BIBASIC SEQUENCES IN BANACH LATTICES

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ABSTRACT. Given a Schauder basic sequence (x_k) in a Banach lattice, we say that (x_k) is bibasic if the expansion of every vector in $[x_k]$ converges not only in norm, but also in order. We prove that, in this definition, order convergence may be replaced with uniform convergence, with order boundedness of the partial sums, or with norm boundedness of finite suprema of the partial sums.

The results in this paper extend and unify those from the pioneering paper *Order Schauder bases in Banach lattices* by A. Gumenchuk, O. Karlova, and M. Popov. In particular, we are able to characterize bibasic sequences in terms of the bibasis inequality, a result they obtained under certain additional assumptions.

After establishing the aforementioned characterizations of bibasic sequences, we embark on a deeper study of their properties. We show, for example, that they are independent of ambient space, stable under small perturbations, and preserved under sequentially uniformly continuous norm isomorphic embeddings. After this we consider several special kinds of bibasic sequences, including permutable sequences, i.e., sequences for which every permutation is bibasic, and absolute sequences, i.e., sequences where expansions remain convergent after we replace every term with its modulus. We provide several equivalent characterizations of absolute sequences, showing how they relate to bibases and to further modifications of the basis inequality.

We further consider bibasic sequences with unique order expansions. We show that this property does generally depend on ambient space, but not for the inclusion of c_0 into ℓ_∞. We also show that small perturbations of bibases with unique order expansions have unique order expansions, but this is not true if "bibases" is replaced with "bibasic sequences".

In the final section, we consider uo-bibasic sequences, which are obtained by replacing order convergence with uo-convergence in the definition of a bibasic sequence. We show that such sequences are very common.

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1. Preliminaries

Schauder bases and decompositions. In this subsection, we collect notation and basic facts about Schauder bases and decompositions. For details, we refer the reader to [LT77, Sin70, Sin81]. A sequence (x_k) in a Banach space X is said to be a ***(Schauder) basis*** of X if every vector x in X admits a unique decomposition $x=\sum_{k=1}^\infty \alpha_k x_k$, where the series converges in norm. For each n, we define the n-th ***basis projection*** $P_n\colon X\to X$ via $P_n\bigl(\sum_{k=1}^\infty \alpha_k x_k\bigr)=\sum_{k=1}^n \alpha_k x_k$. We define the n-th ***coordinate functional*** x_n^* via $x_n^*\bigl(\sum_{k=1}^\infty \alpha_k x_k\bigr)=\alpha_n$. It is known that the P_n's are uniformly bounded; the number $K=\sup_n\lVert P_n\rVert$ is called the ***basis constant*** of (x_k). A sequence (x_k) in X is called a ***(Schauder) basic sequence*** if it is a basis for its closed linear span $[x_k]$; in this case the P_n's and x_n^*'s are defined on $[x_k]$. It is a standard fact that a sequence (x_k) of non-zero vectors in X is basic iff there exists $K\geqslant 1$ such that

$$\Bigl\lVert\sum_{k=1}^n \alpha_k x_k\Bigr\rVert \leqslant K\Bigl\lVert\sum_{k=1}^m \alpha_k x_k\Bigr\rVert \tag{1}$$

for every $n\leqslant m$ and all scalars $\alpha_1,\dots,\alpha_m$; the least value of the constant K is the basis constant of (x_k).

More generally, suppose that (X_k) is a sequence of closed non-zero subspaces of a Banach space X; let $[X_k]$ be the closed linear span of $\bigcup_{k=1}^\infty X_k$. We say that (X_k) is a ***(Schauder) decomposition*** of $[X_k]$ if every x in $[X_k]$ admits a unique expansion $x=\sum_{k=1}^\infty x_k$, where $x_k\in X_k$ for each k and the series converges in norm. As before, we define the ***canonical projections*** $P_n\colon[X_k]\to[X_k]$ via $P_nx=\sum_{k=1}^n x_k$. These projections are uniformly bounded; moreover, a sequence (X_k) of closed non-zero subspaces of X is a Schauder decomposition iff there exists a constant $K\geqslant 1$ such that

$$\Bigl\lVert\sum_{k=1}^n x_k\Bigr\rVert \leqslant K\Bigl\lVert\sum_{k=1}^m x_k\Bigr\rVert \tag{2}$$

whenever $n\leqslant m$ and $x_k\in X_k$ for all $k=1,\dots,m$; see, e.g., Theorem 15.5 in [Sin81, p. 502]. Clearly, every basic sequence (x_k) induces a Schauder decomposition with $X_k=\operatorname{span}x_k$. We refer the reader to [Sin81, §15] or [LT77, 1.g] for further information on Schauder decompositions. Note that unlike [Sin81, LT77], we do not assume that $[X_k]=X$. The reason is that in this paper X will generally be a Banach lattice, but we will not require $[X_k]$ to form a sublattice.

For our purposes, it is important to note that (1) and (2) may be re-written as follows:

$$\bigvee_{n=1}^{m}\Bigl\|\sum_{k=1}^{n}\alpha_k x_k\Bigr\| \leqslant K\Bigl\|\sum_{k=1}^{m}\alpha_k x_k\Bigr\| \quad\text{and}\quad \bigvee_{n=1}^{m}\Bigl\|\sum_{k=1}^{n}x_k\Bigr\| \leqslant K\Bigl\|\sum_{k=1}^{m}x_k\Bigr\|.$$

Uniform and order convergence. Let X be an Archimedean vector lattice. A net (x_α) ***converges uniformly*** to x, denoted $x_\alpha \xrightarrow{\mathrm{u}} x$, if there exists $e \in X_+$ such that for every $\varepsilon > 0$ there exists α_0 such that $|x_\alpha - x| \leqslant \varepsilon e$ whenever $\alpha \geqslant \alpha_0$. We say that (x_α) ***converges in order*** to x and write $x_\alpha \xrightarrow{\mathrm{o}} x$ if there exists a net (u_γ) (which may have a different index set) such that $u_\gamma \downarrow 0$ and for every γ there exists α_0 such that $|x_\alpha - x| \leqslant u_\gamma$ whenever $\alpha \geqslant \alpha_0$. A sequence (x_n) is said to ***σ-order converge*** to x, written $x_n \xrightarrow{\sigma\mathrm{o}} x$, if there exists a sequence (u_n) such that $u_n \downarrow 0$ and $|x_n - x| \leqslant u_n$ for every n. In some of the literature, σ-order convergence is called "order convergence for sequences". It is easy to see that

$$x_n \xrightarrow{\mathrm{u}} x \quad\Rightarrow\quad x_n \xrightarrow{\sigma\mathrm{o}} x \quad\Rightarrow\quad x_n \xrightarrow{\mathrm{o}} x.$$

Although order convergence and σ-order convergence disagree in general, they agree for sequences in σ-order complete vector lattices. Clearly, uniform convergence implies order convergence; in Banach lattices, uniform convergence implies norm convergence.

Lemma 1.1. *Let (x_k) be a sequence in a Banach lattice X such that the series $\sum_{k=1}^{\infty}\|x_k\|$ converges. Then $x_k \xrightarrow{\mathrm{u}} 0$ and the series $\sum_{k=1}^{\infty} x_k$ converges both in norm and uniformly. In particular, every norm convergent sequence in X has a subsequence which converges uniformly and, therefore, in order.*

Proof. Find a sequence (λ_k) such that $1 \leqslant \lambda_k \uparrow \infty$ and $\sum_{k=1}^{\infty}\lambda_k\|x_k\| < \infty$. Put $u = \sum_{k=1}^{\infty}\lambda_k|x_k|$. Then $|x_k| \leqslant \frac{1}{\lambda_k}u$ for every k, hence $x_k \xrightarrow{\mathrm{u}} 0$. Clearly, the series $\sum_{k=1}^{\infty} x_k$ converges in norm; let x be the sum. Then

$$\Bigl|x - \sum_{k=1}^{n}x_k\Bigr| \leqslant \sum_{k=n+1}^{\infty}|x_k| \leqslant \frac{1}{\lambda_n}\sum_{k=n+1}^{\infty}\lambda_k|x_k| \leqslant \frac{1}{\lambda_n}u,$$

so that the series converges uniformly. □

A Banach lattice X is said to be ***order continuous*** if $x_\alpha \xrightarrow{\mathrm{o}} 0$ implies $x_\alpha \xrightarrow{\|\cdot\|} 0$ for every net (x_α) in X. It follows from the Meyer-Nieberg Theorem that a Banach lattice is order continuous iff $x_n \xrightarrow{\mathrm{o}} 0$ implies $x_n \xrightarrow{\|\cdot\|} 0$ for every sequence (x_n) in X. We say that X is ***σ-order continuous*** if $x_n \xrightarrow{\sigma\mathrm{o}} 0$ implies $x_n \xrightarrow{\|\cdot\|} 0$. It can be easily seen that a Banach lattice is order continuous iff uniform convergence agrees with order convergence on nets iff uniform convergence agrees with order convergence on sequences; a Banach lattice is σ-order continuous iff uniform convergence agrees with σ-order convergence on sequences; see, e.g. [BW80].

We next provide two standard examples to illustrate the varied relationships between uniform, norm, and order convergence.

Example 1.2. Let $X = L_p(\mu)$ where μ is a measure and $1 \leqslant p < \infty$. Then X is order continuous and a sequence (f_k) converges in order to f iff (f_k) is order bounded and (f_k) converges to f almost everywhere (a.e.).

Example 1.3. Let $X = C[0,1]$. It is easy to see that uniform convergence agrees with norm convergence. For each $k \in \mathbb{N}$, let $f_k \in X$ be such that $f_k(0) = 1$, f_k is linear on $[0, \frac{1}{2^k}]$, and f_k vanishes on $[\frac{1}{2^k}, 1]$. Then $f_k \downarrow 0$, hence $f_k \xrightarrow{\mathrm{o}} 0$. However, (f_k) does not converge to zero in norm.

We will write $\sum_{k=1}^{\infty} x_k$, ${}^{\mathrm{o}}\sum_{k=1}^{\infty} x_k$, ${}^{\sigma\mathrm{o}}\sum_{k=1}^{\infty} x_k$, and ${}^{\mathrm{u}}\sum_{k=1}^{\infty} x_k$ for the norm, order, σ-order, and uniformly convergent series, respectively. It follows from the last part of Lemma 1.1 that if both $\sum_{k=1}^{\infty} x_k$ and ${}^{\mathrm{o}}\sum_{k=1}^{\infty} x_k$ converge then they have the same sum. Replacing norm convergence in the definition of a Schauder basis with uniform, order, or σ-order convergence, one obtains the concepts of a uniform, order, and σ-order basis, respectively, in a vector lattice. Note that the concepts of an order and a σ-order basis agree in σ-order complete vector lattices; the concepts of a σ-order and a uniform basis agree in σ-order continuous Banach lattices. Although such bases will not be the focus of this paper, we provide some simple examples that will be used later on.

Example 1.4. Let $X = \ell_p$ with $1 \leqslant p < \infty$ or $X = c_0$. The standard unit vector sequence (e_k) is a Schauder basis, an order basis, and a uniform basis. Note that (e_k) is an order basis in ℓ_∞, though it is neither a Schauder basis nor a uniform basis.

Example 1.5. Let $X = C[0,1]$ and consider the Schauder system (x_k) in $C[0,1]$ as described in, e.g., [LT77, p. 3]. Since uniform and norm convergence agree in $C[0,1]$, (x_k) is a uniform basis. However, it is not an order basis. Indeed, it can be easily verified that there is a sequence of coefficients (α_k) such that the sequence (f_k) in Example 1.3 is a tail of the sequence of partial sums for the series $\sum_{k=1}^{\infty} \alpha_k x_k$. It follows that this series converges in order to zero. Hence, zero has non-unique order expansions.

Example 1.6. Let $X = \ell_1$, put $x_1 = e_1$ and $x_k = -e_{k-1} + e_k$ when $k > 1$. It is easy to see that (x_k) is a Schauder basis of ℓ_1. We claim that it is neither a uniform basis nor an order basis. Consider the series $x = \sum_{k=1}^{\infty} \frac{x_k}{k}$. This series converges in norm, but its partial sums are not order bounded, hence it fails to converge uniformly or in order. It follows that (x_k) is neither an order basis nor a uniform basis because otherwise the uniform and the order expansion of x would have to agree with its norm expansion.

Martingale inequalities. We recall two classical martingale inequalities. Let (f_k) be a martingale in $L_1(P)$ for some probability measure P; let (d_k) be its difference sequence, i.e., $f_n = \sum_{k=1}^n d_k$ for every n. Put $f^* = \sup_k |f_k|$ and $S(f) = \bigl(\sum_{k=1}^\infty d_k^2\bigr)^{\frac{1}{2}}$; these functions are computed pointwise and are called the maximal and the square function of (f_k), respectively. Let $1 \leqslant p < \infty$. It is easy to see that $\|f_k\|_{L_p} \leqslant \|f_{k+1}\|_{L_p}$. If (f_k) is norm bounded in $L_p(P)$ for some $1 < p < \infty$, then Doob's inequality asserts that

$$\|f^*\|_{L_p} \leqslant q \sup_k \|f_k\|_{L_p} \tag{3}$$

where $q = p^*$; see, e.g., Theorem 26.3 in [JP03]. Furthermore, Burkholder-Gundy-Davis inequality asserts that for every $1 \leqslant p < \infty$,

$$C_1 \|S(f)\|_{L_p} \leqslant \|f^*\|_{L_p} \leqslant C_2 \|S(f)\|_{L_p}, \tag{4}$$

where C_1 and C_2 depend only on p; see, e.g., [Dav70].

2. The bibasis theorem

The present paper will center around the following theorem:

Theorem 2.1. *Let (x_k) be a Schauder basic sequence in a Banach lattice X. TFAE:*

(i) $x = \sum_{k=1}^\infty \alpha_k x_k$ *implies* $x = {}^{\mathrm{u}}\sum_{k=1}^\infty \alpha_k x_k$ *for every sequence* (α_k)*;*
(ii) $x = \sum_{k=1}^\infty \alpha_k x_k$ *implies* $x = {}^{\sigma\mathrm{o}}\sum_{k=1}^\infty \alpha_k x_k$ *for every sequence* (α_k)*;*
(iii) $x = \sum_{k=1}^\infty \alpha_k x_k$ *implies* $x = {}^{\mathrm{o}}\sum_{k=1}^\infty \alpha_k x_k$ *for every sequence* (α_k)*;*
(iv) *If* $\sum\limits_{k=1}^\infty \alpha_k x_k$ *converges then its partial sums* $\sum\limits_{k=1}^n \alpha_k x_k$ *are order bounded;*
(v) *If* $\sum\limits_{k=1}^\infty \alpha_k x_k$ *converges then the sequence* $\Bigl(\bigvee\limits_{n=1}^m \Bigl|\sum\limits_{k=1}^n \alpha_k x_k\Bigr|\Bigr)_{m=1}^\infty$ *is norm bounded;*
(vi) $\exists M \geqslant 1 \quad \forall m \in \mathbb{N} \quad \forall \alpha_1, \dots, \alpha_m \in \mathbb{R}$

$$\Bigl\|\bigvee_{n=1}^m \Bigl|\sum_{k=1}^n \alpha_k x_k\Bigr|\Bigr\| \leqslant M \Bigl\|\sum_{k=1}^m \alpha_k x_k\Bigr\|.$$

A basic sequence (x_k) satisfying any (and, therefore, all) of the equivalent conditions in the theorem will be referred to as a ***bibasic sequence*** (this is not to be confused with the concept of a *bibasic system* in the theory of biorthogonal systems; see, e.g., [Sin81, p. 85]). If, in addition, $[x_k] = X$ we will call it a ***bibasis***. The condition in

(vi) will be referred to as the ***bibasis inequality***, and the least value of M for which this inequality is satisfied will be called the ***bibasis constant*** of (x_k). It is easy to see that (i)⇒(ii)⇒(iii)⇒(iv)⇒(v). Instead of proving the rest of the theorem directly, we will deduce it as an immediate corollary of a more general fact in Section 3. Before that, we present a few remarks and corollaries.

Remark 2.2. In the theorem, we assume that (x_k) is a Schauder basic sequence. However, a sequence of non-zero vectors which satisfies the bibasis inequality also satisfies the basis inequality (1), hence is automatically a Schauder basic sequence. Thus, a sequence in $X \setminus \{0\}$ is bibasic iff it satisfies the bibasis inequality.

We emphasize that only X, not $[x_k]$, is assumed to be a lattice. It follows from (v) that the concept of a bibasic sequence does not depend on the ambient space X:

Corollary 2.3. *Let Y be a closed sublattice of a Banach lattice X, and (x_k) a sequence in Y. Then (x_k) is bibasic in Y iff it is bibasic in X.*

The following is immediate.

Corollary 2.4. *(x_k) is a bibasis iff it is both a Schauder basis and a uniform basis. If X is σ-order continuous then (x_k) is a bibasis iff it is both a Schauder basis and a σ-order basis.*

Question 2.5. It is not known whether every uniform basis of a Banach lattice is automatically a Schauder basis; cf [GKP15, Problem 1.3].

Example 2.6. It is now somewhat easier to see that the sequence (x_k) in Example 1.6 is not a uniform basis. Indeed, taking $\alpha_k = 1$ as $k = 1, \dots, m$, it is clear that the bibasis inequality fails, hence (x_k) is not a bibasis and, therefore, is not a uniform basis by Corollary 2.4.

Remark 2.7. It follows from (v) or (vi) that every basic sequence in an AM-space is bibasic. In particular, the Schauder system of $C[0,1]$ in Example 1.5 is a bibasis, even though it is not an order basis.

Question 2.8. Let X be a Banach lattice and suppose that every basic sequence in X is bibasic. Does this imply that X is lattice isomorphic to an AM-space?

Remark 2.9. The concepts of a bibasis and a bibasic sequence were originally introduced in [GKP15]. Formally speaking, the definition in [GKP15] is slightly different: they defined a bibasis as a sequence which is both a Schauder basis and a σ-order basis. For example, the Schauder system of $C[0,1]$ is a bibasis in our sense, but not in the sense of [GKP15]. However, in [GKP15] the authors only consider σ-order continuous spaces, and in this case the two definitions agree by Corollary 2.4, so that all the results of [GKP15] remain valid for our definition.

In [GKP15], it was proved that if X is σ-order continuous then every bibasis satisfies the bibasis inequality, which corresponds to the implication (ii)⇒(vi) of Theorem 2.1. They also proved (vi)⇒(ii) under certain additional assumptions. Thus, our Theorem 2.1 improves the results of [GKP15]: we make no assumptions on X, we add conditions (i), (iii), (iv), and (v), and our proof is shorter.

Question 2.10. We do not know whether the definition of a bibasis in [GKP15] implies our definition in an arbitrary Banach lattice. Equivalently, if (x_k) is a Schauder basis and a σ-order basis, do the coefficients in the norm and in the order expansions of the same vector agree?

Example 2.11. Let $X = L_p[0,1]$; let (h_k) be the Haar system in $L_1[0,1]$ as described in [LT77, p. 3]. In particular, (h_k) is a monotone Schauder basis in X. It was shown in [GKP15] that the Haar system (h_k) fails to be a bibasis in $L_1[0,1]$. It was also shown there that (h_k) is a bibasis when $1 < p < \infty$, and its bibasis constant satisfies $M_p \geqslant \left(1 + \frac{1}{2^p - 2}\right)^{\frac{1}{p}}$. We present an alternative approach to this problem. Let $1 < p < \infty$. Fix $\alpha_1, \dots, \alpha_m$, put $x_n = \sum_{k=1}^n \alpha_k h_k$ as $n = 1, \dots, m$. It is easy to see that $(x_n)_{n=1}^m$ is a martingale. By Doob's Inequality (3),

$$\Big\| \bigvee_{n=1}^{m} |x_n| \Big\|_{L_p} \leqslant q \|x_m\|_{L_p}.$$

This yields that (h_k) satisfies the bibasis inequality with $M_p = q$. Furthermore, it was shown in [Burk91, p. 15] that the constant q is sharp in Doob's inequality even for dyadic martingales. Since dyadic martingales are of the form $\left(\sum_{k=1}^n \alpha_k h_k\right)_{n=1}^{\infty}$, it follows that the bibasis constant of (h_k) in $L_p[0,1]$ equals q. This argument also shows that every martingale difference sequence in $L_p(P)$ with $1 < p < \infty$ is bibasic.

We finish this section with a comment about ambient space. As observed in Corollary 2.3, the concept of a bibasic sequence does not depend on the ambient space because, according to parts (v) and (vi) of Theorem 2.1, bibasic sequences may be characterized in terms of the norm and lattice operations only, and the latter do not depend on the ambient space. Parts (ii), (iii), and (iv) characterize bibasic sequences in terms of order convergence, σ-order convergence, and order boundedness; these three concepts may depend on ambient space. For example, the standard unit vector basis (e_k) of c_0 is neither order convergent nor even order bounded, yet it is order null when viewed as a sequence in ℓ_∞. So it is somewhat surprising that order convergence and order boundedness of bibasic expansions in (ii), (iii), and (iv) do not depend on the ambient space. This leaves (i): does uniform convergence depend on the ambient space? It is easy to see that uniform convergence in a sublattice implies uniform convergence in the entire space; however, the converse is false in the category of vector lattices. For example, the sequence $\frac{1}{k}e_k$ is uniformly null in ℓ_∞, but not in ℓ_1. Nevertheless, the following proposition shows that uniform convergence does not depend on the ambient space in the category of Banach lattices.

Proposition 2.12. *Let Y be a closed sublattice of a Banach lattice X and (x_k) a sequence in Y. Then $x_k \xrightarrow{\mathrm{u}} 0$ in Y iff $x_k \xrightarrow{\mathrm{u}} 0$ in X.*

Proof. The forward implication is trivial. Suppose $x_k \xrightarrow{\mathrm{u}} 0$ in X. Find $e \in X_+$ such that (x_k) converges to zero uniformly relative to e. WLOG, scaling everything, we may assume that $\|e\| = 1$. For every n there exists k_n such that $|x_k| \leqslant \frac{1}{n^3}e$ for all $k \geqslant k_n$. WLOG, (k_n) is an increasing sequence. For every n, put $v_n = \bigvee_{k=k_n}^{k_{n+1}-1}|x_k|$. Then $v_n \leqslant \frac{1}{n^3}e$ and, therefore, $\|v_n\| \leqslant \frac{1}{n^3}$. It follows that the series $w := \sum_{n=1}^\infty n v_n$ converges and $w \in Y$. It is left to show that (x_k) converges to zero uniformly relative to w. Let $n \in \mathbb{N}$. Take any $k \geqslant k_n$. Find $m \geqslant n$ such that $k_m \leqslant k < k_{m+1}$. Then $|x_k| \leqslant v_m \leqslant \frac{1}{m}w \leqslant \frac{1}{n}w$. □

The preceding proposition fails for nets. Consider the double sequence $x_{n,m} = \frac{1}{n}e_m$ in ℓ_∞. It follows from $x_{n,m} \leqslant \frac{1}{n}\mathbb{1}$ that $x_{n,m} \xrightarrow{\mathrm{u}} 0$ in ℓ_∞. However, viewed as a net in c_0, its tails are not order bounded, hence it fails to converge uniformly to zero.

It is also worth mentioning that the preceding proposition remains valid for uniformly closed sublattices of uniformly complete vector lattices, with essentially the same proof.

3. Bidecompositions

From now on, when possible, we will work in the language of decompositions. In particular, the results apply to basic sequences. However, we find the language of decompositions more natural and clear for our purposes.

Theorem 3.1. *Let X be a Banach lattice and $(X_k) \subseteq X$ a Schauder decomposition of $[X_k]$. Let $P_n\colon [X_k] \to [X_k]$ be the n-th canonical projection. TFAE:*

(i) *For all $x \in [X_k]$, $P_n x \xrightarrow{\mathrm{u}} x$;*
(ii) *For all $x \in [X_k]$, $P_n x \xrightarrow{\sigma\mathrm{o}} x$;*
(iii) *For all $x \in [X_k]$, $P_n x \xrightarrow{\mathrm{o}} x$;*
(iv) *For all $x \in [X_k]$, $(P_n x)$ is order bounded in X;*

(v) *For all* $x \in [X_k]$, $\big(\bigvee_{n=1}^m |P_n x|\big)_{m=1}^\infty$ *is norm bounded;*

(vi) *There exists* $M \geqslant 1$ *such that for any* $m \in \mathbb{N}$ *and any* $x_1 \in X_1, \ldots, x_m \in X_m$ *one has*

$$\Big\| \bigvee_{n=1}^m \Big| \sum_{k=1}^n x_k \Big| \Big\| \leqslant M \Big\| \sum_{k=1}^m x_k \Big\|. \tag{5}$$

Proof. (i)⇒(ii)⇒(iii)⇒(iv)⇒(v) is straightforward.

(v)⇒(vi): For every $i \in \mathbb{N}$, put

$$F_i = \Big\{ x \in [X_k] \,:\, \sup_m \Big\| \bigvee_{n=1}^m |P_n x| \Big\| \leqslant i \Big\} = \bigcap_{m=1}^\infty \Big\{ x \in [X_k] \,:\, \Big\| \bigvee_{n=1}^m |P_n x| \Big\| \leqslant i \Big\}.$$

Continuity of the canonical projections and lattice operations yields that each F_i is closed. It follows from (v) that $\bigcup_{i=1}^\infty F_i = [X_k]$. By Baire Category theorem, there exists i_0 such that F_{i_0} has non-empty interior relative to $[X_k]$. That is, there exists $x_0 \in F_{i_0}$ and $\varepsilon > 0$ such that $x \in F_{i_0}$ whenever $x \in [X_k]$ and $\|x - x_0\| \leqslant \varepsilon$. Let $x \in [X_k]$ with $\|x\| \leqslant 1$. For each $n \in \mathbb{N}$, the triangle inequality yields $\varepsilon |P_n x| \leqslant |P_n x_0| + |P_n(x_0 + \varepsilon x)|$. It follows that

$$\varepsilon \bigvee_{n=1}^m |P_n x| \leqslant \bigvee_{n=1}^m |P_n x_0| + \bigvee_{n=1}^m |P_n(x_0 + \varepsilon x)|$$

for each $m \in \mathbb{N}$, so that $\varepsilon \Big\| \bigvee_{n=1}^m |P_n x| \Big\| \leqslant 2i_0$. This yields

$$\Big\| \bigvee_{n=1}^m |P_n x| \Big\| \leqslant \frac{2i_0}{\varepsilon} \|x\|$$

for all $x \in [X_k]$ and $m \in \mathbb{N}$. Now given m and $x_1 \in X_1, \ldots, x_m \in X_m$, define $x = \sum_{k=1}^m x_k$ to get (vi) with $M = \frac{2i_0}{\varepsilon}$.

(vi)⇒(i): Let $x \in [X_k]$ and let (x_k) be the unique sequence such that $x_k \in X_k$ for every k and $P_n x = \sum_{k=1}^n x_k \xrightarrow{\|\cdot\|} x$. Then there is a subsequence $(P_{n_m} x)$ such that $P_{n_m} x \xrightarrow{\mathrm{u}} x$. WLOG, passing to a further subsequence and using that $(P_n x)$ is norm Cauchy, we may assume that $\big\| \sum_{k=n_m+1}^i x_k \big\| < \frac{1}{2^m}$ whenever $i > n_m$.

For every $m \in \mathbb{N}$, define

$$u_m = \bigvee_{i=n_m+1}^{n_{m+1}} \Big| \sum_{k=n_m+1}^i x_k \Big|.$$

Applying (vi) to the sequence $(0, \ldots, 0, x_{n_m+1}, x_{n_m+2}, \ldots, x_{n_{m+1}})$ with n_m zeros at the beginning yields

$$\|u_m\| \leqslant M \Big\| \sum_{k=n_m+1}^{n_{m+1}} x_k \Big\| < \frac{M}{2^m}.$$

Define $u=\sum_{m=1}^{\infty} m u_m$; it follows from $u_m \leqslant \frac{u}{m}$ that $u_m \xrightarrow{\mathrm{u}} 0$.

Therefore, $|P_{n_m}x - x| + u_m \xrightarrow{\mathrm{u}} 0$. It follows that there is a vector $e > 0$ with the property that for any $\varepsilon > 0$ there exists m_0 such that $|P_{n_m}x - x| + u_m \leqslant \varepsilon e$ whenever $m \geqslant m_0$. Fix $\varepsilon > 0$, and find the required m_0. Let $i \in \mathbb{N}$ with $i > n_{m_0}$. Then we can find $m \geqslant m_0$ such that $n_m < i \leqslant n_{m+1}$, so that

$$|P_i x - x| \leqslant |P_{n_m}x - x| + |P_i x - P_{n_m}x| = |P_{n_m}x - x| + \Bigl|\sum_{k=n_m+1}^{i} x_k\Bigr| \leqslant |P_{n_m}x - x| + u_m \leqslant \varepsilon e.$$

This shows that $P_i x \xrightarrow{\mathrm{u}} x$. □

A Schauder decomposition (X_k) satisfying the equivalent conditions of Theorem 3.1 will be referred to as a ***bidecomposition***, and the least value of M in (vi) will be called the ***bidecomposition constant*** of (X_k). Note that each X_k (as well as $[X_k]$) is a closed subspace of X which need not be a sublattice.

As in Corollary 2.3, it follows immediately from (v) that the definition of a bidecomposition does not depend on ambient space. In particular, (X_k) is a bidecomposition in X iff it is a bidecomposition in X^{**}.

The following is an analogue of Remark 2.2:

Corollary 3.2. *Let (X_k) be a sequence of closed non-zero subspaces of a Banach lattice X. Then (X_k) is a bidecomposition iff it satisfies* (vi)*.*

Clearly, every bibasic sequence (x_k) induces a bidecomposition (X_k) with $X_k = \operatorname{span} x_k$, hence Theorem 2.1 is a special case of Theorem 3.1. Furthermore, let (X_k) be a bidecomposition; for each k, pick a non-zero vector $x_k \in X_k$. By Theorem 3.1(vi), the resulting basic sequence (x_k) satisfies the bibasis inequality, hence is bibasic. To show a partial converse, we will use the following known fact; see Theorems 15.21(b) and 15.22(4) in [Sin81], pp. 543 and 546, respectively.

Theorem 3.3 ([Sin81])**.** *Let (X_k) be a sequence of closed non-zero subspaces of a Banach space X. Suppose that every sequence (x_k) satisfying $0 \neq x_k \in X_k$ is Schauder basic. Then there exists $N \in \mathbb{N}$ such that the sequence $(X_k)_{k \geqslant N}$ is a Schauder decomposition. If $\dim X_k < \infty$ for every k then one may choose $N = 1$.*

We will now prove a similar fact for bidecompositions and bibasic sequences.

Theorem 3.4. *Let (X_k) be a sequence of closed non-zero subspaces of a Banach lattice X. Suppose that every sequence (x_k) satisfying $0 \neq x_k \in X_k$ is bibasic. Then* *there exists $N \in \mathbb{N}$ such that the sequence $(X_k)_{k \geqslant N}$ is a bidecomposition. Moreover, if the sequence (X_k) is a Schauder decomposition or if $\dim X_k < \infty$ for every k then one may choose $N = 1$.*

Proof. By Theorem 3.3, there exists $N \in \mathbb{N}$ such that the sequence $(X_k)_{k \geqslant N}$ is a Schauder decomposition (in the case when (X_k) is already a Schauder decomposition or when $\dim X_k < \infty$ for every k, take $N = 1$). It suffices to show that the sequence $(X_k)_{k \geqslant N}$ satisfies (i) in Theorem 3.1. Let $x \in [X_k]_{k \geqslant N}$. For every $k \geqslant N$, there exists a unique $x_k \in X_k$ such that $x = \sum_{k=N}^{\infty} x_k$. For every $k \in \mathbb{N}$, if $k \geqslant N$ and $x_k \neq 0$ then put $y_k = x_k$ and $\alpha_k = 1$; if $k < N$ or $x_k = 0$, put $\alpha_k = 0$ and let y_k be an arbitrary non-zero element of X_k. By assumption, (y_k) is bibasic. Then $x = \sum_{k=N}^{\infty} x_k = \sum_{k=1}^{\infty} \alpha_k y_k$ and Theorem 2.1(i) guarantees that the series converges uniformly. □

4. Stability of bibasic sequences under perturbations

It was proved in Theorem 3.1 of [GKP15] that if X is σ-order continuous then every block sequence of a bibasic sequence is again bibasic, and the bibasis constant of the block sequence does not exceed that of the original sequence. In particular, every subsequence of a bibasic sequence is again bibasic. This result now follows immediately from the bibasis inequality in Theorem 2.1; moreover, if one uses our definition of a bibasic sequence then the σ-order continuity assumption is not needed:

Corollary 4.1. *Let (x_k) be a bibasic sequence in a Banach lattice. Then every block sequence of (x_k) is bibasic with a bibasis constant that does not exceed that of (x_k). Similarly, every blocking of a bidecomposition is a bidecomposition.*

The following result improves Theorem 3.2 of [GKP15]: we remove the assumption that the space is σ-order continuous and we add an estimate on the bibasis constant. Essentially, we show that a small perturbation of a bibasic sequence causes a small perturbation of the bibasis constant.

Theorem 4.2. *Let (x_k) be a bibasic sequence in a Banach lattice X with basis constant K and bibasis constant M. Let (y_k) be a sequence in X with*

$$2K \sum_{k=1}^{\infty} \frac{\|x_k - y_k\|}{\|x_k\|} =: \theta < 1.$$

Then (y_k) is bibasic with bibasis constant at most $\frac{M+\theta}{1-\theta}$.

Proof. Fix scalars $\alpha_1,\dots,\alpha_m$; put $x=\sum_{k=1}^m\alpha_k x_k$ and $y=\sum_{k=1}^m\alpha_k y_k$. Note that $|\alpha_k|\leqslant\frac{2K\|x\|}{\|x_k\|}$ as $k=1,\dots,m$. Then

$$\|x-y\|\leqslant\sum_{k=1}^m|\alpha_k|\|x_k-y_k\|\leqslant 2K\|x\|\sum_{k=1}^m\frac{\|x_k-y_k\|}{\|x_k\|}\leqslant\theta\|x\|.$$

This implies that $\|x\|\leqslant\|x-y\|+\|y\|\leqslant\theta\|x\|+\|y\|$, so that $\|x\|\leqslant\frac{\|y\|}{1-\theta}$. Define $u:=\sum_{k=1}^\infty\frac{|x_k-y_k|}{\|x_k\|}$. Then $\|u\|\leqslant\frac{\theta}{2K}$. For every $n=1,\dots,m$, we have

$$\Bigl|\sum_{k=1}^n\alpha_k y_k\Bigr|\leqslant\Bigl|\sum_{k=1}^n\alpha_k x_k\Bigr|+\sum_{k=1}^n|\alpha_k|\cdot|x_k-y_k|\leqslant\Bigl|\sum_{k=1}^n\alpha_k x_k\Bigr|+2K\|x\|u.$$

Therefore,

$$\bigvee_{n=1}^m\Bigl|\sum_{k=1}^n\alpha_k y_k\Bigr|\leqslant\bigvee_{n=1}^m\Bigl|\sum_{k=1}^n\alpha_k x_k\Bigr|+2K\|x\|u,$$

which yields, after an application of the bibasis inequality for (x_k),

$$\Bigl\|\bigvee_{n=1}^m\Bigl|\sum_{k=1}^n\alpha_k y_k\Bigr|\Bigr\|\leqslant M\|x\|+2K\|x\|\|u\|\leqslant\frac{M+\theta}{1-\theta}\|y\|.$$

Therefore, (y_k) satisfies the bibasis inequality and the conclusion follows. □

Corollary 4.3. *Let (x_k) be a bibasic sequence in a Banach lattice X. Then every closed infinite-dimensional subspace of $[x_k]$ contains a bibasic sequence.*

Proof. Let Y be a closed infinite-dimensional subspace of $[x_k]$. Using Bessaga-Pełczyński's selection principle (see, e.g., Proposition 1.a.11 in [LT77]), one can find a basic sequence (y_k) in Y and a block sequence (u_k) of (x_k) such that $\|y_k-u_k\|\to 0$ sufficiently fast, so that (y_k) is bibasic by Theorem 4.2 (note that (u_k) is bibasic by Corollary 4.1). □

Remark 4.4. Since disjoint sequences are bibasic, it is clear that every Banach lattice contains a bibasic sequence. It is open whether every closed infinite dimensional subspace of a Banach lattice contains a bibasic sequence. By Corollary 4.3, this is the case when X itself has a bibasis. We will come back to this problem in the final section of the paper.

Example 4.5. *Bibasic sequences are not stable under duality.* Let $X=c_0$ and $x_k=\sum_{n=1}^k e_n$. Being a basis of c_0, (x_k) is a bibasis by Remark 2.7. Its coordinate functionals satisfy $x_k^*=e_k^*-e_{k+1}^*$. As in Examples 1.6 and 2.6, (x_k^*) fails to be bibasic.

5. Stability of bibasic sequences under operators

Suppose that $T\colon X \to Y$ is an isomorphic embedding between Banach lattices. Then, clearly, T maps basic sequences in X to basic sequences in Y. What additional requirements should one impose on T to ensure that T maps bibasic sequences to bibasic sequences? Theorem 3.1 suggests that one look at operators that preserve uniform convergence, or at least turn uniform convergence into order convergence. Inspired by this, we characterize order bounded operators in a way that mimics the bibasis theorem.

Theorem 5.1. *Let $T\colon X \to Y$ be a linear operator between Archimedean vector lattices. TFAE:*

(i) *T is order bounded;*
(ii) *$x_\alpha \xrightarrow{\mathrm{u}} 0$ implies $Tx_\alpha \xrightarrow{\mathrm{u}} 0$ for all nets (x_α) in X;*
(iii) *$x_\alpha \xrightarrow{\mathrm{u}} 0$ implies $Tx_\alpha \xrightarrow{\mathrm{o}} 0$ for all nets (x_α) in X;*
(iv) *$x_\alpha \xrightarrow{\mathrm{u}} 0$ implies that (Tx_α) has an order bounded tail for all nets (x_α) in X.*

Proof. (i)⇒(ii) Suppose that T is order bounded and $x_\alpha \xrightarrow{\mathrm{u}} 0$. Then there exists $e \in X_+$ such that for every $\varepsilon > 0$ there exists α_0 such that $|x_\alpha| \leqslant \varepsilon e$ whenever $\alpha \geqslant \alpha_0$. Find $a \in Y_+$ with $T[-e,e] \subseteq [-a,a]$. Then for ε and α_0 as above, we have $|Tx_\alpha| \leqslant \varepsilon a$ for all $\alpha \geqslant \alpha_0$. This shows that $Tx_\alpha \xrightarrow{\mathrm{u}} 0$.

(ii)⇒(iii)⇒(iv) is trivial.

(iv)⇒(i) Take $e \in X_+$; it suffices to show that $T[0,e]$ is order bounded in Y. Let $\Lambda = \big\{(n,x)\ :\ n \in \mathbb{N},\ x \in [0,e]\big\}$ ordered lexicographically: $(n,x) \leqslant (m,y)$ whenever $n < m$ or $n = m$ and $x \leqslant y$. Clearly, Λ is a directed set. Consider the following net in X indexed by Λ: $v_{(n,x)} = \frac{1}{n}x$. It follows from $0 \leqslant v_{(n,x)} \leqslant \frac{1}{n}e$ that $v_{(n,x)} \xrightarrow{\mathrm{u}} 0$. By assumption, the net $(Tv_{(n,x)})$ has an order bounded tail, i.e., there exist $n_0 \in \mathbb{N}$, $x_0 \in [0,e]$, and $u \in Y_+$ such that $Tv_{(n,x)} \in [-u,u]$ whenever $(n,x) \geqslant (n_0,x_0)$. In particular, $\frac{1}{n_0+1}Tx = Tv_{(n_0+1,x)} \in [-u,u]$ for all $x \in [0,e]$. This shows that $T[0,e]$ is order bounded in Y. □

Remark 5.2. This theorem yields a simple proof of the classical fact that every order bounded (and, in particular, every positive) operator from a Banach lattice to a normed lattice is norm continuous. Indeed, let $T\colon X \to Y$ be such an operator. Suppose that $x_n \xrightarrow{\|\cdot\|} 0$ in X; we need to show that $Tx_n \xrightarrow{\|\cdot\|} 0$ in Y. Suppose not, then, after passing to a subsequence, we can find $\varepsilon > 0$ such that $\|Tx_n\| > \varepsilon$ for all n. Since $x_n \xrightarrow{\|\cdot\|} 0$, passing to a further subsequence, we may assume that $x_n \xrightarrow{\mathrm{u}} 0$. Theorem 5.1 yields that $Tx_n \xrightarrow{\mathrm{u}} 0$ and, therefore, $Tx_n \xrightarrow{\|\cdot\|} 0$, which contradicts $\|Tx_n\| > \varepsilon$ for all n.

We are going to show next that the sequential analogues of the conditions (ii), (iii), and (iv) in Theorem 5.1 are also equivalent, even though they do not imply order boundedness. Moreover, we can consider operators defined on a subspace Z of X instead of all of X. For a sequence (x_n) in Z, the notation $x_n \xrightarrow{\mathrm{u}} 0$ means that the sequence converges to zero uniformly in X.

Proposition 5.3. *Let X and Y be two Archimedean vector lattices, Z a subspace of X, and $T\colon Z \to Y$ a linear operator. TFAE:*

(i) $x_n \xrightarrow{\mathrm{u}} 0$ *implies* $Tx_n \xrightarrow{\mathrm{u}} 0$ *for all sequences* (x_n) *in* Z*;*
(ii) $x_n \xrightarrow{\mathrm{u}} 0$ *implies* $Tx_n \xrightarrow{\mathrm{o}} 0$ *for all sequences* (x_n) *in* Z*;*
(iii) $x_n \xrightarrow{\mathrm{u}} 0$ *implies* (Tx_n) *is order bounded for all sequences* (x_n) *in* Z*.*

Proof. (i)⇒(ii)⇒(iii) is trivial. Suppose that (iii) holds; let (x_n) be a sequence in Z such that $x_n \xrightarrow{\mathrm{u}} 0$ in X. It is easy to see that there is a sequence (λ_n) in $\mathbb{R}_+$ such that $\lambda_n \uparrow \infty$ and $\lambda_n x_n \xrightarrow{\mathrm{u}} 0$. By assumption, the sequence $T(\lambda_n x_n)$ is order bounded. Let $a \in Y_+$ such that $T(\lambda_n x_n) \in [-a, a]$ for every n. It follows that $|Tx_n| \leqslant \frac{1}{\lambda_n} a$, so that $Tx_n \xrightarrow{\mathrm{u}} 0$. □

An operator which satisfies the equivalent conditions of Proposition 5.3 will be called ***sequentially uniformly continuous***. By Theorem 5.1, every order bounded operator is sequentially uniformly continuous. Remark 5.2 shows that every sequentially uniformly continuous operator from a closed subspace of a Banach lattice to a Banach lattice is norm continuous.

Proposition 5.4. *Let $T\colon Z \to Y$ be an operator from a closed subspace Z of a Banach lattice X to a Banach lattice Y. Then T is sequentially uniformly continuous iff the sequence $\bigl(\bigvee_{k=1}^{n} |Tx_k|\bigr)_n$ is norm bounded whenever $x_k \xrightarrow{\mathrm{u}} 0$.*

Proof. The forward implication is an immediate corollary of Proposition 5.3. To prove the converse, suppose that $x_k \xrightarrow{\mathrm{u}} 0$ implies that the sequence $\bigl(\bigvee_{k=1}^{n} |Tx_k|\bigr)_n$ is norm bounded. Then this sequence is also norm bounded in Y^{**}. Since Y^{**} is monotonically complete by, e.g., Proposition 2.4.19(ii) in [MN91], this sequence is order bounded in Y^{**}, hence (Tx_k) is order bounded in Y^{**}. By Proposition 5.3, T is sequentially uniformly continuous as an operator from Z to Y^{**}. Hence, if $x_k \xrightarrow{\mathrm{u}} 0$ then $Tx_k \xrightarrow{\mathrm{u}} 0$ in Y^{**}. Since Y is a closed sublattice in Y^{**}, it follows from Proposition 2.12 that $Tx_k \xrightarrow{\mathrm{u}} 0$ in Y. Therefore, $T\colon Z \to Y$ is sequentially uniformly continuous. □

Example 5.5. *A sequentially uniformly continuous operator which fails to be order bounded.* Let $T\colon c \to c_0$, defined by

$$T\colon (a_1, a_2, \dots) \mapsto (a_\infty, a_\infty - a_1, a_\infty - a_2, \dots), \text{ where } a_\infty = \lim_n a_n.$$

Note that $T\colon (0,\dots,0,1,1,\dots) \mapsto (1,\dots,1,0,0,\dots)$, so that $T[0,\mathbb{1}]$ is not order bounded in c_0, hence T is not order bounded. On the other hand, suppose that $x_n \xrightarrow{\mathrm{u}} 0$ in c. Then $x_n \xrightarrow{\|\cdot\|} 0$ in c. Since T is norm bounded, it follows that $Tx_n \xrightarrow{\|\cdot\|} 0$ in c_0. It is easy to see that this yields $Tx_n \xrightarrow{\mathrm{u}} 0$ in c_0.

Proposition 5.6. *Let (X_k) be a bidecomposition in a Banach lattice X, let $T\colon [X_k] \to Y$ be a sequentially uniformly continuous norm isomorphic embedding into a Banach lattice Y. Then the sequence (TX_k) is a bidecomposition.*

Proof. Since T is a norm isomorphic embedding, (TX_k) is a Schauder decomposition in Y. Suppose that $y \in [TX_k]$ in Y. Then $y = \sum_{k=1}^\infty Tx_k$ for some $x_k \in X_k$. It follows that $y = Tx$, where $x = \sum_{k=1}^\infty x_k$ in X. Since (X_k) is a bidecomposition, we have $x = {}^{\mathrm{u}}\!\sum_{k=1}^\infty x_k$. By assumption, $y = {}^{\mathrm{u}}\!\sum_{k=1}^\infty Tx_k$. □

Corollary 5.7. *Let $T\colon X \to Y$ be an order bounded norm isomorphic embedding between Banach lattices. Then T maps bibasic sequences to bibasic sequences.*

In Theorem 5.1 of [GKP15] the authors observe that $L_1[0,1]$ admits no bibasis; moreover, $L_1[0,1]$ does not even embed by means of a lattice embedding T with σ-order continuous inverse map into a σ-order continuous Banach lattice with a bibasis. A careful analysis of the proof, together with the preceding results of the current paper, reveals that the assumptions may be considerably relaxed:

Theorem 5.8. *There is no norm isomorphic embedding T from $L_1[0,1]$ to a Banach lattice such that* Range T *is contained in the closed linear span of a bibasic sequence (or even of a bi-FDD) and T^{-1}, viewed as an operator from* Range T *to $L_1[0,1]$, is sequentially uniformly continuous.*

6. Unconditional and permutable decompositions

Recall that a Schauder decomposition (X_k) is unconditional if every convergent series $\sum_{k=1}^\infty x_k$ with $x_k \in X_k$ converges unconditionally; see [Sin81, p. 534] or [LT77, 1.g] for properties of unconditional decompositions. By an ***unconditional bidecomposition*** we mean an unconditional Schauder decomposition which is also a bidecomposition.

Proposition 6.1. *A sequence of closed non-zero subspaces (X_k) of a Banach lattice X is an unconditional bidecomposition of $[X_k]$ iff there exists a constant L such that*

$$\sup_{\varepsilon_k=\pm 1}\Bigl\|\bigvee_{n=1}^{m}\Bigl|\sum_{k=1}^{n}\varepsilon_k x_k\Bigr|\Bigr\| \leqslant L\Bigl\|\sum_{k=1}^{m}x_k\Bigr\| \tag{6}$$

for any $m \in \mathbb{N}$ and any $x_1 \in X_1, \dots, x_m \in X_m$.

Proof. Suppose that (X_k) is a bidecomposition. Then

$$\Big\|\bigvee_{n=1}^{m}\Big|\sum_{k=1}^{n}\varepsilon_k x_k\Big|\Big\| \leqslant M\Big\|\sum_{k=1}^{m}\varepsilon_k x_k\Big\| \leqslant MK_u\Big\|\sum_{k=1}^{m}x_k\Big\|,$$

where M is the bidecomposition constant and K_u is the unconditional constant of (X_k). Conversely, suppose that (6) is satisfied for any $x_k \in X_k$ as $k = 1, \dots, m$. This clearly implies the bidecomposition inequality (5), hence (X_k) is a bidecomposition. Furthermore,

$$\sup_{\varepsilon_k=\pm1}\Big\|\sum_{k=1}^{m}\varepsilon_k x_k\Big\| \leqslant \sup_{\varepsilon_k=\pm1}\Big\|\bigvee_{n=1}^{m}\Big|\sum_{k=1}^{n}\varepsilon_k x_k\Big|\Big\| \leqslant L\Big\|\sum_{k=1}^{m}x_k\Big\|,$$

hence (X_k) is unconditional. □

It is easy to see that, analogously to the theory of unconditional decompositions, the supremum over all choices of signs in Proposition 6.1 may be replaced with the supremum over all choices of $\delta_k \in \{0, 1\}$ or the supremum over all β_k with $|\beta_k| \leqslant 1$. However, this analogy breaks if we consider permutations of the index set. Recall that a basic sequence is unconditional iff every permutation of it is again a basic sequence. Clearly, every permutation of an unconditional bibasic sequence is an unconditional basic sequence. However, the following example shows that the bibasis property may be lost after a permutation.

Example 6.2. *A permutation of an unconditional bibasis need not be a bibasis:* Indeed, the Haar system (h_k) in $L_p[0, 1]$ is an unconditional bibasis when $1 < p < \infty$. However, there is a function in $L_\infty[0, 1]$ whose Haar series diverges a.e., and, therefore, cannot converge in order, after a rearrangement of the series; see [KS89, p. 96]. This shows that the corresponding rearrangement of (h_k) fails to be a bibasis.

The preceding example motivates the following definition. A bidecomposition is said to be ***permutable*** if every permutation of it is a bidecomposition. Similarly, a bibasic sequence is ***permutable*** if every permutation of it is bibasic. The preceding example shows that the Haar system in $L_p[0, 1]$ $(1 < p < \infty)$ fails to be permutable. It is easy to see that permutability implies unconditionality.

If (x_k) is an unconditional basic sequence, the supremum of the basis constants over all permutations of (x_k) is finite. We establish a similar result for permutable decompositions, even though the uniform boundedness principle is not applicable in this context.

Theorem 6.3. *Let (X_k) be a permutable bidecomposition. The supremum of the bibasis constants over all permutations of (X_k) is finite.*

Proof. For the sake of contradiction, assume that the supremum is infinite. We claim, then, that the supremum of the bidecomposition constants over all permutations of $(X_k)_{k\geqslant 2}$ is also infinite. Suppose not. Then there is a constant M such that for any distinct $k_1,\dots,k_m$ with $k_i\neq 1$ and $x_{k_i}\in X_{k_i}$ for $i=1,\dots,m$ we have $\Big\|\bigvee_{n=1}^m\big|\sum_{i=1}^n x_{k_i}\big|\Big\|\leqslant M\big\|\sum_{i=1}^m x_{k_i}\big\|$. So if we take any distinct indices $k_1,\dots,k_m$ with $k_{i_0}=1$ for some $i_0\in\{1,\dots,m\}$, and $x_{k_i}\in X_{k_i}$ as $i=1,\dots,m$ then

$$\Big\|\bigvee_{n=1}^m\Big|\sum_{i=1}^n x_{k_i}\Big|\Big\|\leqslant\Big\||x_{k_{i_0}}|+\bigvee_{n=1}^m\Big|\sum_{i\in\{1,\dots,n\}\setminus\{i_0\}}x_{k_i}\Big|\Big\|$$
$$\leqslant\|x_{k_{i_0}}\|+M\Big\|\sum_{i\in\{1,\dots,m\}\setminus\{i_0\}}x_{k_i}\Big\|\leqslant(K_u+MK_u)\Big\|\sum_{i=1}^m x_{k_i}\Big\|,$$

where K_u is the unconditional constant of (X_k). This contradicts the assumption, and, therefore, proves the claim. Proceeding inductively, we deduce that the supremum of the bidecomposition constants over all permutations of $(X_k)_{k\geqslant N}$ is infinite for every N.

Hence, we can find distinct indices $k_1^1,\dots,k_{m_1}^1$ and vectors $x_{k_1^1}\in X_{k_1^1}$, …, $x_{k_{m_1}^1}\in X_{k_{m_1}^1}$ such that

$$\Big\|\bigvee_{n=1}^{m_1}\Big|\sum_{i=1}^n x_{k_i^1}\Big|\Big\|\geqslant\Big\|\sum_{i=1}^{m_1}x_{k_i^1}\Big\|.$$

Let $N_1=\max\{k_1^1,\dots,k_{m_1}^1\}$. Applying the previous paragraph to $(X_k)_{k>N_1}$, we find distinct $k_1^2,\dots,k_{m_2}^2>N_1$ and $x_{k_1^2}\in X_{k_1^2}$, …, $x_{k_{m_2}^2}\in X_{k_{m_2}^2}$ such that

$$\Big\|\bigvee_{n=1}^{m_2}\Big|\sum_{i=1}^n x_{k_i^2}\Big|\Big\|\geqslant 2\Big\|\sum_{i=1}^{m_2}x_{k_i^2}\Big\|.$$

We then repeat the process in the obvious way; the elements of $\mathbb{N}$ that are missed we enumerate as $l_1,l_2,\dots$. The sequence $k_1^1,\dots,k_{m_1}^1,l_1,k_1^2,\dots,k_{m_2}^2,l_2,\dots$ is a permutation of $\mathbb{N}$, say, σ, and it is clear that under this permutation, $(X_{\sigma(k)})$ fails the bidecomposition inequality and, hence, is not a bidecomposition. ☐

Corollary 6.4. *Let (X_k) be a sequence of closed non-zero subspaces of a Banach lattice X. TFAE:*

(i) *(X_k) is a permutable bidecomposition of $[X_k]$;*
(ii) *There is a constant M such that for any sequence (x_k) with $x_k\in X_k$ and any distinct indices $k_1,\dots,k_m$, we have*

$$\Big\|\bigvee_{n=1}^m\Big|\sum_{i=1}^n x_{k_i}\Big|\Big\|\leqslant M\Big\|\sum_{i=1}^m x_{k_i}\Big\|.$$

Proof. (i)⇒(ii) Let M be the supremum of the bidecomposition constants, guaranteed to be finite by Theorem 6.3. Choose a permutation σ with $\sigma(i)=k_i$ as $i=1,\dots,m$. The bidecomposition inequality for $(X_{\sigma(k)})$ yields (ii).

(ii)⇒(i) Let σ be a permutation. Applying (ii) with $m\in\mathbb{N}$ and $k_i=\sigma(i)$ as $i=1,\dots,m$, we conclude that $(X_{\sigma(k)})$ satisfies the bidecomposition inequality, hence is a bidecomposition, which yields (i). ☐

7. Absolute decompositions

Let (X_k) be a permutable bidecomposition, and let P_n^σ denote the n-th canonical projection associated to the permutation σ. By Theorem 3.1(iv), for each $x\in[X_k]$ there exists $u^\sigma\in X_+$ such that $|P_n^\sigma x|\leqslant u^\sigma$ for all n. Motivated by Theorem 6.3, it is natural to wonder if one can choose u^σ independent of σ. It turns out that one cannot; to do so one must further modify the basis inequality. This leads to the following definition, which is of interest in its own right.

Definition 7.1. Let (X_k) be a sequence of closed non-zero subspaces of a Banach lattice X. We say that (X_k) is an ***absolute decomposition*** of $[X_k]$ if there exists a constant $A\geqslant 1$ such that for any $m\in\mathbb{N}$ and any $x_1\in X_1,\dots,x_m\in X_m$,

$$\Bigl\|\sum_{k=1}^{m}|x_k|\Bigr\|\leqslant A\Bigl\|\sum_{k=1}^{m}x_k\Bigr\|. \tag{7}$$

In particular, a sequence (x_k) is ***absolute*** if there exists a constant $A\geqslant 1$ such that for any $m\in\mathbb{N}$ and any $\alpha_1,\dots,\alpha_m$ we have $\bigl\|\sum_{k=1}^m|\alpha_k x_k|\bigr\|\leqslant A\bigl\|\sum_{k=1}^m\alpha_k x_k\bigr\|$.

It is clear that every absolute decomposition is a Schauder decomposition. Every permutation of an absolute decomposition satisfies the bidecomposition inequality; it follows that every absolute decomposition is permutable and, therefore, unconditional. Moreover, one can easily check that the absolute property is stable under permutation.

We next prove an absolute version of Theorem 3.1. Recall that for any sequence (x_k) it follows from $\bigl\|\sum_{k=n}^m x_k\bigr\|\leqslant\bigl\|\sum_{k=n}^m|x_k|\bigr\|$ whenever $n\leqslant m$ that if $\sum_{k=1}^\infty|x_k|$ converges then $\sum_{k=1}^\infty x_k$ converges.

Theorem 7.2. *Let X be a Banach lattice and $(X_k)\subseteq X$ a Schauder decomposition of $[X_k]$. TFAE:*

(i) *(X_k) is absolute;*
(ii) *The convergence of $\sum_{k=1}^\infty x_k$ implies the convergence of $\sum_{k=1}^\infty|x_k|$ for every sequence (x_k) with $x_k\in X_k$;*
(iii) *If $\sum_{k=1}^\infty x_k$ converges then the sequence of partial sums $\bigl(\sum_{k=1}^m|x_k|\bigr)_m$ is order bounded for all $x_k\in X_k$;*

(iv) *If* $\sum_{k=1}^{\infty} x_k$ *converges then the sequence of partial sums* $\bigl(\sum_{k=1}^{m}|x_k|\bigr)_m$ *is norm bounded for all* $x_k \in X_k$*;*

Proof. (i)⇒(ii): Let (x_k) be such that $x_k \in X_k$ for every k and $\sum_{k=1}^{\infty} x_k$ converges. Then for each $n \leqslant m$ the absolute inequality (7) yields that

$$\Bigl\|\sum_{k=n}^{m}|x_k|\Bigr\| \leqslant A\Bigl\|\sum_{k=n}^{m} x_k\Bigr\|.$$

Hence, $\bigl(\sum_{k=1}^{m}|x_k|\bigr)_m$ is Cauchy, so that $\sum_{k=1}^{\infty}|x_k|$ converges.

(ii)⇒(iii)⇒(iv) trivially.

(iv)⇒(i): For every m and $x = \sum_{k=1}^{\infty} x_k$, where $x_k \in X_k$ for every k, we define

$$\varphi_m(x) = \sum_{k=1}^{m}|x_k| = |P_1x| + |P_2x - P_1x| + \cdots + |P_mx - P_{m-1}x|.$$

It is clear that φ_m is continuous, so that the set

$$F_i = \Bigl\{x \in [X_k] \,:\, \forall m \in \mathbb{N}\ \Bigl\|\sum_{k=1}^{m}|x_k|\Bigr\| \leqslant i\Bigr\} = \bigcap_{m=1}^{\infty}\bigl\{x \in [X_k] \,:\, \bigl\|\varphi_m(x)\bigr\| \leqslant i\bigr\}$$

is closed for every $i \in \mathbb{N}$. The rest of the proof is analogous to that of (v)⇒(vi) in Theorem 3.1. □

Remark 7.3. Since the sequence of the partial sums $\bigl(\sum_{k=1}^{m}|x_k|\bigr)$ in (ii) is increasing, one easily sees that the convergence is, in fact, uniform.

Recall the following standard fact:

Lemma 7.4. *For any vectors* $x_1, \dots, x_m$ *in an Archimedean vector lattice we have*

(i) $\sum_{k=1}^{m}|x_k| = \sup\sum_{k=1}^{m}\varepsilon_k x_k = \sup\Bigl|\sum_{k=1}^{m}\varepsilon_k x_k\Bigr|$, *where the supremum is taken over all choices of signs* $\varepsilon_k = \pm 1$*;*

(ii) $\sum_{k=1}^{m}|x_k| \leqslant 2\sup\bigl|\sum_{i=1}^{n} x_{k_i}\bigr|$, *where the supremum is taken over all choices of* $n \leqslant m$ *and* $1 \leqslant k_1 < \cdots < k_n \leqslant m$*.*

Proof. These statements hold for real numbers and, therefore, for elements of every Archimedean vector lattice. □

Theorem 7.2 immediately yields the characterization of absolute decompositions that motivated this section:

Proposition 7.5. *Let* X *be a Banach lattice and* (X_k) *a bidecomposition in* X*. TFAE:*

(i) (X_k) *is absolute;*

(ii) *for each $x \in [X_k]$ there exists $u \geqslant 0$ such that $|P_n^\sigma x| \leqslant u$ for all $n \in \mathbb{N}$ and all permutations σ.*

Proof. Suppose (X_k) is absolute and take $x = \sum_{k=1}^\infty x_k \in [X_k]$. It is clear that $u := \sum_{k=1}^\infty |x_k|$ is as required; Theorem 3.1(iv) is one way to see that (X_k) is permutable.

To prove the converse, let $x \in [X_k]$ and find u as in the statement. Lemma 7.4(ii) yields that for each m, $\sum_{k=1}^m |x_k| \leqslant 2u$. Thus, (X_k) is absolute by Theorem 7.2(iii). □

There are several other natural ways to motivate the concepts of an absolute decomposition and an absolute sequence. In view of Lemma 7.4(i), the absolute inequality (7) may be viewed as the unconditional inequality $\sup\limits_{\varepsilon_k=\pm1} \bigl\|\sum_{k=1}^m \varepsilon_k x_k\bigr\| \leqslant K_u \bigl\|\sum_{k=1}^m x_k\bigr\|$ with the supremum pulled inside the norm: $\bigl\|\sup\limits_{\varepsilon_k=\pm1} \sum_{k=1}^m \varepsilon_k x_k\bigr\| \leqslant A \bigl\|\sum_{k=1}^m x_k\bigr\|$. On the other hand, it is easy to see that a normalized basic sequence (x_k) in a Banach space is equivalent to the unit vector basis of ℓ_1 iff the convergence of $\sum_{k=1}^\infty \alpha_k x_k$ is equivalent to the convergence of $\sum_{k=1}^\infty \|\alpha_k x_k\|$. Replacing norm with modulus, we obtain the definition of an absolute sequence.

We know that "absolute" ⇒ "permutable" ⇒ "unconditional". We will now list several cases where the three concepts are equivalent.

Remark 7.6. For positive basic sequences, being absolute is equivalent to being unconditional. Indeed, let (x_k) be a positive unconditional basic sequence. Fix $\alpha_1, \dots, \alpha_m$. Then

$$\Bigl\|\sum_{k=1}^m |\alpha_k x_k|\Bigr\| = \Bigl\|\sum_{k=1}^m |\alpha_k| x_k\Bigr\| \leqslant K_u \Bigl\|\sum_{k=1}^m \alpha_k x_k\Bigr\|,$$

where K_u is the unconditional constant of (x_k).

Proposition 7.7. *A Schauder decomposition in an AM-space is absolute iff it is unconditional. In particular, a basic sequence in an AM-space is absolute iff it is unconditional.*

Proof. Let (X_k) be an unconditional Schauder decomposition with unconditional constant K_u. Then for $x_1 \in X_1, \dots, x_m \in X_m$ Lemma 7.4(i) yields

$$\Bigl\|\sum_{k=1}^m |x_k|\Bigr\| = \Bigl\|\sup_{\varepsilon_k=\pm1} \Bigl|\sum_{k=1}^m \varepsilon_k x_k\Bigr|\Bigr\| = \sup_{\varepsilon_k=\pm1} \Bigl\|\sum_{k=1}^m \varepsilon_k x_k\Bigr\| \leqslant K_u \Bigl\|\sum_{k=1}^m x_k\Bigr\|.$$

□

Proposition 7.8. *Every basic sequence in a Banach lattice which is equivalent to the unit vector basis of ℓ_1 is absolute. In particular, a basis of ℓ_1 is absolute iff it is unconditional.*

Proof. Suppose that (x_k) is a basic sequence in a Banach lattice such that (x_k) is M-equivalent to the unit vector basis (e_k) of ℓ_1. In particular, (x_k) is seminormalized, so that there exists $C > 0$ such that $\|x_k\| \leqslant C$ for every k. For every $\alpha_1, \dots, \alpha_m$, we have

$$\Bigl\|\sum_{k=1}^{m}|\alpha_k x_k|\Bigr\| \leqslant C\sum_{k=1}^{m}|\alpha_k| = C\Bigl\|\sum_{k=1}^{m}\alpha_k e_k\Bigr\|_{\ell_1} \leqslant CM\Bigl\|\sum_{k=1}^{m}\alpha_k x_k\Bigr\|.$$

Up to equivalence, ℓ_1 has only one normalized unconditional basis; see [LT77, Proposition 2.b.9]. Hence, given any unconditional basis (x_k) of ℓ_1, $\bigl(\frac{x_k}{\|x_k\|}\bigr)$ is equivalent to the unit vector basis of ℓ_1 and, therefore, is absolute. It follows that (x_k) is absolute. □

Question 7.9. Does there exist a Banach lattice with a bibasis but no conditional bibasis?

Proposition 7.10. *Suppose that (x_k) is an absolute basic sequence. If $\bigl(|x_k|\bigr)$ is a basic sequence then it is dominated by (x_k).*

Proof. Fix $\alpha_1, \dots, \alpha_m$. Let $I_+ = \{k : \alpha_k \geqslant 0\}$ and $I_- = \{k : \alpha_k < 0\}$. Then

$$\begin{aligned}\Bigl\|\sum_{k=1}^{m}\alpha_k|x_k|\Bigr\| &= \Bigl\|\sum_{k\in I_+}|\alpha_k x_k| - \sum_{k\in I_-}|\alpha_k x_k|\Bigr\| \leqslant \Bigl\|\sum_{k\in I_+}|\alpha_k x_k|\Bigr\| + \Bigl\|\sum_{k\in I_-}|\alpha_k x_k|\Bigr\| \\ &\leqslant A\Bigl\|\sum_{k\in I_+}\alpha_k x_k\Bigr\| + A\Bigl\|\sum_{k\in I_-}\alpha_k x_k\Bigr\| \leqslant 2AK_u\Bigl\|\sum_{k=1}^{m}\alpha_k x_k\Bigr\|,\end{aligned}$$

where A is the absolute constant, and K_u is the unconditional constant of (x_k). □

Example 7.11. In general, even if (x_k) is an absolute basis, the sequence $\bigl(|x_k|\bigr)$ need not be basic. For example, take $X = \ell_p$ $(1 \leqslant p < \infty)$, and put $x_1 = e_1 + e_2$, $x_2 = e_1 - e_2$, and $x_k = e_k$ whenever $k > 2$.

Example 7.12. *An absolute sequence (x_k) such that the sequence $(|x_k|)$ is conditional basic, hence not equivalent to (x_k).* Let $X = \ell_\infty$ and let x_k be the k-th row of the following infinite matrix:

$$\begin{array}{|cc|cccc|cccccccc|cccccccccccccccc|cccccccccc}
1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \dots \\
\vdots & & \vdots & & & & \vdots & & & & & & & & \vdots & & & & & & & & & & & & & & & & \vdots & & & & & & & & &
\end{array}$$

Each x_k is made up of "blocks", where initial blocks are zeros and further blocks are discrete finite Rademacher vectors. For any $m \in \mathbb{N}$ and any $\alpha_1, \dots, \alpha_m$, there is a column in the matrix whose first m entries match the signs of $\alpha_1, \dots, \alpha_m$. This yields $\left\|\sum_{k=1}^m \alpha_k x_k\right\| = \sum_{k=1}^m |\alpha_k| = \left\|\sum_{k=1}^m |\alpha_k x_k|\right\|$. It follows that (x_k) is a 1-unconditional 1-absolute basic sequence equivalent to the unit vector basis of ℓ_1. On the other hand, it can be easily verified that $\bigl(|x_k|\bigr)$ is a conditional basic sequence.

Remark 7.13. In [BB09], the authors define a ***lattice decomposition*** of a Banach lattice X as a Schauder decomposition (X_k) such that $X = [X_k]$ and for every k, the operator $Q_k = P_k - P_{k-1}$, which is a projection onto X_k, is a lattice homomorphism (we take $P_0 = 0$). More generally, let (X_k) be a Schauder decomposition of X with $X = [X_k]$ such that $Q_k \geqslant 0$ for every k. Such a decomposition is absolute with absolute constant $A = 1$. Indeed, let $x = \sum_{k=1}^m x_k$, where $x_k \in X_k$. Then

$$\sum_{k=1}^m |x_k| = \sum_{k=1}^m |Q_k x| \leqslant \sum_{k=1}^m Q_k |x| \leqslant \sum_{k=1}^\infty Q_k |x| = |x|,$$

so that $\left\|\sum_{k=1}^m |x_k|\right\| \leqslant \left\|\sum_{k=1}^m x_k\right\|$.

It can be easily verified that if (x_k) is a basis such that $Q_k \geqslant 0$ for every k then X is atomic and (x_k) is a disjoint sequence of atoms. We don't have a good understanding of the structure of those Banach lattices X which admit FDDs (X_k) with Q_k positive for each k. In particular, does X have atoms? Is there a disjoint sequence such that each X_k is the span of a block of this sequence? Notice that if the Q_k's are lattice homomorphisms then both these questions have positive answers.

8. Permutable and absolute sequences in L_p spaces

Suppose $1 \leqslant p < \infty$. We mentioned in Example 2.11 that the Haar basis (h_k) of $L_p[0,1]$ is a bibasis iff $p > 1$. In this case, it follows that every block sequence of it is bibasic. In particular, if $p > 1$ then the Rademacher sequence (r_k), being a block sequence of (h_k), is a bibasic sequence. The latter statement remains valid for $p = 1$:

Proposition 8.1. *Let $1 \leqslant p < \infty$. The Rademacher sequence (r_k) is a bibasic sequence in $L_p[0,1]$. Furthermore, it is permutable but not absolute.*

Proof. Let (x_k) be a permutation of the Rademacher sequence. Fix scalars $\alpha_1, \dots, \alpha_m$ and let $f_n = \sum_{k=1}^n \alpha_k x_k$ as $n = 1, \dots, m$. It is easy to see that $(f_n)_{n=1}^m$ is a martingale with difference sequence $d_k = \alpha_k x_k$. The associated square function is $S(f) = \bigl(\sum_{k=1}^m \alpha_k^2\bigr)^{\frac{1}{2}} \mathbb{1}$. Applying Burkholder-Gundy-Davis inequality (4) followed by Khintchine's inequality, we get

$$\Bigl\|\bigvee_{n=1}^m \Bigl|\sum_{k=1}^n \alpha_k x_k\Bigr|\Bigr\|_{L_p} \leqslant C\Bigl(\sum_{k=1}^m \alpha_k^2\Bigr)^{\frac{1}{2}} \leqslant C' \Bigl\|\sum_{k=1}^m \alpha_k x_k\Bigr\|_{L_p}.$$

Hence (x_k) is a bibasic sequence, which yields that (r_k) is a permutable bibasic sequence.

Furthermore, it follows from $|r_k| = \mathbb{1}$ that $\left\|\sum_{k=1}^m |\alpha_k r_k|\right\|_{L_p} = \sum_{k=1}^m |\alpha_k|$, while Khintchine's inequality yields that $\left\|\sum_{k=1}^m \alpha_k r_k\right\|_{L_p} \sim \left(\sum_{k=1}^m \alpha_k^2\right)^{\frac{1}{2}}$. As these two quantities are not equivalent, we conclude that (r_k) is not absolute. □

The fact that the Rademacher sequence is bibasic in $L_1[0,1]$ may be generalized as follows.

Proposition 8.2. *Let (x_k) be a block sequence of the Haar basis (h_k). If (x_k) is unconditional then it is bibasic in $L_1[0,1]$.*

Proof. Fix scalars $\alpha_1, \dots, \alpha_m$ and let $f_n = \sum_{k=1}^n \alpha_k x_k$ as $n = 1, \dots, m$. Since (h_k) is a martingale difference sequence, so is (x_k), hence (f_n) is a martingale. Applying Burkholder-Gundy-Davis inequality (4), we get

$$\left\|\bigvee_{n=1}^m \left|\sum_{k=1}^n \alpha_k x_k\right|\right\| \sim \left\|\left(\sum_{k=1}^m |\alpha_k x_k|^2\right)^{\frac{1}{2}}\right\|.$$

By Khintchine's inequality, there is a constant C such that

$$\left(\sum_{k=1}^m |\alpha_k x_k|^2\right)^{\frac{1}{2}} \leqslant C \frac{1}{2^m} \sum_{\varepsilon_k = \pm 1} \left|\sum_{k=1}^m \varepsilon_k \alpha_k x_k\right|.$$

Indeed, this inequality is true for real numbers, hence it remains valid for vectors in X. It follows that

$$\left\|\left(\sum_{k=1}^m |\alpha_k x_k|^2\right)^{\frac{1}{2}}\right\| \leqslant C \frac{1}{2^m} \sum_{\varepsilon_k = \pm 1} \left\|\sum_{k=1}^m \varepsilon_k \alpha_k x_k\right\| \leqslant C K_u \left\|\sum_{k=1}^m \alpha_k x_k\right\|,$$

where K_u is the unconditional constant of (x_k). Therefore, (x_k) is bibasic. □

It is known that every closed infinite-dimensional subspace of $L_1[0,1]$ contains an unconditional basic sequence; see, [Ros73, Corollary 12] or [LT79, p. 38].

Corollary 8.3. *Every closed infinite-dimensional subspace of an AL-space contains an unconditional bibasic sequence.*

Proof. Let X be a closed infinite-dimensional subspace of an AL-space L. WLOG we may take $L = L_1[0,1]$. Indeed, it is easy to see that we may assume WLOG that X is separable. Replacing L with the closed sublattice generated by X, we may assume WLOG that L is separable. It is well-known that, up to a lattice isometry, L is one of the following: ℓ_1, $L_1[0,1]$, $\ell_1 \oplus_1 L_1[0,1]$, or $\ell_1^m \oplus_1 L_1[0,1]$; see, e.g., [LW76] or Section 2.7 in [MN91]. All these spaces can be lattice isometrically embedded into $L_1[0,1]$, so we may assume that $L = L_1[0,1]$.

Case 1: X is non-reflexive. Since $L_1[0,1]$ is a KB-space, X contains no isomorphic copy of c_0. By Theorem 1.c.5 in [LT79], X contains an isomorphic copy of ℓ_1, and, therefore, X contains a basic sequence which is equivalent to the unit vector basis of ℓ_1. By Proposition 7.8, it is absolute; in particular, it is unconditional and bibasic.

Case 2: X is reflexive. Fix a normalized unconditional basic sequence (x_k) in X. Since X is reflexive, (x_k) is weakly null. Passing to a subsequence and using Bessaga-Pełczyński's selection principle, we find a block sequence (u_k) of the Haar basis (h_k) such that $\|x_k - u_k\| \to 0$ sufficiently fast so that (u_k) is equivalent to (x_k). It follows that (u_k) is unconditional and, therefore, bibasic by Proposition 8.2. Theorem 4.2 now yields that, after passing to further subsequences if necessary, (x_k) is bibasic. □

Proposition 8.4. *A normalized basic sequence in an AL-space is absolute iff it is equivalent to the unit vector basis of ℓ_1.*

Proof. Let (x_k) be a normalized basic sequence in an AL-space. For every $\alpha_1,\dots,\alpha_m$, we have

$$\Bigl\|\sum_{k=1}^m |\alpha_k x_k|\Bigr\| = \sum_{k=1}^m |\alpha_k| = \Bigl\|\sum_{k=1}^m \alpha_k e_k\Bigr\|,$$

where (e_k) is the standard unit vector basis of ℓ_1. It follows that (x_k) is absolute iff it is equivalent to (e_k). □

It is also true that every normalized absolute sequence in $L_p(\mu)$ is equivalent to the unit vector basis of ℓ_p. Our proof is based on the proof of Theorem 2 in [JS15].

Theorem 8.5. *Every normalized absolute sequence in $L_p(\mu)$ ($1 \leqslant p < \infty$) is equivalent to the unit vector basis (e_k) of ℓ_p.*

Proof. Let (x_k) be a normalized absolute sequence in $L_p(\mu)$. Fix $\alpha_1,\dots,\alpha_m$. Put $f_k = \alpha_k x_k$. Being absolute, (x_k) is unconditional, so that $\Bigl\|\sum_{k=1}^m f_k\Bigr\| \sim \Bigl\|\sum_{k=1}^m \varepsilon_k f_k\Bigr\|$ for every choice of signs $\varepsilon_k = \pm 1$. Using Fubini's Theorem and Khintchine's inequality, we get

$$\begin{aligned}\Bigl\|\sum_{k=1}^m f_k\Bigr\|^p &\sim \operatorname*{Ave}_{\varepsilon_k=\pm1}\Bigl\|\sum_{k=1}^m \varepsilon_k f_k\Bigr\|^p = \int_{t=0}^1 \Bigl\|\sum_{k=1}^m r_k(t) f_k\Bigr\|^p dt \\ &= \int_{t=0}^1 \int_\Omega \Bigl|\sum_{k=1}^m r_k(t) f_k(\omega)\Bigr|^p d\omega\, dt \lesssim \int_\Omega \Bigl(\sum_{k=1}^m |f_k(\omega)|^2\Bigr)^{\frac{p}{2}} d\omega = \Bigl\|\Bigl(\sum_{k=1}^m |f_k|^2\Bigr)^{\frac12}\Bigr\|^p.\end{aligned}$$

Thus, $\left\|\sum_{k=1}^m f_k\right\| \lesssim \left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\|$. On the other hand, since (x_k) is absolute, we have

$$\left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\| \leqslant \left\|\sum_{k=1}^m |f_k|\right\| \lesssim \left\|\sum_{k=1}^m f_k\right\|,$$

so that $\left\|\sum_{k=1}^m f_k\right\| \sim \left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\|$.

If $1 \leqslant p \leqslant 2$, we have

$$\left\|\sum_{k=1}^m f_k\right\| \lesssim \left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\| \leqslant \left\|\left(\sum_{k=1}^m |f_k|^p\right)^{\frac{1}{p}}\right\| \leqslant \left\|\sum_{k=1}^m |f_k|\right\| \lesssim \left\|\sum_{k=1}^m f_k\right\|,$$

so that

$$\left\|\sum_{k=1}^m \alpha_k x_k\right\| = \left\|\sum_{k=1}^m f_k\right\| \sim \left\|\left(\sum_{k=1}^m |f_k|^p\right)^{\frac{1}{p}}\right\| = \left(\sum_{k=1}^m |\alpha_k|^p\right)^{\frac{1}{p}}.$$

Now suppose that $2 < p < \infty$. Then

$$\left\|\sum_{k=1}^m f_k\right\| \sim \left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\| \gtrsim \left\|\left(\sum_{k=1}^m |f_k|^p\right)^{\frac{1}{p}}\right\| = \left(\sum_{k=1}^m |\alpha_k|^p\right)^{\frac{1}{p}}.$$

To prove the opposite inequality, let θ be such that $\frac{1}{2} = \frac{\theta}{1} + \frac{1-\theta}{p}$. Using Hölder's inequality in the form $\left(\int f\right)^{\lambda}\left(\int g\right)^{\mu} \geqslant \int f^{\lambda} g^{\mu}$ where $\lambda, \mu \geqslant 0$ with $\lambda + \mu = 1$, and f and g are two positive integrable functions, we get

$$\begin{aligned}\left\|\sum_{k=1}^m |f_k|\right\|^{p\theta} \cdot \left\|\left(\sum_{k=1}^m |f_k|^p\right)^{\frac{1}{p}}\right\|^{p(1-\theta)} &= \left(\int \left(\sum_{k=1}^m |f_k|\right)^p\right)^{\theta} \cdot \left(\int \sum_{k=1}^m |f_k|^p\right)^{1-\theta} \\ \geqslant \int \left(\sum_{k=1}^m |f_k|\right)^{p\theta} \cdot \left(\sum_{k=1}^m |f_k|^p\right)^{1-\theta} &= \int \left[\left(\sum_{k=1}^m |f_k|\right)^{2\theta} \cdot \left(\sum_{k=1}^m |f_k|^p\right)^{\frac{2(1-\theta)}{p}}\right]^{\frac{p}{2}} \\ &\geqslant \int \left[\sum_{k=1}^m |f_k|^{2\theta} \cdot |f_k|^{2(1-\theta)}\right]^{\frac{p}{2}} = \int \left[\sum_{k=1}^m |f_k|^2\right]^{\frac{p}{2}},\end{aligned}$$

which yields

$$\left\|\left(\sum_{k=1}^m |f_k|^2\right)^{\frac{1}{2}}\right\| \leqslant \left\|\sum_{k=1}^m |f_k|\right\|^{\theta} \cdot \left\|\left(\sum_{k=1}^m |f_k|^p\right)^{\frac{1}{p}}\right\|^{1-\theta} \sim \left\|\sum_{k=1}^m f_k\right\|^{\theta} \cdot \left(\sum_{k=1}^m |\alpha_k|^p\right)^{\frac{1-\theta}{p}}.$$

It follows that

$$\left\|\sum_{k=1}^m f_k\right\| \lesssim \left\|\sum_{k=1}^m f_k\right\|^{\theta} \cdot \left(\sum_{k=1}^m |\alpha_k|^p\right)^{\frac{1-\theta}{p}},$$

and, therefore, $\left\|\sum_{k=1}^m f_k\right\| \lesssim \left(\sum_{k=1}^m |\alpha_k|^p\right)^{\frac{1}{p}}$. □

Remark 8.6. Proposition 8.1 shows that "absolute" cannot be replaced with "permutable" in Theorem 8.5 when $p \neq 2$.

Example 8.7. Let R be the subspace spanned by the Rademacher sequence in $L_p[0,1]$, $1 \leqslant p < \infty$ and $p \neq 2$. Then R contains no absolute sequence. Indeed, if (x_k) is a sequence in R which is absolute as a sequence in $L_p[0,1]$ then (x_k) is equivalent to the unit vector basis of ℓ_p by Theorem 8.5. However, R is isomorphic to ℓ_2, hence it contains no isomorphic copy of ℓ_p.

Example 8.8. *A permutable bibasic sequence in ℓ_p with $p \neq 2$ which is not equivalent to the unit vector basis; in particular, it is not absolute.* We construct such a sequence as a discretization of the Rademacher sequence. Fix $1 \leqslant p < \infty$ with $p \neq 2$. Let $s \in \mathbb{N}$. There is a natural lattice isometric embedding $T\colon \ell_p^{2^s} \to L_p[0,1]$ via $Te_i = f_i/\|f_i\|$, where $f_i = \mathbb{1}_{[\frac{i-1}{2^s},\frac{i}{2^s}]}$. Then $\operatorname{Range} T$ consists of all dyadic functions of level up to s in $L_p[0,1]$. In particular, it contains the first s terms of the Rademacher sequence. Let $z_k = T^{-1} r_k$ as $k = 1,\dots,s$. Then this finite sequence is C_p-equivalent to the unit vector basis of ℓ_2^s. Also, as in Proposition 8.1,

$$\Bigl\|\bigvee_{n=1}^{s}\Bigl|\sum_{k=1}^{n}\alpha_k z_k\Bigr|\Bigr\|_{\ell_p^{2^s}} = \Bigl\|\bigvee_{n=1}^{s}\Bigl|\sum_{k=1}^{n}\alpha_k r_k\Bigr|\Bigr\|_{L_p[0,1]} \leqslant C_p'\Bigl\|\sum_{k=1}^{s}\alpha_k z_k\Bigr\|_{\ell_p^{2^s}}. \tag{8}$$

Relabel z_k as $z_k^{(s)}$.

Now view ℓ_p as $\Bigl(\bigoplus_{s=1}^{\infty}\ell_p^{2^s}\Bigr)_{\ell_p}$. Merge the sequences $(z_k^{(s)})_{k=1}^{s}$ into a sequence in ℓ_p; denote it (x_j). We claim that (x_j) is bibasic. It suffices to verify the bibasis inequality. Let x be a finite linear combination of x_j's. We may assume that x is of the form $\sum_{s=1}^{m}\sum_{k=1}^{s}\alpha_k^{(s)} z_k^{(s)}$. Since the inner blocks have disjoint supports, the left hand side of the bibasis inequality may be written as

$$\Bigl\|\sum_{s=1}^{m}\bigvee_{n=1}^{s}\Bigl|\sum_{k=1}^{n}\alpha_k^{(s)} z_k^{(s)}\Bigr|\Bigr\| = \Biggl(\sum_{s=1}^{m}\Bigl\|\bigvee_{n=1}^{s}\Bigl|\sum_{k=1}^{n}\alpha_k^{(s)} z_k^{(s)}\Bigr|\Bigr\|^p\Biggr)^{\frac{1}{p}}$$

$$\leqslant C_p'\Biggl(\sum_{s=1}^{m}\Bigl\|\sum_{k=1}^{s}\alpha_k^{(s)} z_k^{(s)}\Bigr\|^p\Biggr)^{\frac{1}{p}} = C_p'\|x\|.$$

Hence, (x_j) is bibasic. On the other hand, since $(z_k^{(s)})_{k=1}^{s}$ is C_p-equivalent to the unit vector basis of ℓ_2^s for every s, the sequence (x_j) is not equivalent to the unit vector basis of ℓ_p.

Finally, we sketch the proof that (x_j) is permutable. We will use Proposition 6.4. As in the proof of Proposition 8.1, the estimate in (8) remains valid if we permute the sequence $\big(z_k^{(s)}\big)_{k=1}^{s}$. Fix indices $k_1,\dots,k_m$ and coefficients $\alpha_1,\dots,\alpha_m$. Then

$$\bigvee_{n=1}^{m}\Big|\sum_{i=1}^{n}\alpha_i x_{k_i}\Big| = \sum_{s=1}^{\infty}\bigvee_{n=1}^{m}\Big|\sum_{\substack{i=1,\dots,n\\ 2^s\leqslant k_i<2^{s+1}}}\alpha_i x_{k_i}\Big|$$

where the sum over s has only finitely many non-zero terms. Moreover, the terms are pair-wise disjoint, so that, using the permuted version of (8), we get

$$\begin{aligned}\Big\|\bigvee_{n=1}^{m}\Big|\sum_{i=1}^{n}\alpha_i x_{k_i}\Big|\Big\| &= \Bigg(\sum_{s=1}^{\infty}\Big\|\bigvee_{n=1}^{m}\Big|\sum_{\substack{i=1,\dots,n\\ 2^s\leqslant k_i<2^{s+1}}}\alpha_i x_{k_i}\Big|\Big\|^p\Bigg)^{\frac{1}{p}}\\ &\leqslant C_p'\Bigg(\sum_{s=1}^{\infty}\Big\|\sum_{\substack{i=1,\dots,m\\ 2^s\leqslant k_i<2^{s+1}}}\alpha_i x_{k_i}\Big\|^p\Bigg)^{\frac{1}{p}} = C_p'\Big\|\sum_{i=1}^{m}\alpha_i x_{k_i}\Big\|.\end{aligned}$$

The basic sequence, constructed in the previous example, is clearly not a basis. Recall that every permutable basis in ℓ_1 is unconditional and, therefore, absolute by Proposition 7.8. This motivates the following question:

Question 8.9. Is there a basis of ℓ_p $(1<p<\infty)$ which is permutable but not absolute?

Example 8.10. *The Walsh sequence and Krengel's operator.* While the preceding example deals with a discretization of the Rademacher sequence, in this example we consider the Walsh sequence and its discretization. Let (w_k) be the Walsh sequence in $L_2[0,1]$ with its standard enumeration as in, e.g., [Wade82]. Then (w_k) is an orthonormal basis of $L_2[0,1]$. It is shown in [Hunt70] (see, also, [Wade82, p. 631]) that there is a constant M such that for every $f\in L_2[0,1]$ with Walsh-Fourier expansion $f=\sum_{k=0}^{\infty}\alpha_k w_k$ one has $\Big\|\bigvee_{n=0}^{\infty}\big|\sum_{k=0}^{n}\alpha_k w_k\big|\Big\|\leqslant M\|f\|$. It follows from Bibasis Theorem 2.1 that (w_k) is a bibasis.

It is easy to see that (w_k) is not absolute. Indeed, since it is an orthonormal basis of $L_2[0,1]$, we have $\Big\|\sum_{k=0}^{n-1}w_k\Big\|=\sqrt{n}$. On the other hand, it follows from $|w_k|=\mathbb{1}$ that $\Big\|\sum_{k=0}^{n-1}|w_k|\Big\|=n$. Hence, the two expressions are not equivalent.

The Walsh sequence is closely related to the classical example of an operator $T\colon\ell_2\to\ell_2$ which is not order bounded, due to Krengel; see, e.g., Example 5.6 in [AB06]. The operator is defined as follows. For each $n=0,1,2,\dots$, define the Hadamar matrix H_n as follows: $H_0=(1)$, $H_{n+1}=\left[\begin{smallmatrix}H_n & H_n\\ H_n & -H_n\end{smallmatrix}\right]$.

Put $T_n=2^{-\frac{n}{2}}H_n$. Viewed as an operator on $\ell_2^{2^n}$, T_n is a surjective isometry. We view $\ell_2=\bigoplus_{n=0}^{\infty}\ell_2^{2^n}$ with $T=\bigoplus_{n=0}^{\infty}T_n$. Then T is a surjective isometry, and, therefore, the sequence (Te_k) is an orthonormal basis in ℓ_2. Is it bibasic? Since T fails to be order bounded (it even fails to be sequentially uniformly continuous), we cannot apply Theorem 5.6.

Let W_n be the matrix obtained from H_n be ordering its columns by the number of sign changes; W_n is called the Walsh matrix of order n. The columns of W_n viewed as a sequence in $\ell_2^{2^n}$ correspond to $(w_k)_{k=0}^{2^n-1}$. Replacing H_n in the construction of T with W_n, we obtain another surjective isometry on ℓ_2; denote it by S. The sequence (Se_k) is a permutation of (Te_k). We claim that (Se_k) is a bibasis. Note that this sequence comes in pairwise disjoint blocks. Within the n-th block, the sequence (Se_k) may be identified with $(w_k)_{k=0}^{2^n-1}$, hence it satisfies the bibasis inequality with constant M as before. Thus, (Se_k) is a bibasis.

Since W_n is obtained by permuting columns in H_n, we have $H_n = W_nP_n$ for some permutation matrix P_n. Since both H_n and W_n are symmetric, we have $H_n = P_n^TW_n$, so that $T = US$, where U is a permutation of the standard basis in ℓ_2. It follows that $Te_k = U(Se_k)$. Since U is clearly a surjective isometry and a lattice homomorphism, Theorem 5.6 yields that (Te_k) is a bibasis.

Now let $1 < p < \infty$ with $p \neq 2$. In this case, the Walsh sequence (w_k) forms a conditional basis in $L_p[0,1]$; see, e.g., [PR13, p. 6] or [Mul05, pp. 23–24]. It was shown in [Sjol69] that (w_k) satisfies the bibasis inequality, hence it is a bibasis. As in the preceding paragraphs, one can construct a discretized version of (w_k) in ℓ_p; it is easy to see that it is a conditional bibasis of ℓ_p. Our investigation leaves open the existence of a conditional bibasis in ℓ_1, ℓ_2, and $L_2[0,1]$.

9. Bibasic sequences with unique order expansions

A sequence (x_k) in a Banach lattice X is said to have ***unique order expansions*** if ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k = {}^{\mathrm{o}}\sum_{k=1}^{\infty} \beta_k x_k$ implies that $\alpha_k = \beta_k$ for every k. Clearly, this is equivalent to zero having a unique order expansion. In particular, (x_k) is an order basis if every vector has a unique order expansion. It is easy to see that every bibasic sequence in an order continuous Banach lattice has unique order expansions.

Remark 9.1. In the definition of unique order expansions one must choose whether to use order or σ-order-convergence. We will work with order convergence; the reader who prefers σ-order convergence can make the appropriate modifications. For bibases, we do not know if the choice of order convergence matters:

Question 9.2. Suppose (x_k) is a bibasis with unique σ-order expansions. Does (x_k) have unique order expansions?

Example 9.3. Let $X = c$, the space of all convergent sequences. Let $e_0 = (1, 1, \dots)$. Then $(e_n)_{n\geqslant 0}$ is a basis and, therefore, a bibasis of c. However, $e_0 = {}^{\mathrm{o}}\sum_{k=1}^{\infty} e_k$, hence e_0 has multiple order expansions. Notice, in contrast, that the basis $(x_k)_{k\geqslant 1}$ of c with $x_k = (0, \dots, 0, 1, 1, 1, \dots)$ has unique order expansions.

Example 9.4. *Uniqueness of order expansions depends on the ambient space.* Let (x_k) be the Schauder system in $C[0, 1]$. Since $C[0, 1]$ is an AM-space, (x_k) is a bibasis. Yet, it fails to have unique order expansions by Example 1.5. We are going to construct a Banach lattice X such that $C[0, 1]$ is a closed sublattice of X and (x_k) has unique order expansions relative to X.

For a compact Hausdorff space K, we put $c_0(K)$ to be the space of real-valued functions f on K such that the set $\{|f| > \varepsilon\}$ is finite for every $\varepsilon > 0$. In particular, $c_0(K)$ contains all the functions with finite support. One defines $CD_0(K)$ as the space of functions of the form $f + g$ where $f \in C(K)$ and $g \in c_0(K)$. It is known that $CD_0(K)$ is an AM-space; $C(K)$ is a norm closed sublattice of $CD_0(K)$. We refer the reader to [AW93, Tro04] and references therein for basic properties of $CD_0(K)$-spaces.

Put $X = CD_0[0, 1]$. We claim that order expansions of (x_k) with respect to $CD_0[0, 1]$ are unique. Suppose that ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k = 0$ in $CD_0[0, 1]$. Let $s_n = \sum_{k=1}^{n} \alpha_k x_k$. Note that $s_n(0) = \alpha_1$ for every n. It follows that $\left|\alpha_1 \mathbb{1}_{\{0\}}\right| \leqslant |s_n| \xrightarrow{\mathrm{o}} 0$ in $CD_0[0, 1]$, hence $\alpha_1 = 0$. Therefore, $s_n(1) = \alpha_2$ and, therefore, $\left|\alpha_2 \mathbb{1}_{\{1\}}\right| \leqslant |s_n|$ for every $n \geqslant 2$. It follows from $s_n \xrightarrow{\mathrm{o}} 0$ in $CD_0[0, 1]$ that $\alpha_2 = 0$. It follows that $s_n(\frac{1}{2}) = \alpha_3$ and, therefore, $\left|\alpha_3 \mathbb{1}_{\{\frac{1}{2}\}}\right| \leqslant |s_n|$ for all $n \geqslant 3$. This yields $\alpha_3 = 0$. Proceeding inductively, $\alpha_k = 0$ for all k.

While in general the concept of a bibasic sequence with unique order expansions may depend on the ambient space, here we present an interesting example where it does not. Recall that every basic sequence in c_0 is bibasic with unique order expansions.

Theorem 9.5. *Let (x_k) be a basic sequence in c_0. Viewed as a sequence in ℓ_∞, it is bibasic with unique order expansions.*

Proof. Clearly, (x_k) is bibasic in ℓ_∞. Suppose that there exists a sequence (α_k) of coefficients, not all of them zero, such that ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k = 0$ in ℓ_∞. WLOG, ${}^{\mathrm{o}}\sum_{k=1}^{\infty} x_k = 0$; otherwise, pass to the subsequence of those x_k's for which $\alpha_k \neq 0$ and replace x_k with $\alpha_k x_k$.

Put $s_n = \sum_{k=1}^{n} x_k$. Then $s_n \xrightarrow{\mathrm{o}} 0$ in ℓ_∞. In particular, (s_n) converges to zero coordinate-wise and (s_n) is order bounded in ℓ_∞ and, therefore, norm bounded. Since (x_k) is basic, the zero vector has no non-trivial norm expansions, so that (s_n) does not converge to zero. It follows that there exists $\delta > 0$ such that $\|s_n\| > \delta$ for infinitely many values of n.

Fix a sequence (ε_m) in $(0, \delta/2)$ such that $\varepsilon_m \to 0$. We will use a variant of a "gliding hump" technique to find an "almost disjoint" subsequence of (s_n). Let $P_n\colon \ell_\infty \to \ell_\infty$ be the projection onto the first n coordinates; let $Q_n = I - P_n$.

Choose n_1 so that $\|s_{n_1}\| > \delta$. Since s_{n_1} is in c_0, there exists k_1 such that $\|Q_{k_1} s_{n_1}\| < \varepsilon_1$. Put $v_1 = P_{k_1} s_{n_1}$, then $\operatorname{supp} v_1 \subseteq [1, k_1]$ and $\|s_{n_1} - v_1\| < \varepsilon_1$. Since (s_n) converges to zero coordinate-wise, we can find $n_2 > n_1$ such that $\|P_{k_1} s_{n_2}\| < \varepsilon_2$ and $\|s_{n_2}\| > \delta$. Since $s_{n_2} \in c_0$, find k_2 such that $\|Q_{k_2} s_{n_2}\| < \varepsilon_2$. Put $v_2 = P_{k_2} Q_{k_1} s_{n_2}$. Then $\operatorname{supp} v_2 \subseteq [k_1 + 1, k_2]$ and $\|s_{n_2} - v_2\| < \varepsilon_2$.

Proceeding inductively, we produce a subsequence (s_{n_m}) of (s_n) and a sequence (v_m) such that $\|s_{n_m}\| > \delta$, $\|s_{n_m} - v_m\| < \varepsilon_m$, and $\operatorname{supp} v_m < \operatorname{supp} v_{m+1}$ for every m. It follows from $\varepsilon_m < \delta/2$ that $\|v_m\| > \delta/2$. Being a disjoint seminormalized sequence in c_0, (v_m) is basic and is equivalent to (e_m). Passing to a further subsequence if necessary, we may assume that (s_{n_m}) is also basic and is equivalent to (v_m) and, therefore, to (e_m). Hence, there is an isomorphic embedding $T\colon c_0 \to c_0$ with $Te_m = s_{n_m}$. Put $W = \operatorname{Range} T = [s_{n_m}]$.

Put $y_1 = s_{n_1}$ and $y_m = s_{n_m} - s_{n_{m-1}}$ when $m > 1$. Then $y_m \in W$ and $y_m = \sum_{k=n_{m-1}+1}^{n_m} x_k$ for every m. It follows that (y_m) is a block sequence of (x_k), hence is a basic sequence. It follows from $T^{-1} y_1 = e_1$ and $T^{-1} y_m = e_m - e_{m-1}$ that the sequence $(e_1, e_2 - e_1, e_3 - e_2, \dots)$ is basic. This is a contradiction because this sequence fails the basis inequality. Indeed, for every m, we have

$$e_1 + \tfrac{m-1}{m}(e_2 - e_1) + \tfrac{m-2}{m}(e_3 - e_2) + \dots + \tfrac{1}{m}(e_m - e_{m-1}) = \tfrac{1}{m}(e_1 + \dots + e_m),$$

hence this vector has norm $\frac{1}{m}$, while $\|e_1\| = 1$. □

Remark 9.6. Example 9.3 shows that one cannot replace c_0 with c in the above theorem.

Example 9.7. *For bibasic sequences, uniqueness of order expansions is not always preserved under small perturbations.* Let $X = c$. Put $y_1 = (0, 1, 1, \dots)$ and $y_k = e_k$ as $k \geqslant 2$. Clearly, (y_k) is basic; since c is an AM-space, it follows that (y_k) is bibasic. However, it fails to have unique order expansions as $y_1 = {}^{\mathrm{o}}\!\sum_{k=2}^{\infty} y_k$. Let $x_1 = y_1 + \varepsilon e_1$ and $x_k = y_k$ when $k \geqslant 2$. Picking $\varepsilon > 0$ sufficiently small, (y_k) is a small perturbation of (x_k), yet (x_k) has unique order expansions. By amplifying this example to the c_0-sum of infinitely many copies of c with the ε-perturbation in the n-th copy of (x_k) going to zero sufficiently fast, and then re-enumerating the resulting sequence, we can produce a normalized bibasic sequence (x_k) with unique order expansions such that for every $\delta > 0$ one can find a bibasic sequence (y_k) such that $\sum_{k=1}^{\infty} \|x_k - y_k\| < \delta$ and, nevertheless, (y_k) fails to have unique order expansions.

We next identify a combination of conditions which guarantees stability under small perturbation. For a bibasic sequence (x_k) in a Banach lattice X, we write $[x_k]^o$ for the set of all vectors $x \in X$ which admit an order expansion of the form $x = {}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k$. Since (x_k) is bibasic, it is immediate that $[x_k] \subseteq [x_k]^o$. We say that a sequence (x_k) is ***sester-basic*** if it is bibasic, has unique order expansions, and $[x_k] = [x_k]^o$. Clearly, every basis with unique order expansions is sester-basic; if X is order continuous then every bibasic sequence is sester-basic. The proof of the following proposition is straightforward.

Proposition 9.8. *For a sequence (x_k), TFAE:*

(i) *(x_k) is sester-basic;*
(ii) *(x_k) is bibasic and $\sum_{k=1}^{\infty} \alpha_k x_k$ converges whenever ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k$ converges;*
(iii) *(x_k) is basic, and $\sum_{k=1}^{\infty} \alpha_k x_k$ converges iff ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k$ converges.*

Proposition 9.9. *Let (x_k) be a sester-basic sequence in a Banach lattice X with basis constant K; let (y_k) be a sequence in X such that*

$$2K \sum_{k=1}^{\infty} \frac{\|x_k - y_k\|}{\|x_k\|} < 1.$$

Then (y_k) is sester-basic.

Proof. By Principle of Small Perturbations and Theorem 4.2, (y_k) is bibasic and equivalent to (x_k). WLOG (x_k) is normalized, hence (y_k) is semi-normalized. Suppose ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k y_k$ converges; it suffices to show that $\sum_{k=1}^{\infty} \alpha_k y_k$ converges. The partial sums $\left(\sum_{k=1}^{n} \alpha_k y_k\right)_n$ are order bounded, hence $(\alpha_k y_k)$ is order and norm bounded, hence (α_k) is bounded. Put $z := \sum_{k=1}^{\infty} \alpha_k (x_k - y_k)$. This series converges absolutely. Lemma 1.1 yields that it converges uniformly and, therefore, in order. It follows that the series ${}^{\mathrm{o}}\sum_{k=1}^{\infty} \alpha_k x_k$ converges, which implies the convergence of $\sum_{k=1}^{\infty} \alpha_k x_k$. Since $(y_k) \sim (x_k)$, the series $\sum_{k=1}^{\infty} \alpha_k y_k$ converges. □

Example 9.10. *The property of being sester-basic depends on the ambient space.* The standard unit vector basis (e_k) of c_0 is sester-basic in c_0 but not in ℓ_∞ because in ℓ_∞ we have $c_0 = [e_k] \neq [e_k]^o = \ell_\infty$. This example also shows that the inclusion $[x_k] \subseteq [x_k]^o$ may be proper.

Question 9.11. It was observed in Example 1.5 that the Schauder system in $C[0,1]$ is not a sester-basis. Does $C[0,1]$ admit a sester-basis? More generally, does every Banach lattice with a bibasis admit a sester-basis?

Question 9.12. Does every block sequence of a bibasis with unique order expansions again have unique order expansions? If so, is it sester-basic? More generally, is every block sequence of a sester-basic sequence again sester-basic?

10. Uo-bibasic sequences

Recall that a net (x_α) in an Archimedean vector lattice X is said to ***uo-converge*** to x if $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{o}} 0$ for every $u \geqslant 0$; in this case, we write $x_\alpha \xrightarrow{\mathrm{uo}} x$. Clearly, order convergence implies uo-convergence; the two convergences agree for order bounded nets. If Y is a regular sublattice of X (in particular, if Y is an ideal of X or if Y is a closed sublattice and X is an order continuous Banach lattice) and (x_α) is a net in Y then $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in Y iff $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in X. For sequences in $L_p(\mu)$ spaces with μ semi-finite, uo-convergence agrees with convergence almost everywhere. We refer the reader to [GTX17] and references therein for background on uo-convergence; see also [Pap64, Frem04].

Motivated by the definition of a bibasic sequence, we say that a sequence (x_k) in a Banach lattice X is ***uo-bibasic*** if it is basic and for each $x \in [x_k]$ the sequence of partial sums of x uo-converges to x. It is clear that every bibasic sequence is uo-bibasic.

Example 10.1. In an atomic Banach lattice uo-convergence agrees with point-wise convergence, so every basic sequence is uo-bibasic. In particular, the class of uo-bibasic sequences is much larger than the class of bibasic sequences, even in ℓ_p ($p < \infty$).

Example 10.2. The Haar basis (h_k) in its standard ordering is a uo-bibasis in $L_p[0,1]$ when $1 \leqslant p < \infty$. In the case when $p > 1$, it follows from the fact that (h_k) is a bibasis; when $p = 1$ the statement follows from, e.g., Theorem 4 in [KS89, p. 68].

In general, we do not know whether the property of being a uo-bibasic sequence depends on the ambient space. However, if Y is a closed regular sublattice of X and (x_k) is a sequence in Y, it is clear that it is uo-bibasic in Y iff it is uo-bibasic in X.

It can be easily verified that a block sequence of a uo-bibasic sequence is again uo-bibasic. We next show that uo-bibasic sequences are stable under small perturbations; the proof is analogous to that of Theorem 3.2 in [GKP15].

Proposition 10.3. *Let (x_k) be a uo-bibasic sequence in a Banach lattice X with basis constant K; let (y_k) be a sequence in X such that $2K \sum_{k=1}^{\infty} \frac{\|x_k - y_k\|}{\|x_k\|} < 1$. Then (y_k) is uo-bibasic.*

Proof. By the usual Principle of Small Perturbations for basic sequences, (y_k) is basic and is equivalent to (x_k). WLOG, (x_k) is normalized. Suppose that $y = \sum_{k=1}^{\infty} \alpha_k y_k$. We need to show that $y = {}^{\mathrm{uo}}\sum_{k=1}^{\infty} \alpha_k y_k$. Since $(y_k) \sim (x_k)$, the series $x := \sum_{k=1}^{\infty} \alpha_k x_k$ converges in norm. Since (x_k) is uo-bibasic, we have $x = {}^{\mathrm{uo}}\sum_{k=1}^{\infty} \alpha_k x_k$. Since (x_k) is normalized, the sequence (α_k) is bounded. It follows that the series $\sum_{k=1}^{\infty} |\alpha_k||x_k - y_k|$ converges. Put $v_n = \sum_{k=n+1}^{\infty} |\alpha_k||x_k - y_k|$. Then $v_n \downarrow 0$. It follows from

$$\Big|y - \sum_{k=1}^{n} \alpha_k y_k\Big| \leqslant \Big|x - \sum_{k=1}^{n} \alpha_k x_k\Big| + v_n \xrightarrow{\mathrm{uo}} 0$$

that $y = {}^{\mathrm{uo}}\sum_{k=1}^{\infty} \alpha_k y_k$. □

Question 10.4. Does every closed infinite dimensional subspace of a Banach lattice contain a uo-bibasic sequence?

To finish, we answer the above question in a large class of Banach lattices.

Theorem 10.5. *Every closed infinite dimensional subspace of an order continuous Banach lattice contains an unconditional uo-bibasic sequence.*

Proof. Let Y be a closed infinite dimensional subspace of an order continuous Banach lattice X; we will show that Y contains an unconditional uo-bibasic sequence. WLOG, Y is separable. Let $\overline{S(Y)}$ be the closed sublattice generated by Y in X. It is easy to see that $\overline{S(Y)}$ is separable and regular in X. Therefore, replacing X with $\overline{S(Y)}$, we may assume that X is separable. It follows that X has a weak unit. We may then continuously embed X as a norm dense ideal into $L_1(\mu)$ for some probability measure μ; see Theorem 1.b.14 in [LT79]. Following the proof of Proposition 1.c.8 in [LT79], we reduce to the following two cases:

Case 1: The norms $\|\cdot\|_X$ and $\|\cdot\|_{L_1(\mu)}$ are equivalent on Y. In this case we may view Y as a closed subspace of $L_1(\mu)$. By Corollary 8.3, Y contains an unconditional basic sequence (y_k) which is bibasic in $L_1(\mu)$. It is left to show that (y_k) is uo-bibasic in X. Let $y = \sum_{k=1}^{\infty} \alpha_k y_k$, where the series converges in norm (it does not matter in which norm because $\|\cdot\|_X$ and $\|\cdot\|_{L_1(\mu)}$ are equivalent on Y). Since (y_k) is uo–bibasic in $L_1(\mu)$, we have $y = {}^{\mathrm{uo}}\sum_{k=1}^{\infty} \alpha_k y_k$ in $L_1(\mu)$. Since X is an ideal in $L_1(\mu)$, we conclude that $y = {}^{\mathrm{uo}}\sum_{k=1}^{\infty} \alpha_k y_k$ in X.

Case 2: There is a sequence (y_k) in Y and a disjoint sequence (x_k) in X such that $\|y_k\|_X = 1$ for all k and $\|y_k - x_k\|_X \to 0$. Being disjoint, (x_k) is unconditional and bibasic in X. Passing to a subsequence, if necessary, and applying the Principle of Small Perturbations, we conclude that (y_k) is unconditional and bibasic and, therefore, uo-bibasic in X. □

We do not know if every closed infinite dimensional subspace of an order continuous Banach lattice contains a bibasic sequence. We also don't know if such subspaces contain permutable uo-bibasic sequences, i.e., unconditional basic sequences such that every permutation is uo-bibasic.

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