SIMPLE CONSTRUCTIONS OF FBL(A) AND FBL[E]

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Abstract. We show that the free Banach lattice FBL(A) may be constructed as the completion of FVL(A) with respect to the maximal lattice seminorm ν on FVL(A) with $\nu(a) \leqslant 1$ for all $a \in A$. We present a similar construction for the free Banach lattice FBL[E] generated by a Banach space E.

1. Preliminaries

The free vector lattice over a set A, denoted by FVL(A), goes back to [Bir42]. More recently, a free Banach lattice FBL(A) has been introduced and investigated; see [dPW15, ART18]. It has been folklore knowledge (and was implicitly mentioned in [dPW15, ART18]) that the norm of FBL(A) is, in some sense, the greatest lattice norm one can put on FVL(A). In this note, we make this idea into a formal statement and provide a direct proof. This yields an alternative way of constructing FBL(A) and FBL[E].

Let A be a subset of a vector lattice X. We say that X is a free vector lattice over A if every function $\varphi\colon A \to Y$, where Y is an arbitrary vector lattice, extends uniquely to a lattice homomorphism $\tilde{\varphi}\colon X \to Y$. For every set A there is a vector lattice X which contains A and is free over A. It is easy to see that if X_1 and X_2 are both free over A then there exists a lattice isomorphism between X_1 and X_2 which fixes A. So a free vector lattice over X is determined uniquely up to a lattice isomorphism; we denote it by FVL(A).

We outline below a construction of FVL(A) and some of its basic properties; we refer the reader to [Ble73, dPW15] for further details on

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free vector lattices. Given a set A. For every $a\in A$, we write δ_a for the "evaluation functional" of a in the following sense: $\delta_a\colon\mathbb{R}^A\to\mathbb{R}$ with $\delta_a(x)=x(a)$ for $x\colon A\to\mathbb{R}$. Then $\mathrm{FVL}(A)$ may be identified with the sublattice of $\mathbb{R}^{\mathbb{R}^A}$ generated by $\{\delta_a\ :\ a\in A\}$. Identifying a with δ_a, one may view A as a subset of $\mathrm{FVL}(A)$. Since $\mathrm{FVL}(A)$ is a sublattice $\mathbb{R}^{\mathbb{R}^A}$, and the latter is Archimedean, $\mathrm{FVL}(A)$ is also Archimedean.

It is easy to see that if $a_1,\dots,a_n\in A$ then $\mathrm{FVL}\bigl(\{a_1,\dots,a_n\}\bigr)$ may be identified with the sublattice of $\mathrm{FVL}(A)$ generated by $a_1,\dots,a_n$. For every $f\in\mathrm{FVL}(A)$ there exists a finite subset $\{a_1,\dots,a_n\}$ of A such that f belongs to the sublattice of $\mathrm{FVL}(A)$ generated by $a_1,\dots,a_n$. Furthermore, if A itself is finite, say, $A=\{a_1,\dots,a_n\}$, then $\bigvee_{k=1}^n\lvert a_k\rvert$ is a strong unit in $\mathrm{FVL}(A)$.

By a lattice-linear expression, we mean an expression formed by finitely many variables and linear and lattice operations. For example, $F(t_1,t_2,t_3)=t_1\wedge t_2+t_1\vee(2t_3)$ is a lattice-linear expression. Clearly, a lattice-linear expression $F(t_1,\dots,t_n)$ induces a positively homogeneous function from $\mathbb{R}^n$ to $\mathbb{R}$. On the other hand, if X is an Archimedean vector lattice and $x_1,\dots,x_n\in X$, plugging $x_1,\dots,x_n$ into F instead of $t_1,\dots,t_n$, we can define $F(x_1,\dots,x_n)$ as an element of X in a natural way. We say that $F(x_1,\dots,x_n)$ is a lattice-linear combination of $x_1,\dots,x_n$. If two lattice-linear expressions F and G agree as functions from $\mathbb{R}^n$ to $\mathbb{R}$ then $F(x_1,\dots,x_n)=G(x_1,\dots,x_n)$. Actually, the calculus of lattice-linear expressions in X is a restriction of Krivine's function calculus; see, e.g., [BdPvR91, Proposition 3.6]. Observe that the sublattice of X generated by $x_1,\dots,x_n$ is exactly the set of all lattice-linear combinations of $x_1,\dots,x_n$. $\mathrm{FVL}(A)$ may be interpreted as the set of all formal lattice-linear expressions of elements of A, where we identify two expressions if they agree as functions from $\mathbb{R}^n$ to $\mathbb{R}$. For example, we identify $a_1+(a_2\vee a_3)$ and $(a_1+a_2)\vee(a_1+a_3)$. Formally speaking, $\mathrm{FVL}(A)$ consists of equivalence classes of lattice-linear expressions.

In [dPW15], the concept of a free Banach lattice was introduced. Let A be a subset of a Banach lattice X. We say that X is a free Banach lattice over a set A if every function $\varphi\colon A\to Y$, where Y is an arbitrary Banach lattice, satisfying $\sup_{a\in A}\bigl\lVert\varphi(a)\bigr\rVert\leqslant 1$ extends uniquely to a lattice homomorphism $\tilde{\varphi}\colon X \to Y$ with $\|\tilde{\varphi}\| \leqslant 1$. It was shown in [dPW15] that for every set A there is a Banach lattice X which contains A and is free over A. Again, it is easy to see that such a Banach lattice is unique up to a lattice isometry which fixes A; we denote it by FBL(A). It is easy to see that A is a subset of the unit sphere of FBL(A).

An alternative way of constructing FBL(A) was recently obtained in [ART18]. In [ART18], the authors also prove that for every Banach space E there exists a Banach lattice FBL[E] such that E is a closed subspace of FBL[E] and every bounded operator $T\colon E \to Y$, where Y is an arbitrary Banach lattice, extends uniquely to a lattice homomorphism $\widetilde{T}\colon$ FBL[E] $\to Y$ with $\|\widetilde{T}\| = \|T\|$. It is easy to see that FBL[E] is unique up to a lattice isometry preserving E. Furthermore, it can be easily verified that $\mathrm{FBL}(A) = \mathrm{FBL}\big[\ell_1(A)\big]$ for any set A.

In this note, we present constructions of FBL(A) and FBL[E] that are somewhat easier than those in [dPW15, ART18].

2. A construction of FBL(A)

Theorem 2.1. *There exists a maximal lattice seminorm ν on* FVL(A) *with $\nu(a) \leqslant 1$ for all $a \in A$. It is a lattice norm, and the completion of* FVL(A) *with respect to it is* FBL(A).

Proof. As before, we identify FVL(A) with the sublattice of $\mathbb{R}^{\mathbb{R}^A}$ generated by $\{\delta_a \,:\, a \in A\}$; by identifying $a \in A$ with $\delta_a \in \mathrm{FVL}(A)$, we may view A as a subset of FVL(A). Let $\mathcal{N}$ be the set of all lattice seminorms ν on FVL(A) such that $\nu(\delta_a) \leqslant 1$ for every $a \in A$.

Let $x \in \mathbb{R}^A$ such that $|x(a)| \leqslant 1$ for all $a \in A$. For $f \in \mathrm{FVL}(A)$, put $\nu_x(f) = \big|f(x)\big|$. It can be easily verified that $\nu_x \in \mathcal{N}$.

For $f \in \mathrm{FVL}(A)$, put $\|f\| = \sup\limits_{\nu\in\mathcal{N}} \nu(f)$. We claim that this is a lattice norm on FVL(A).

First, observe that $\|f\|$ is finite. Find $a_1,\dots,a_n \in A$ such that $f \in \mathrm{FVL}\big(\{a_1,\dots,a_n\}\big)$. Since $|\delta_{a_1}| + \dots + |\delta_{a_n}|$ is a strong unit in $\mathrm{FVL}\big(\{a_1,\dots,a_n\}\big)$, there exists $\lambda \in \mathbb{R}_+$ such that

$$|f| \leqslant \lambda\big(|\delta_{a_1}| + \dots + |\delta_{a_n}|\big).$$

It follows that $\nu(f)\leqslant\lambda n$ for every $\nu\in\mathcal{N}$, hence $\|f\|\leqslant\lambda n<\infty$.

It is straightforward that $\|\cdot\|$ is positively homogeneous. To verify the triangle inequality, let $f,g\in\mathrm{FVL}(A)$ and fix $\varepsilon>0$. There exists $\nu\in\mathcal{N}$ such that

$$\|f+g\|-\varepsilon<\nu(f+g)\leqslant\nu(f)+\nu(g)\leqslant\|f\|+\|g\|.$$

It follows that $\|f+g\|\leqslant\|f\|+\|g\|$.

Suppose that $f\neq 0$. Find $a_1,\dots,a_n\in A$ such that $f\in\mathrm{FVL}\bigl(\{a_1,\dots,a_n\}\bigr)$. It follows that there is $x\in\mathbb{R}^A$ such that $f(x)\neq 0$ and $\operatorname{supp}x\subseteq\{a_1,\dots,a_n\}$. Without loss of generality, scaling x if necessary, $\lvert x(a)\rvert\leqslant 1$ for all $a\in A$. Then $\nu_x(f)=\bigl\lvert f(x)\bigr\rvert>0$, hence $\|f\|\neq 0$.

If $\lvert f\rvert\leqslant\lvert g\rvert$ in $\mathrm{FVL}(A)$ then $\nu(f)\leqslant\nu(g)$ for every $\nu\in\mathcal{N}$, hence $\|f\|\leqslant\|g\|$. Thus, $\|\cdot\|$ is a lattice norm on $\mathrm{FVL}(A)$.

Let X be the completion of $\bigl(\mathrm{FVL}(A),\|\cdot\|\bigr)$. We claim that X is a free Banach lattice over A.

Let $\varphi\colon A\to Y$, where Y is a Banach lattice and $\sup_{a\in A}\|\varphi(a)\|\leqslant 1$. Then φ extends to a lattice homomorphism $\hat{\varphi}\colon\mathrm{FVL}(A)\to Y$. It suffices to show that $\|\hat{\varphi}\|\leqslant 1$; it would follow that $\hat{\varphi}$ extends to a contractive lattice homomorphism $\tilde{\varphi}$ from X to Y. For $f\in\mathrm{FVL}(A)$, put $\nu(f)=\bigl\|\hat{\varphi}(f)\bigr\|$. It is easy to see that $\nu\in\mathcal{N}$, hence $\bigl\|\hat{\varphi}(f)\bigr\|=\nu(f)\leqslant\|f\|$.

Uniqueness of the extension follows from the fact that any lattice homomorphism extension of φ to X has to agree with $\hat{\varphi}$ on $\mathrm{FVL}(A)$, hence with $\tilde{\varphi}$ on X as $\mathrm{FVL}(A)$ is dense in X. □

It follows that the FBL norm is the greatest norm on $\mathrm{FVL}(A)$ such that $\|a\|\leqslant 1$ for every $a\in A$.

3. A construction of FBL[E]

For a Banach lattice E and a vector $x\in E$, the evaluation functional $\hat{x}\in E^{**}$ is defined by $\hat{x}(x^*)=x^*(x)$ for $x^*\in X^*$. In particular, $\hat{x}$ is a function from E^* to $\mathbb{R}$, i.e., an element of $\mathbb{R}^{E^*}$.

Theorem 3.1. *Let E be a Banach space; let L be the sublattice of $\mathbb{R}^{E^*}$ generated by $\{\hat{x} : x\in E\}$. There is a maximal lattice seminorm ν*

on L satisfying $\nu(\hat{x}) \leqslant \|x\|$ for all $x \in E$. It is a lattice norm; the completion of L with respect to it is FBL[E].

Proof. It is easy to see that the map $x \in E \mapsto \hat{x} \in L$ is a linear embedding, so that we may view E as a linear subspace of L. Let $\mathcal{M}$ be the set of all lattice seminorms ν on L such that $\nu(\hat{x}) \leqslant \|x\|$ for all $x \in E$.

For every $x^* \in B_{E^*}$, define $\nu_{x^*}(f) = \big|f(x^*)\big|$ for $f \in L$. It is easy to see that ν_{x^*} is a seminorm on L. Note that ν_{x^*} is a lattice seminorm because $|f|(x^*) = \big|f(x^*)\big|$. Furthermore, if $x \in E$ then $\nu_{x^*}(\hat{x}) = \big|\hat{x}(x^*)\big| = \big|x^*(x)\big| \leqslant \|x\|$, so that $\nu_{x^*} \in \mathcal{M}$.

For $f \in L$, define $|||f||| = \sup_{\nu\in\mathcal{M}} \nu(f)$. We claim that $|||\cdot|||$ is a lattice norm on L. First, we will show that it is finite. Let $f \in L$. Then f is a lattice-linear expression of $\hat{x}_1, \dots, \hat{x}_n$ for some $x_1, \dots, x_n$ in E. Since lattice-linear functions are positively homogeneous, we may assume without loss of generality that $x_1, \dots, x_n \in B_E$. Clearly, $|\hat{x}_1| + \cdots + |\hat{x}_n|$ is a strong unit in the sublattice of L generated by $\hat{x}_1, \dots, \hat{x}_n$. It follows that $|f| \leqslant \lambda\big(|\hat{x}_1| + \cdots + |\hat{x}_n|\big)$ for some $\lambda > 0$, so that $\nu(f) \leqslant \lambda n$ for every $\nu \in \mathcal{M}$, hence $|||f||| \leqslant \lambda n < \infty$.

It is straightforward that $|||\cdot|||$ is a lattice seminorm on L. Suppose that $0 \neq f \in L$. Then $f(x^*) \neq 0$ for some $x^* \in E^*$. Since f is positively homogeneous, we may assume without loss of generality that $x^* \in B_{E^*}$. Then $|||f||| \geqslant \nu_{x^*}(f) = \big|f(x^*)\big| > 0$. Thus, $|||\cdot|||$ is a lattice norm on L.

Note also that $|||\hat{x}||| = \|x\|$ for all $x \in E$, so that we may view $|||\cdot|||$ as an extension of $\|\cdot\|$ from E to L. Indeed, $\nu(\hat{x}) \leqslant \|x\|$ for every $\nu \in \mathcal{M}$, hence $|||\hat{x}||| \leqslant \|x\|$. On the other hand, let $x^* \in B_{E^*}$ such that $x^*(x) = \|x\|$; then $|||\hat{x}||| \geqslant \nu_{x^*}(\hat{x}) = \big|\hat{x}(x^*)\big| = \|x\|$.

Let X be the completion of $\big(L, |||\cdot|||\big)$. We claim that $X = \mathrm{FBL}[E]$.

Let $T\colon E \to Y$ be a linear operator from E to an arbitrary Banach lattice Y with $\|T\| = 1$. We define $\widehat{T}\colon L \to Y$ as follows. Let $f \in L$. Then f is a lattice-linear combination of $\hat{x}_1, \dots, \hat{x}_n$ for some $x_1, \dots, x_n \in E$. Without loss of generality, $x_1, \dots, x_n$ are linearly independent in E. We define $\widehat{T}f$ to be the same lattice-linear combination of $Tx_1, \dots, Tx_n$ in Y. That is, suppose that $f = F(\hat{x}_1, \dots, \hat{x}_n)$ for some formal lattice-linear expression $F(t_1, \dots, t_n)$; we then put

$\widehat{T}f = F(Tx_1, \dots, Tx_n)$. Note that $\widehat{T}f$ is well-defined, i.e., does not depend on a particular choice of a lattice-linear combination representing f. Indeed, suppose that $f = G(\hat{x}_1, \dots, \hat{x}_n)$, where $G(t_1, \dots, t_n)$ is another formal lattice-linear expressions. Since L is a sublattice of $\mathbb{R}^{E^*}$, the lattice operations in L are point-wise, hence

$$f(x^*) = F\bigl(\hat{x}_1(x^*), \dots, \hat{x}_n(x^*)\bigr) = F\bigl(x^*(x_1), \dots, x^*(x_n)\bigr)$$

in $\mathbb{R}$ for every $x^* \in X^*$. Similarly, $f(x^*) = G\bigl(x^*(x_1), \dots, x^*(x_n)\bigr)$. Since $x_1, \dots, x_n$ are linearly independent, this means that $F(t_1, \dots, t_n) = G(t_1, \dots, t_n)$ for all $t_1, \dots, t_n \in \mathbb{R}$. Therefore, $\widehat{T}$ is well-defined.

The definition of $\widehat{T}$ immediately yields that it is a lattice homomorphism. Clearly, $\widehat{T}$ extends T in the sense that $\widehat{T}\hat{x} = Tx$ for every $x \in E$. It follows that $\|\widehat{T}\| \geqslant 1$. We claim that that $\|\widehat{T}\| = 1$. Indeed, for $f \in L$, define $\nu(f) = \|\widehat{T}f\|$ in Y. It is easy to see that ν is a lattice seminorm on L. For every $x \in X$, one has $\nu(\hat{x}) = \|\widehat{T}\hat{x}\| = \|Tx\| \leqslant \|x\|$. It follows that $\nu \in \mathcal{M}$, so that $\|\widehat{T}f\| = \nu(f) \leqslant |||f|||$, so that $\|\widehat{T}\| \leqslant 1$. It follows that $\widehat{T}$ extends to a contractive lattice homomorphism $\widetilde{T}\colon X \to Y$.

Again, uniqueness of the extension follows from the fact that any contractive lattice homomorphism that extends T to X has to agree with $\widehat{T}$ on L and, therefore, with $\widetilde{T}$ on X. □

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