Options and Efficiency in Spaces of Bounded Claims

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Abstract

In a seminal contribution, Ross (1976) showed that a static finite state-space market can be completed by supplementing the primitive securities with ordinary call and put options. Galvani (2009) extends this result to norm separable L_p-spaces, with $1 \leqslant p < \infty$. This study concludes that options maintain the same spanning power in the space of bounded payoffs topologized by the duality with the space of the state price densities. In particular, under mild assumptions on the probability space, options written on a claim that is a.s. equal to an injective function complete the market.
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1. Introduction

In the finite dimensional setting, Ross (1976) showed that options on an injective contingent claim complete a static securities market in the same

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way that adding Arrow securities would in an incomplete Arrow–Debreu economy.[2] This finding supports the view that the market structure necessary to span all contingent claims needs not to involve a complex set of securities, but rather a large number of ordinary call or put options.[3] (Galvani, 2009, Theorem 1) proved that options maintain the same spanning power in the L_p-spaces, with $1 \leqslant p < \infty$, that are defined over a separable measure algebra of the state-space. A similar result holds with respect to notion of approximation offered by the pointwise convergence of sequences for spaces of measurable functions (Galvani, 2009, Corollary 7). In addition, underlyers for which options obtain market completeness are shown to be dense in the L_p-spaces (Galvani, 2009, Corollaries 6 and 7).[4]

This work analyzes the spanning power of options in spaces of bounded random variables. We identify the space of contingent claims with the space $L_\infty(P)$ of bounded measurable functions that are defined over a probability space. This space is equipped with the weak-star (or w^*) topology defined by the duality with the space of the state-price densities, $L_1(P)$. We choose to equip the L_∞-space with the weak* topology to maintain the equivalence between market completeness and the uniqueness of a strictly positive state price density, under suitable no-arbitrage conditions.[5]

[2]Baptista (2003, 2005) discussed the multi-period model in the finite-dimensional framework.

[3]However, it might be the case that options are not replicated by any portfolio of primitive securities (Aliprantis and Tourky (2002); Baptista (2007)).

[4]Galvani (2005, 2007a,b) discuss the generalization of Ross' spanning proposition for continuous underlying asset in the space of continuous payoffs and in the L_p-spaces.

[5]See the discussion of Artzner and Heath's paradox in Jarrow et al (1999).

Options are said to complete the space of contingent claims $L_\infty(P)$ whenever finite-component portfolios of plain call and put options form a weak* dense subspace of the L_∞-space. One of the results of this study is that the ability of ordinary put and call options to complete the space $L_\infty(P)$ is equivalent to the weak* separability of this space. An implication of this result is that under very mild assumptions—namely that the probability measure on the state-space is atomless—the familiar space $L_\infty[0,1]$ is essentially the only L_∞-space for which an attempt to generalize Ross' spanning proposition is not futile.

This work also sheds some light on the class of underlying assets for which options complete the market. A general result is that these underlyers are pervasive in the sense that they form a weak* dense subset of a weak* separable L_∞-space that is defined on an atomless probability space. A more compelling characterization can be obtained when the state-space is assumed to be a completely separable metric space equipped with the completion of its Borel σ-algebra and, once more, measured by an atomless probability. In this case options on a claim that is a.s. equal to an injective function (i.e., that is a.s. injective) complete the market. This amounts to a direct generalization of Ross' finite-dimensional spanning result for a class of L_∞-spaces that are extremely common in the extant literature. Also, this result extends the findings of Section 3 in Galvani (2009) to $L_\infty(P)$.

The structure of the paper is the following. The next section provides some background. Section 3 presents our main results. A few concluding remarks can be found in Section 4.

2. Background

Throughout this section, (Ω, Σ, P) will be a non-atomic probability space. We write $L_p(P)$ for $L_p(\Omega, \Sigma, P)$ as $1 \leqslant p \leqslant +\infty$. We denote by Σ_P the measure algebra associated to the space equipped with metric induced by the norm of $L_1(P)$. We write $L_p[0,1]$ for the L_p space corresponding to the Lebesgue measure on $[0,1]$. By the weak* (or w^*) topology on $L_\infty(P)$ we mean the topology induced by $L_1(P)$. That is, $f_n \xrightarrow{w^*} f$ in $L_\infty(P)$ if

$$\int_\Omega (f_n - f)x \, dP \to 0$$

as $n \to \infty$ for every x in $L_1(P)$. Recall that the weak* topology of $L_\infty(P)$ is weaker than the norm topology. In particular, if $f_n \to f$ in the norm of $L_\infty(P)$ then $f_n \xrightarrow{w^*} f$, and a subset of $L_\infty(P)$ which is dense in the norm topology is also dense in the weak* topology.

For each claim x in $L_\infty(P)$ we define the option space of x by:

$$\mathcal{O}_x = \operatorname{span}\big\{(x-k)^+ \; : \; k \in \mathbb{R}\big\}.$$

More generally, if J is a countable subset of $L_\infty(P)$, we put

$$\mathcal{O}_J = \operatorname{span}\big\{(x-k)^+ \; : \; x \in J, \; k \in \mathbb{R}\big\},$$

where $\mathcal{O}_J$ is the option space of the set J.[6]

An element x in $L_\infty(P)$ is called a.s. injective if it is a.s. equal to an injective function on Ω, or, more precisely, if x has an injective representative.

[6] Because we can always make positive a claim in $L_\infty(P)$ by adding a constant, in this work we do not need to treat separately the case of options with positive strike prices (c.f., Galvani (2009)).

The following fact is an extension (with the same proof) of Lemma 2 of Galvani (2009) to the case of $L_\infty[0,1]$.

Lemma 1. *The set of all a.s. injective elements in* $L_\infty[0,1]$ *is dense in* $L_\infty[0,1]$ *in the norm and, therefore, in the weak* topology.*

The following fact will be used hereafter.

Lemma 2. *If* M *is a norm dense sublattice of* $L_1[0,1]$ *and* $\mathbf{1} \in M \subseteq L_\infty[0,1]$ *then* M *is weak* dense in* $L_\infty[0,1]$.

Proof. Let $f \in L_\infty[0,1]$, show that there exists a sequence (x_n) in M such that $f_n \xrightarrow{w^*} f$. Since $f = f^+ - f^-$, we may assume without loss of generality that $f \geqslant 0$. Let $\lambda = \|f\|_\infty$. By assumption, there exists a sequence (g_n) in M such that $g_n \xrightarrow{\|\cdot\|_1} f$. Put $f_n = (g_n \wedge \lambda\mathbf{1})^+$, then $0 \leqslant f_n \leqslant \lambda\mathbf{1}$ and $f_n \xrightarrow{\|\cdot\|_1} f$. Also, since M is a sublattice containing $\mathbf{1}$, we have $(f_n) \subseteq M$.

It is left to show that $f_n \xrightarrow{w^*} f$ in $L_\infty[0,1]$. Indeed, take any $x \in L_1[0,1]$. Take any $\varepsilon > 0$. There exists $K \in \mathbb{R}_+$ such that $\int_A |x|\,dP < \frac{\varepsilon}{4\lambda}$, where $A = \{t : |x(t)| > K\}$. We write A^C for the complement of A in $[0,1]$. There exists n_0 such that $\|f_n - f\|_1 < \frac{\varepsilon}{2K}$ whenever $n \geqslant n_0$. It follows that

$$\Big|\int_\Omega (f_n - f)x\,dP\Big| \leqslant \int_\Omega |f_n - f||x|\,dP = \int_A |f_n - f||x|\,dP + \int_{A^C} |f_n - f||x|\,dP$$
$$\leqslant 2\lambda \int_A |x|\,dP + K\int_{A^C} |f_n - f|\,dP \leqslant 2\lambda\frac{\varepsilon}{4\lambda} + K\|f_n - f\|_1 < \varepsilon.$$

□

(Ross, 1976, Theorem 4) proved that options on an injective claim complete the Euclidean space. This result can be generalized to the space $L_\infty[0,1]$, as stated hereafter.

Corollary 3. *If $x \in L_\infty[0,1]$ is a.s. injective then $\mathcal{O}_x$ is weak* dense in $L_\infty[0,1]$.*

Corollary 3 follows immediately from Lemma 2 and Theorem 1 in Galvani (2009).

We conclude this introductory section by stating few classical facts.

Theorem 4. *Suppose that (Ω, Σ, P) is a non-atomic probability space. The following statements are equivalent.*

1. *Σ_P is separable;*
2. *Σ_P is isomorphic to the measure algebra of the Lebesgue measure on $[0,1]$;*
3. *$L_1(P)$ is separable.*
4. *$L_\infty(P)$ is weak* separable.*

In this case, $L_p(P)$ is lattice isometric as a Banach lattice to $L_p[0,1]$ whenever $1 \leqslant p \leqslant +\infty$.

Proof. The equivalences (1)⇔(2)⇔(3) and the statement about $L_p(P)$ can be found in Section 13.5 of Aliprantis and Border (2005); c.f., also, Theorem 15.4 in Royden (1988). The Banach space $L_1(P)$ is weakly compactly generated, see Definition 11.1 in Fabian et al (2001). Hence by Amir–Lindenstrauss Theorem (see Theorem 11.3 in the same book), the density character of $L_1(P)$ and the weak* density character of $L_\infty(P)$ coincide.[7] This gives (3)⇔(4). □

[7]The density character of a topological space X is the minimal cardinality of a dense subset of X.

3. Option Spanning

Let $\mathcal{C}(P)$ be the set of all $x \in L_\infty(P)$ such that $\mathcal{O}_x$ is weak* dense in $L_\infty(P)$. The following can be viewed as an extension of Theorem 4.

Theorem 5. *Suppose that* (Ω, Σ, P) *is a non-atomic probability space. The following statements are equivalent.*

1. $\mathcal{C}(P)$ *is norm dense in* $L_\infty(P)$*;*
2. $\mathcal{C}(P)$ *is weak* dense in* $L_\infty(P)$*;*
3. *there exists* x *in* $L_\infty(P)$ *such that* $\mathcal{O}_x$ *is weak* dense in* $L_\infty(P)$*;*
4. $\mathcal{O}_J$ *is weak* dense in* $L_\infty(P)$ *for some countable set* J *in* $L_\infty(P)$*;*
5. $L_\infty(P)$ *is weak* separable.*

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are trivial.

(4)$\Rightarrow$(5) Suppose that $\mathcal{O}_J$ is weak* dense in $L_\infty(P)$ for some countable set J in $L_\infty(P)$. Similarly to the definition of $\mathcal{O}_J$, let $\mathcal{O}_{\mathbb{Q}J}$ be the set of all the linear combinations with rational coefficients of $(x-k)^+$ where x runs through J and k runs through $\mathbb{Q}$. Clearly, $\mathcal{O}_{\mathbb{Q}J}$ is countable. Since lattice operations are continuous in $L_\infty(P)$, we have $\mathcal{O}_{\mathbb{Q}J}$ is norm dense in $\mathcal{O}_J$. It follows that $\mathcal{O}_{\mathbb{Q}J}$ is weak* dense in $L_\infty(P)$.

(5)$\Rightarrow$(1) Suppose that $L_\infty(P)$ is weak* separable. By Theorem 4, we may assume without loss of generality that P is the Lebesgue measure on $[0,1]$. By Corollary 3, $\mathcal{C}(P)$ contains all a.s. injective elements of $L_\infty[0,1]$; now Lemma 1 completes the proof. $\square$

Theorem 5 establishes the equivalence between the weak* separability of the L_∞-space and options' ability to complete the market. An obvious generalization of the proof shows that only separable L_∞-spaces can be completed by options, so that a generalization of Ross' spanning proposition to nonseparable L_∞-spaces is unfeasible. To make an example, options fail to complete the familiar space of claims $L_\infty[0,1]$ equipped with the norm of the essential supremum.[8] Moreover, the theorem shows that the class of underlying assets for which options complete the market are pervasive, in the sense that they are dense in the space of contingent claims.

Theorem 5, coupled with Theorem 4, also indicates that $L_\infty[0,1]$ is essentially the only L_∞-space for which options might obtain the allocative efficiency of a complete market structure. This uniqueness is defined up to lattice homeomorphisms that preserve the constants, i.e. the riskfree asset's payoff.

The following result, our last before the discussion, characterizes the underlying claims for which options complete $L_\infty(P)$ in terms of injectivity, much as done in Ross (1976) for the finite dimensional case. Of course, since we demand to identify these underlyers by means of a pointwise relationship, we must allow some latitude in what is taken as to be the standard state-space structure, which until now, besides separability, has been left unconstrained. When we add the requirement that the space of states of nature is a complete and separable topological space which is measured by a non-atomic Borel probability, options on a.s. injective claims are shown to complete $L_\infty(P)$. In line with the density results proposed in Theorem 5, these underlying assets form a weak* dense subset of the space of contingent claims.

[8]The space $L_\infty[0,1]$ is non-separable in the norm topology, see, e.g., (Fabian et al, 2001, Proposition 1.27).

Theorem 6. *Let Ω be an uncountable complete separable metric space and P a non-atomic probability Borel measure on Ω.*[9] *Then*

1. *If $x \in L_\infty(P)$ is a.s. injective then $\mathcal{O}_x$ is weak* dense in $L_\infty(P)$;*
2. *The set of all a.s. injective functions is norm dense in $L_\infty(P)$.*

Proof. By (Royden, 1988, Theorem 15.16), our probability space is isomorphic to $[0,1]$ with the Lebesgue measure. Let $\varphi\colon [0,1] \to \Omega$ be such an isomorphism. For f in $L_\infty(P)$, put $Hf = f \circ \varphi$. Clearly, H is a lattice isometry from $L_\infty(P)$ onto $L_\infty[0,1]$. Moreover, suppose that f is an element of $L_\infty(P)$ which is a.s. injective. Without loss of generality, we may consider f to be itself injective. Since φ is one-to-one, Hf is an a.s. injective element of $L_\infty[0,1]$. Therefore, H takes a.s. injective elements of $L_\infty(P)$ into a.s. injective elements of $L_\infty[0,1]$. Similarly, H^{-1} takes a.s. injective elements of $L_\infty[0,1]$ into a.s. injective elements of $L_\infty(P)$. Now (2) and (1) follow from Lemma 1 and Corollary 3, respectively. $\square$

4. Brief Discussion of the Results

In the framework proposed by Green and Jarrow (1987), and Nachman (1987, 1989), a payoff x is efficient with respect to a collection N of at most countable many claims whenever $\sigma(x)$ and $\sigma(N)$ coincide, where $\sigma(N)$ is the σ-algebra generated by the claims in N. A derivative written on one or on more than one portfolios of the securities in N can be identified with a $\sigma(N)$-measurable function. Hence, an efficient asset has the role of summarizing all information that is payoff-relevant for derivatives written on the portfolios

[9] Following (Royden, 1988, p. 406), we assume that Borel measures are complete.

of the N securities. (Nachman, 1989, Corollary 5) proved that options on an asset that is efficient for a collection of N securities are pointwise dense in the space of $\sigma(N)$-measurable claims. In particular, when all claims are also p-integrable, then options on an efficient asset x complete the space of p-integrable and $\sigma(N)$-measurable claims with respect to the L_p-norm. In this terminology, this work shows that an a.s. injective claim is efficient with respect to the entire space of all contingent claims when the state-space is a completely separable probability space equipped with a non-atomic Borel probability. This is because it can be easily shown that an a.s. injective function generates the whole Borel σ-algebra. Hence, among other results, this article presents an extension of Nachman's spanning propositions to spaces of bounded claims.

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