathcal{A}}$ is assigned a weight. For a set $A \in \mathcal{A}$ we put

$$\lambda(A) = \operatorname{st}\left(\sum_{p \in P_{\mathcal{A}},\ p \subseteq {}^*A} w(p)\right).$$

It can be easily verified that λ is a standard finitely additive measure and $\lambda_{|\mathcal{C}} = \mu$. □

Obviously, the same reasoning can be used to prove Horn-Tarsky Theorem for a Banach-valued measure, but again we would need the range of the measure be relatively compact so that we could take standard parts when defining λ.

7. Ergodic transformations

Let $(\mathcal{X}, \mathcal{A})$ again be a measurable space, and let $\tau\colon \mathcal{X} \to \mathcal{X}$ be a ***measurable transformation***, i.e., $\tau^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. We say that a set A is ***τ-invariant*** if $\tau^{-1}(A) = A$. If P is an ifp for $(\mathcal{X}, \mathcal{A})$, we say that P is ***τ-invariant*** if $\tau^{-1}(p) \in P$ for each $p \in P$. In this case τ^{-1} induces a permutation of elements of P. Let μ be a probability measure on $(\mathcal{X}, \mathcal{A})$. Recall that a measurable transformation τ is said to be ***measure-preserving*** if $\mu\big(\tau^{-1}(A)\big) = \mu(A)$ for each $A \in \mathcal{A}$, and ***ergodic*** if each τ-invariant set has measure zero or one. If P is an ifp, we write $P_+ = \{p \in P \mid {}^*\mu(p) > 0\}$.

7.1. **Proposition.** *If τ is ergodic and measure preserving, and P is a τ-invariant ifp, then P_+ is hyperfinite and τ-invariant, all the elements of P_+ have the same measure, and τ^{-1} induces a cyclic permutation of elements of P_+.*

Proof. Fix $p \in P_+$ and consider the sequence $\{\tau^{-k}(p) \mid k \in {}^*\mathbb{N}\}$, i.e., the orbit of p in P under τ^{-1}. Since τ is measure-preserving, all the sets in the orbit have the same measure. If all of them were distinct, hence disjoint, this would contradict to μ being a probability measure. Thus, we have $\tau^{-n}(p) = \tau^{-m}(p)$ for some distinct m and n. Without loss of generality we can assume that $m = 0$, i.e., $\tau^{-n}(p) = p$ for some $n \in {}^*\mathbb{N}$. Then $A = \bigcup_{k=0}^{n-1} \tau^{-k}(p)$ is a τ-invariant set of positive measure. Since τ is ergodic, we have $\mu(A) = 1$. This implies that every element of P_+ is contained in A, so that $P_+ = \{\tau^{-k}(p) \mid k = 0, \dots, n-1\}$. It follows immediately that P_+ is hyperfinite and that τ acts as a cyclic permutation of the elements of P_+. □

Unfortunately, one cannot always find an invariant ifp. For example, it is easy to see that no ifp is invarial under a strongly mixing transformation. Recall, that a measure-preserving transformation τ is called a ***strong mixing*** if $\mu\big(A \cap \tau^{-n}(B)\big) \to \mu(A)\mu(B)$ for any two $A, B \in \mathcal{A}$. Indeed, if P were a τ-invariant ifp, then by Proposition 7.1 P has to be hyperfinite, and τ^{-1} induces a cyclic permutation of the elements of P_+. But then $\mu\big(p_1 \cap \tau^{-n}(p_2)\big)$ does not converge for any $p_1, p_2 \in P$, a contradiction.

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E-mail address: `vladimir@math.utexas.edu`