BOUNDED INDECOMPOSABLE SEMIGROUPS OF NON-NEGATIVE MATRICES

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Abstract. A semigroup $\mathfrak{S}$ of non-negative $n \times n$ matrices is indecomposable if for every pair $i, j \leqslant n$ there exists $S \in \mathfrak{S}$ such that $(S)_{ij} \neq 0$. We show that if there is a pair k, l such that $\{(S)_{kl} : S \in \mathfrak{S}\}$ is bounded then, after a positive diagonal similarity, all the entries are in $[0,1]$. We also provide quantitative versions of this result, as well as extensions to infinite-dimensional cases.

1. Introduction

The following general type of question has been of interest in various contexts, including linear representations of groups and semigroups: if something about a group or a semigroup $\mathfrak{S}$ is "small" in some sense, then is $\mathfrak{S}$ itself small? For example, it is well known that if $\mathfrak{S}$ is an irreducible group of matrices, and if the trace functional takes a finite number of values on $\mathfrak{S}$, then $\mathfrak{S}$ is itself finite (*irreducible* means no common proper non-trivial invariant subspaces). Okninsky in [Ok98, Proposition 4.9] generalizes this to irreducible semigroups. A further extension to another version of "smallness" is given in [RR08]: it replaces the trace functional with any nontrivial linear functional. Yet another measure of "smallness" is boundedness. For example, if the values of a nontrivial linear functional on an irreducible semigroup $\mathfrak{S}$ form a bounded set, then $\mathfrak{S}$ itself is bounded. For this and other instances of this local-to-global phenomena see [RR08].

In this paper we discuss another variation on this question which is more suitable in the positivity setting. We consider semigroups of non-negative matrices, replace the irreducibility assumption on $\mathfrak{S}$ with the weaker hypothesis of indecomposablity, i.e., no common invariant ideals, and ask: if a non-negative linear functional has bounded values on $\mathfrak{S}$, then is $\mathfrak{S}$ necessarily bounded? The version of this problem in which "smallness" is interpreted as finiteness has also been studied in [LMR].

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Throughout this paper, all the matrices are taken over $\mathbb{R}$. For two $n \times n$ matrices A and B, we write $A \leqslant B$ if $(A)_{ij} \leqslant (B)_{ij}$ for every pair $i, j \leqslant n$. A matrix A is ***non-negative*** if $A \geqslant 0$ and ***positive*** if $(A)_{ij} > 0$ for every i, j.[1] We will write $M_n^+(\mathbb{R})$ for all non-negative $n \times n$ real matrices. By E_{ij} we will denote the ij-th elementary matrix.

A (multiplicative) semigroup $\mathfrak{S}$ of $M_n^+(\mathbb{R})$ is said to be ***indecomposable*** if for every $i, j \leqslant n$ there exists $S \in \mathfrak{S}$ with $(S)_{ij} > 0$. The following two lemmas are straightforward and standard; see [Min88] for more details.

Proposition 1. *Let $\mathfrak{S}$ be a semigroup in $M_n^+(\mathbb{R})$. Then the following statements are equivalent.*

(i) *$\mathfrak{S}$ is indecomposable;*
(ii) *$\mathfrak{S}$ has no common non-trivial proper invariant ideals (i.e., subspaces spanned by a subset of the standard basis);*
(iii) *No permutation of the basis reduces $\mathfrak{S}$ to the block form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$.*

Remark 2. Let $A, D \in M_n(\mathbb{R})$ such that D is diagonal and invertible, $A = (a_{ij})$ and $D = \operatorname{diag}(d_1, \dots, d_n)$. Then the ij-th entry of $D^{-1}AD$ equals $\frac{d_j}{d_i} a_{ij}$. In particular, the diagonal entries of A and of $D^{-1}AD$ agree.

Remark 3. Let $\mathfrak{S}$ be a semigroup in $M_n^+(\mathbb{R})$. Since $M_n(\mathbb{R})$ is finite-dimensional, the following are equivalent:

(i) $\mathfrak{S}$ is norm bounded;
(ii) $\mathfrak{S}$ is bounded entry-wise, i.e., $\sup\{(S)_{ij} : S \in \mathfrak{S}\} < +\infty$ for every pair $i, j \leqslant n$;
(iii) $\mathfrak{S}$ is order bounded, i.e., there exists $T \in M_n(\mathbb{R})$ such that $S \leqslant T$ for every $S \in \mathfrak{S}$. In this case, we write $\mathfrak{S} \leqslant T$.

In this case, $\sup \mathfrak{S}$ is defined. That is, $\sup \mathfrak{S}$ is the matrix whose ij-th entry is $\sup\{(S)_{ij} : S \in \mathfrak{S}\}$.

Lemma 4. *Let $\mathfrak{S}$ be a bounded semigroup in $M_n^+(\mathbb{R})$ and D a diagonal matrix with positive diagonal entries. Then $D^{-1}\mathfrak{S}D$ is again a bounded semigroup and $\sup(D^{-1}\mathfrak{S}D) = D^{-1}(\sup \mathfrak{S})D$.*

A matrix $T = (t_{ij})$ will be called ***compressed*** if $T \geqslant 0$ and $t_{ij}t_{jk} \leqslant t_{ik}$ for all i, j, and k. The following observations are straightforward.

[1]Note that in Banach lattice theory, $A \geqslant 0$ is usually termed "positive", while $\forall i, j\ (A)_{ij} > 0$ is termed "strictly positive".

Lemma 5. (i) *If $T = (t_{ij})$ is compressed then $t_{ii} \leqslant 1$ for all i.*

(ii) *If $\mathfrak{S}$ is a bounded semigroup in $M_n^+(\mathbb{R})$ then $T = \sup \mathfrak{S}$ is compressed. In this case, $\mathfrak{S}$ is indecomposable iff T is positive.*

(iii) *Let T be a compressed matrix and D a diagonal matrix with positive diagonal entries. Then $D^{-1}TD$ is compressed.*

2. Main results

Let $\mathfrak{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$. In this section we show that if $\mathfrak{S}$ is bounded then, after a positive diagonal similarity, all its entries are bounded by 1. Moreover, it suffices to assume only that the set $\bigl\{(S)_{ij} : S \in \mathfrak{S}\bigr\}$ is bounded for some pair (i, j). Next, we will show that if the diagonal entries in $\mathfrak{S}$ are (uniformly) bounded by some $\epsilon > 0$, then, after a positive diagonal similarity, all the entries are uniformly bounded by $\sqrt[n]{\varepsilon}$.

Given $r > 0$, we write $M_n\bigl([0, r]\bigr)$ for the set of all $n \times n$ matrices with entries in $[0, r]$.

Lemma 6. *Suppose that $r \geqslant 1$ and $T \in M_n\bigl([0, r]\bigr)$ is compressed. Then there exists $D = \operatorname{diag}(d_m)_{m=1}^n$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}TD \in M_n([0, 1])$.*

Proof. Let $T = (t_{ij})$. Since T is compressed, $t_{ii} \leqslant 1$ for all i. We will inductively construct $(d_m)_{m=1}^n$. For every $m \leqslant n$ we will put

$$D_m = \operatorname{diag}(d_1, \dots, d_m, 1, 1, \dots, 1).$$

Note that for every $m > 1$, $D_m^{-1}TD_m$ can be obtained from $D_{m-1}^{-1}TD_{m-1}$ by scaling the m-th column of the latter by d_m and the m-th row by $\frac{1}{d_m}$. It follows that the upper left $m \times m$ corners of $D_k^{-1}TD_k$ are the same for all $k \geqslant m$ and agree with the upper left $m \times m$ corner of $D^{-1}TD$. Therefore, it suffices to show that the $m \times m$ upper left corner of $D_m^{-1}TD_m$ is in $M_m\bigl([0, 1]\bigr)$ for every $m = 1, \dots, n$.

Put $d_1 = 1$. Suppose that $d_1, \dots, d_{m-1}$ have already been constructed (in the interval $[\frac{1}{r}, r]$) so that $U := D_{m-1}^{-1}TD_{m-1}$ is in $M_n([0, r])$ and its $(m-1) \times (m-1)$ upper left corner is in $M_{m-1}([0, 1])$. Put $U = (u_{ij})$. Once we assign a value to d_m, we will write $V = D_m^{-1}TD_m$, $V = (v_{ij})$. Put

$$a = \max_{i=1,\dots,m-1} u_{im} \qquad \text{and} \qquad b = \max_{j=1,\dots,m-1} u_{mj}.$$

Suppose first that both a and b are less then or equal to 1. In this case, the $m \times m$ upper left corner of U is already in $M_m([0, 1])$. Take $d_m = 1$; then $V = U$. Suppose now that $\max\{a, b\} > 1$.

Case $a \geqslant b$. Then $1 < a \leqslant r$ and there exists $k < m$ such that $u_{km} = a$. In this case, we put $d_m = \frac{1}{a}$, then $\frac{1}{r} \leqslant d_m < 1$. Since the m-th column of V is obtained by dividing the m-th column of U by a (except v_{mm} which equals $t_{mm} \leqslant 1$), we have $v_{im} \leqslant 1$ as $i = 1, \dots, m$ and $v_{im} \leqslant u_{im} \leqslant r$ as $i > m$. Also, $v_{km} = 1$. Since V is compressed, for every $j \neq m$ we have

$$v_{mj} = v_{km} v_{mj} \leqslant v_{kj} = u_{kj}$$

because $k < m$. It follows that $v_{mj} \leqslant 1$ for $j = 1, \dots, m$ and $v_{mj} \leqslant r$ for every $j > m$. Hence, V is in $M_n([0, r])$ and its $m \times m$ upper left corner is in $M_m([0, 1])$.

Case $b > a$. This case is similar. We have $1 < b \leqslant r$ and there exists $k < m$ such that $u_{mk} = b$. Put $d_m = b$. Since the m-th row of V is obtained by dividing the m-th row of U by b (except v_{mm} which equals $t_{mm} \leqslant 1$), we have $v_{mj} \leqslant 1$ as $j = 1, \dots, m$ and $v_{mj} \leqslant u_{mj} \leqslant r$ as $j > m$. Also, $v_{mk} = 1$. Since V is compressed, for every $i \neq m$ we have

$$v_{im} = v_{im} v_{mk} \leqslant v_{ik} = u_{ik}$$

as before. It follows that $v_{im} \leqslant 1$ for $i = 1, \dots, m$ and $v_{im} \leqslant r$ for $i > m$. Hence, V is in $M_n([0, r])$ and its $m \times m$ upper left corner is in $M_m([0, 1])$. ☐

Theorem 7. *Let $r \geqslant 1$ and $\mathfrak{S}$ be a semigroup in $M_n\big([0, r]\big)$. Then there exists $D = \operatorname{diag}(d_m)_{m=1}^n$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}\mathfrak{S}D \in M_n\big([0, 1]\big)$.*

Proof. Let $T = \sup \mathfrak{S}$. Then T is compressed. Let D be as in Lemma 6. Now Lemma 4 yields $D^{-1}\mathfrak{S}D \leqslant D^{-1}TD$, and the result follows. ☐

Proposition 8. *Let $\mathfrak{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$. Suppose that there exists a non-zero non-negative functional $\phi \in \big(M_n(\mathbb{R})\big)^*$ such that the set $\{\phi(S) : S \in \mathfrak{S}\}$ is bounded. Then $\mathfrak{S}$ is bounded.*

Proof. Write

$$\phi(A) = \sum_{i,j=1}^{n} c_{ij} a_{ij}, \quad A = (a_{ij}),$$

where $c_{ij} \geqslant 0$. Since ϕ is non-zero, there exist k, l such that $c_{kl} \neq 0$. Since $\phi(A) \geqslant c_{kl} a_{kl}$ for every positive matrix $A = (a_{ij})$, the set $\{(S)_{kl} : S \in \mathfrak{S}\}$ is bounded.

To finish the proof, it suffices to show that the set $\{(S)_{ij} \colon S \in \mathfrak{S}\}$ is bounded for every pair of indices $i, j \leqslant n$. Suppose that this statement is not true: there exist two indices $i, j \leqslant n$ and a sequence (S_m) in $\mathfrak{S}$ such that $(S_m)_{ij} \to \infty$ as $m \to \infty$. There are two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathfrak{S}$ such that $a_{ki} \neq 0$ and $b_{jl} \neq 0$. Then

$$a_{ki}(S_m)_{ij} b_{jl} \leqslant \big(AS_mB\big)_{kl} \leqslant \sup\{(S)_{kl} \colon S \in \mathcal{S}\} < \infty$$

holds for every $m \in \mathbb{N}$, which is impossible. $\square$

Combining Theorem 7 with Proposition 8, we immediately get the following results.

Corollary 9. *Let $\mathfrak{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ such that $\varphi(\mathfrak{S})$ is bounded for some positive functional $\phi \in \big(M_n(\mathbb{R})\big)^*$. Then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathfrak{S}D \subseteq M_n\big([0,1]\big)$.*

Corollary 10. *Let $\mathfrak{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ such that the set $\big\{(S)_{ij} \,:\, S \in \mathfrak{S}\big\}$ is bounded for some pair (i,j). Then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathfrak{S}D \subseteq M_n\big([0,1]\big)$.*

We have proved that if $\mathfrak{S}$ is bounded at a single entry then, after a positive diagonal similarity, all its entries are bounded by 1. Next, we will try to replace "bounded" with "small". We will show that if all the diagonal entries of $\mathfrak{S}$ are small then all the entries are small (after a positive diagonal similarity).

Lemma 11. *Let $T = (t_{ij})$ be an $n \times n$ positive compressed matrix and $\varepsilon > 0$. If $t_{ii} < \varepsilon$ for all $i \leqslant n$ then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}TD \in M_n\big([0, \sqrt[n]{\varepsilon}]\big)$.*

Proof. If $\varepsilon = 1$, the result follows immediately from Lemma 6. So we assume for the rest of the proof that $\varepsilon < 1$. Let

$$\delta = \inf_D \big\{ \max_{i,j} (D^{-1}TD)_{ij} \big\}, \tag{1}$$

where the infimum is taken over all diagonal matrices D with positive diagonal entries. Let $t_{\max} = \max_{i,j} t_{ij}$ and $t_{\min} = \min_{i,j} t_{ij}$. Note that $t_{\min} > 0$ as T is positive. Put

$$\mathcal{D} = \Big\{ \operatorname{diag}(d_1, \dots, d_n) \,:\, 1 \leqslant d_i \leqslant \frac{t_{\max}}{t_{\min}} \text{ for all } i = 1, \dots, n \Big\}.$$

We claim that the infimum in (1) can be taken over all $D \in \mathcal{D}$. Indeed, let D be a diagonal matrix with positive diagonal entries. Scaling D by a positive scalar we may assume that $\min d_i = 1$ without changing $D^{-1}TD$. Let $i_0 \leqslant n$ be such that $d_{i_0} = 1$; put $V = D^{-1}TD$, $V = (v_{ij})$. If there is a pair (i,j) such that $v_{ij} > t_{\max}$ then replacing D with I will only decrease $\max\limits_{i,j}(D^{-1}TD)_{ij}$; and $I \in \mathcal{D}$. So we may assume that $v_{ij} \leqslant t_{\max}$ for all i and j. Then, for every j we have

$$t_{\max} \geqslant v_{i_0 j} = t_{i_0 j} \frac{d_j}{d_{i_0}} \geqslant t_{\min} d_j,$$

so that $d_j \leqslant \frac{t_{\max}}{t_{\min}}$, hence $D \in \mathcal{D}$. This completes the proof of the claim.

Since $\mathcal{D}$ is compact, it follows that the infimum in (1) is, actually, attained at some D. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ and put $V = D^{-1}TD$, $V = (v_{ij})$. Then $\delta = \max_{i,j} v_{ij}$. Moreover, we may choose D so that the number of occurrences of δ in V is the smallest possible. Note that V is compressed by Lemma 5. It is left to show that $\delta \leqslant \sqrt[n]{\varepsilon}$. Suppose that, on the contrary, $\delta > \sqrt[n]{\varepsilon}$.

It follows that $\delta > \varepsilon$, so that δ never occurs on the diagonal of V. Hence, after a permutation of the basis, we may assume that $v_{12} = \delta$. We claim that $v_{2j} = \delta$ for some j. Indeed, otherwise, we could slightly decrease d_2 so that the non-diagonal entries in the second row of V increase but stay below δ, but then the non-diagonal entries in the second column of V would decrease, so that v_{12} would become less then δ; but this would contradict our assumption that V has the smallest possible number of occurrences of δ. Since δ never occurs on the diagonal of V we know that $j \neq 2$. Note also that $j \neq 1$ as, otherwise,

$$\delta^n \leqslant \delta^2 = v_{12}v_{21} \leqslant v_{11} = t_{11} \leqslant \varepsilon$$

would contradict our assumption that $\delta > \sqrt[n]{\varepsilon}$. Thus, $j > 2$. Again, by a permutation of the basis vectors $\vec{e}_3, \dots, \vec{e}_n$, we may assume that $j = 3$, so that $v_{23} = \delta$.

As in the preceding paragraph, we observe that $v_{3j} = \delta$ for some j. Again, we must have $j > 3$ because

$$\begin{aligned}
&\text{if } j = 1 \text{ then } \delta^n \leqslant \delta^3 = v_{12}v_{23}v_{31} \leqslant v_{11} = t_{11} \leqslant \varepsilon,\\
&\text{if } j = 2 \text{ then } \delta^n \leqslant \delta^2 = v_{23}v_{32} \leqslant v_{22} = t_{22} \leqslant \varepsilon,\\
&\text{if } j = 3 \text{ then } \delta^n \leqslant \delta = v_{33} = t_{33} \leqslant \varepsilon;
\end{aligned}$$

each case contradicts $\delta > \sqrt[n]{\varepsilon}$. Again, by a permutation of the basis vectors $\vec{e}_4, \dots, \vec{e}_n$, we may assume that $j = 4$, so that $v_{34} = \delta$.

Proceeding inductively, we show that for each $m \leqslant n$ we have (after a permutation of the basis) $v_{12} = \dots = v_{m-1,m} = \delta$, and that $v_{mj} = \delta$ for some j. Furthermore, $j > m$ as, otherwise, we would get $\delta^n \leqslant \delta^m \leqslant \varepsilon$. But this leads to a contradiction for $m = n$ as $j > n$ is impossible. $\square$

Theorem 12. *Let $\mathfrak{S}$ be an indecomposable semigroup in $M_n^+(\mathbb{R})$ and $\varepsilon > 0$. If all the diagonal entries in all the matrices in $\mathfrak{S}$ are less than or equal to ε then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}\mathfrak{S}D \subseteq M_n\big([0, \sqrt[n]{\varepsilon}]\big)$.*

Proof. By Proposition 8, $\mathfrak{S}$ is bounded. Let $T = \sup \mathfrak{S}$. Then T is positive and compressed by Lemma 5. By Lemma 11, there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}TD \in M_n\big([0, \sqrt[n]{\varepsilon}]\big)$. By Lemma 4, $D^{-1}\mathfrak{S}D \leqslant D^{-1}TD$, so that $D^{-1}\mathfrak{S}D \subseteq M_n\big([0, \sqrt[n]{\varepsilon}]\big)$. $\square$

The following example shows that the estimate obtained in Theorem 12 is sharp.

Example 13. Take any $\varepsilon \in (0,1]$ and put $\delta = \sqrt[n]{\varepsilon}$. Let

$$P = \begin{bmatrix} 0 & \delta & 0 & 0 & \dots & 0 \\ 0 & 0 & \delta & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \delta & 0 \\ 0 & 0 & \dots & \dots & 0 & \delta \\ \delta & 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

Let $\mathfrak{S} = \{P^k \ : \ k = 1, 2, \dots\}$. Clearly, $\mathfrak{S}$ is an indecomposable semigroup. The diagonal elements of P^k are all zeros for each $1 \leqslant k \leqslant n-1$, and $P^n = \delta^n I = \varepsilon I$. Also, $P^{k+n} = P^k P^n = \delta^n P^k \leqslant P^k$. Thus, the maximal value for every diagonal element over all the matrices in $\mathfrak{S}$ is ε. On the other hand, $(P)_{i,i+1} = (P)_{n,1} = \delta = \sqrt[n]{\varepsilon}$ for all $1 \leqslant i < n$. It is clear that this bound cannot be decreased by a positive diagonal similarity.

It might be natural to ask whether the assumption about the smallness of the diagonal entries in Theorem 11 could be replaced with smallness of some other functionals. For example, could it be sufficient to assume that a certain entry is small in all the matrices of $\mathfrak{S}$? The following example shows that the answer is negative.

Example 14. Let $\varepsilon > 0$. Generate a semigroup $\mathfrak{S}$ by the following matrices:

$$A = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $\mathfrak{S}$ is indecomposable. Also, it can be easily checked that

$$B^2 = C^2 = AC = BA = DA = DB = CD = 0,$$

$AB \leqslant B$, $CA \leqslant C$, $BC \leqslant A$, $CB \leqslant D$, $BD = B$, $DC = C$, $A^2 \leqslant A$, and $D^2 = D$. Hence, $(S)_{11} \leqslant \varepsilon$, $(S)_{12} \leqslant \varepsilon$, and $(S)_{21} \leqslant \varepsilon$ for all $S \in \mathfrak{S}$. Nevertheless, $(D)_{22} = 1$, and this cannot be made any smaller by applying a diagonal similarity since a diagonal similarity does not change diagonal entries of matrices.

3. An extension to infinite matrices

In this section, we extend Lemma 6 and Theorem 7 to the infinite-dimensional case. We start with extending our terminology. By an ***infinite non-negative matrix*** we will mean a double sequence $S = (s_{ij})_{i,j=1}^{\infty}$ with $s_{ij} \geqslant 0$ for all $i, j \in \mathbb{N}$. The set of all such matrices will be denoted $M_\infty^+(\mathbb{R})$. We will write $S \in M_\infty\bigl([0,r]\bigr)$ if $s_{ij} \leqslant r$ for some fixed $r > 0$. We say that $T = (t_{ij}) \in M_\infty^+(\mathbb{R})$ is ***compressed*** if $t_{ik}t_{kj} \leqslant t_{ij}$ for all i, j, and k. For $S, T, R \in M_\infty^+(\mathbb{R})$, we write $R = ST$ if $r_{ij} = \sum_{k=1}^\infty s_{ik}t_{kj}$ for every i and j in $\mathbb{N}$ (in particular, the series converges). A subset $\mathfrak{S}$ of $M_\infty^+(\mathbb{R})$ will be called a ***semigroup*** if ST exists and belongs to $\mathfrak{S}$ whenever $S, T \in \mathfrak{S}$. It is easy to see that in this case the multiplication is associative on $\mathfrak{S}$. A semigroup $\mathfrak{S}$ in $M_\infty^+(\mathbb{R})$ is said to be ***indecomposable*** if for every $i, j \in \mathbb{N}$ there exists $A \in \mathfrak{S}$ such that $(A)_{ij} \neq 0$. A semigroup $\mathfrak{S}$ in $M_\infty^+(\mathbb{R})$ is said to ***bounded entry-wise*** if $t_{ij} := \sup\{(S)_{ij} \,:\, S \in \mathfrak{S}\}$ is finite for every pair i, j. In this case, we write $T = \sup \mathfrak{S}$ where $T = (t_{ij})$. It is easy to see that, in this case, T is compressed. Then $\mathfrak{S}$ is indecomposable iff $t_{ij} > 0$ for all i and j.

The following lemma is straightforward.

Lemma 15. *Suppose that $D = \operatorname{diag}(d_m)_{m=1}^\infty \in M_\infty^+(\mathbb{R})$ such that $d_m > 0$ for all i.*

(i) $D^{-1} := \operatorname{diag}(d_m^{-1}) \in M_\infty^+(\mathbb{R})$ *and* $D^{-1}D = DD^{-1} = I$.

(ii) *For every $A \in M_\infty^+(\mathbb{R})$, $A = (a_{ij})$, the matrix $D^{-1}AD \in M_\infty^+(\mathbb{R})$ and its ij-th entry equals $a_{ij}\frac{d_j}{d_i}$. If S is compressed then so is $D^{-1}AD$.*

(iii) *If $\mathfrak{S} \subset M_\infty^+(\mathbb{R})$ is an entry-wise bounded semigroup then so is $D^{-1}\mathfrak{S}D$. Moreover, if $\mathfrak{S}$ is bounded entry-wise, then $D^{-1}\mathfrak{S}D$ is bounded entry-wise and* $\sup D^{-1}\mathfrak{S}D = D^{-1}\sup \mathfrak{S}D$.

The proofs of Lemma 16 and Theorem 17 repeat almost verbatim the proofs of Lemma 6 and Theorem 7; just replace n with ∞.

Lemma 16. *Suppose that $r \geqslant 1$ and $T \in M_\infty\bigl([0,r]\bigr)$ is compressed. Then there exists $D = \operatorname{diag}(d_m)$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}TD \in M_\infty([0,1])$.*

Theorem 17. *Let $r \geqslant 1$ and $\mathfrak{S}$ be a semigroup in $M_\infty\bigl([0,r]\bigr)$. Then there exists $D = \operatorname{diag}(d_m)$ with $(d_m) \subset [\frac{1}{r}, r]$ such that $D^{-1}\mathfrak{S}D \in M_\infty([0,1])$.*

Next, we will prove an analogue of Corollary 10.

Lemma 18. *Suppose that $T \in M_\infty^+(\mathbb{R})$ is positive and compressed. Then there exists $D = \operatorname{diag}(d_m)$ with $d_m > 0$ for all m such that $D^{-1}TD \in M_\infty([0,1])$.*

Proof. Put $d_1 = 1$. Inductively define positive numbers $d_2, d_3 \dots$, so that, for $D = \operatorname{diag}(d_m)$, $V = D^{-1}TD$, $V = (v_{ij})$, we have for every m:

(i) $v_{im} \leqslant 1$ whenever $i \leqslant m$, and

(ii) there exists $i_m < m$ such that $v_{i_m m} = 1$.

Indeed, once $d_1,\dots,d_{m-1}$ are chosen, put $D_{m-1}=\operatorname{diag}(d_1,\dots,d_{m-1},1,1,\dots)$, and take d_m to be the reciprocal of the maximal element in the m-th column of $D_{m-1}^{-1}TD_{m-1}$ above the diagonal.

By Lemma 15, V is compressed and $v_{ii}=t_{ii}\leqslant 1$ for every i. To show that $V\in M_n([0,1])$, we will prove by induction on m that the $m\times m$ upper-left corner of V is in $M_m([0,1])$. For $m=1$ we have $v_{11}=t_{11}\leqslant 1$. Suppose that $m>1$ and the $(m-1)\times(m-1)$ upper-left corner of V is in $M_{m-1}([0,1])$. Take $i,j\leqslant m$; we will show that $v_{ij}\leqslant 1$. If $i,j<m$, there is nothing to prove. If $j=m$ then we are done by (i). Suppose that $j<i=m$. Note that $v_{i_m,m}=1$ by (ii). By Lemma 15, V is compressed, so that

$$v_{ij}=v_{mj}=v_{i_m,m}v_{mj}\leqslant v_{i_m,j}\leqslant 1$$

by the induction hypothesis, as $i_m<m$ and $j<m$. $\square$

Theorem 19. *Let $\mathfrak{S}$ be an indecomposable semigroup in $M_\infty^+(\mathbb{R})$. If there exist $k,l\in\mathbb{N}$ such that the set $\big\{(S)_{kl}\ :\ S\in\mathfrak{S}\big\}$ is bounded then there exists $D=\operatorname{diag}(d_m)$ with $d_m>0$ for all m such that $D^{-1}\mathfrak{S}D\in M_\infty([0,1])$.*

Proof. First, we will show that $\mathfrak{S}$ is entry-wise bounded. Suppose not. Then there exist $i,j\in\mathbb{N}$ and a sequence (S_n) in $\mathfrak{S}$ such that $(S_n)_{ij}\to+\infty$ as $n\to+\infty$. Since $\mathfrak{S}$ is indecomposable, there exist $A,B\in\mathfrak{S}$ such that $(A)_{ki}\neq 0$ and $(B)_{jl}\neq 0$. Then $(AS_nB)_{kl}\geqslant (A)_{ki}(S_n)_{ij}(B)_{jl}\to+\infty$; a contradiction. Hence, $\mathfrak{S}$ is entry-wise bounded.

Put $T=\sup\mathfrak{S}$, then T is compressed. Let D be as in Lemma 18. Lemma 15 yields $D^{-1}\mathfrak{S}D\leqslant D^{-1}TD$, so that $D^{-1}\mathfrak{S}D\in M_\infty([0,1])$. $\square$

We would like to mention an immediate application to discrete Banach lattices. For the relevant terminology and more details, we refer the reader to [AA02, LT77]. Suppose that X is a Banach lattice where the order is generated by a 1-unconditional basis (e_n), that is, $\sum_{n=1}^\infty\alpha_ne_n\leqslant\sum_{n=1}^\infty\beta_ne_n$ iff $\alpha_n\leqslant\beta_n$ for all n (for example, X could be ℓ_p with $1\leqslant p<1$ or c_0). By scaling the vectors of the basis, we may usually assume without loss of generality that the basis is normalized, i.e., $\|e_i\|=1$ for every i.

Recall that an operator $T\colon X\to X$ determines an infinite matrix t_{ij} via $Te_j=\sum_{i=1}^\infty t_{ij}e_i$. The product of any two bounded operators agrees with the matrix product of their infinite matrices. An operator T is said to be ***positive*** if $Tx\geqslant 0$ whenever $x\geqslant 0$ or, equivalently, if its matrix is non-negative. In this case, T is automatically bounded (see, e.g., [AA02]).

An operator D is called ***diagonal*** if its infinite matrix is diagonal. Suppose $D = \mathrm{diag}(d_m)_{m=1}^{\infty}$, then D is bounded iff the sequence (d_m) is bounded; D is invertible with $D^{-1} = \mathrm{diag}(d_m^{-1})$ as long as $\inf d_m > 0$.

Corollary 20. *Let X be a Banach lattice with the order given by a 1-unconditional basis, and $\mathfrak{S}$ a semigroup of positive operators on X. If there exists $r \geqslant 1$ such that $(S)_{ij} \leqslant r$ for all $S \in \mathfrak{S}$ and $i, j \in \mathbb{N}$ then there exists $D = \mathrm{diag}(d_m)$ with $(d_m) \subset [\frac{1}{r}, r]$ such that all the entries of $D^{-1}\mathfrak{S}D$ are in $[0, 1]$.*

Corollary 21. *Let X be a Banach lattice with the order given by a 1-unconditional normalized basis, and $\mathfrak{S}$ a semigroup of positive operators on X. If $\mathfrak{S} \leqslant T$ for some bounded operator T, then there exists an invertible positive diagonal operator D such that all the entries of $D^{-1}\mathfrak{S}D$ are in $[0, 1]$.*

Proof. The matrix of T is entry-wise bounded because $t_{ij}e_i \leqslant Te_j$ yields $t_{ij} \leqslant \|T\|$ for all i, j. Now apply Corollary 20 with $r = \|T\|$. □

The following example shows that Theorem 19 cannot be extended to bounded operators on discrete Banach lattices.

Example 22. *An entry-wise bounded semigroup $\mathfrak{S}$ of bounded positive operators on ℓ_1 such that there is no positive diagonal operator $D\colon \ell_1 \to \ell_1$ with all the entries of $D^{-1}\mathfrak{S}D$ in $[0, 1]$.*

Let $\mathfrak{S} = \left\{\frac{i}{j}E_{ij} \ : \ i, j \in \mathbb{N}\right\} \cup \{0\}$. It can be easily verified that $\mathfrak{S}$ is an entry-wise bounded semigroup $\mathfrak{S}$ of bounded positive operators on ℓ_1. Suppose that $D = \mathrm{diag}(d_i)_{i=1}^{\infty}$ such that $d_i > 0$ for all i and all the entries of $D^{-1}\mathfrak{S}D$ are in $[0, 1]$. In particular, for any $i \in \mathbb{N}$ we have $iE_{i1} \in \mathfrak{S}$ and the $(i, 1)$th entry of $D^{-1}(iE_{i1})D$ is $i\frac{d_1}{d_i}$, so that $d_i \to +\infty$ as $i \to +\infty$. It follows that D is not an operator on ℓ_1.

Note that $\mathfrak{S}$ is the semigroup generated by nE_{n1} and $\frac{1}{n}E_{1n}$ for all $n \in \mathbb{N}$. If, instead, we generate $\mathfrak{S}$ by E_{11}, $\frac{1}{2}E_{12}$, $3E_{13}$, $\frac{1}{4}E_{14}$,... and $2E_{21}$, $\frac{1}{3}E_{31}$, $4E_{41}$, etc, then neither D nor D^{-1} can be chosen to be bounded so that $D^{-1}\mathfrak{S}D$ in $[0, 1]$.

Finally, we should mention that there seems to be no reasonable extension of Theorem 12 to the infinite-dimensional case because the estimate there essentially depends on the dimension.

4. The continuous case

Let K be a compact Hausdorff space, and μ be a Borel measure on K. Then $K \times K$ equipped with the product topology is also a compact Hausdorff space. As usual, $C(K)$

and $C(K \times K)$ will stand for the spaces of all real-valued continuous functions on K and $K \times K$, respectively. It is well known that these spaces are Banach lattices with respect to point-wise order. We equip $C(K \times K)$ with convolution defined via

$$(S * T)(x, y) = \int_K S(x, t)T(t, y)\, d\mu$$

for $S, T \in C(K \times K)$.

Let us introduce some terminology. Let $\mathcal{F}$ be a subset of $C(K \times K)$. We say that $\mathcal{F}$ is a convolution semigroup in $C(K \times K)$ if it is closed under convolution. We say that $\mathcal{F}$ is ***equicontinuous*** if for every $a, b \in K$ and every $\varepsilon > 0$ there exists a neighborhood V of (a, b) in $K \times K$ such that $\big|S(x, y) - S(a, b)\big| < \varepsilon$ for every $S \in \mathcal{F}$ and every pair $(x, y) \in V$. We say that $\mathcal{F}$ is ***bounded*** at some (a, b) in $K \times K$ if the set $\big\{S(a, b) \,:\, S \in \mathcal{F}\big\}$ is bounded. We write $\ker \mathcal{F}$ for the set of all pairs (x, y) such that $S(x, y) = 0$ for all $S \in \mathcal{F}$. That is, $\ker \mathcal{F}$ is the intersection of the kernels of all members of $\mathcal{F}$. It follows that $\ker \mathcal{F}$ is closed.

Theorem 23. *Suppose that μ is positive on the non-empty open subsets of K. Let $\mathfrak{S}$ be an equicontinuous convolution semigroup of non-negative functions in $C(K \times K)$ such that*

(i) $\ker \mathfrak{S}$ *contains no non-empty open sets, and*
(ii) *there exists $(a_1, a_2) \in K \times K$ such that $\mathfrak{S}$ is bounded at (a_1, a_2).*

Then $\mathfrak{S}$ is relatively norm compact in $C(K \times K)$.

Proof. Equicontinuity of $\mathfrak{S}$ implies that there exists a neighborhood U of (a_1, a_2) such that $\mathfrak{S}$ is uniformly bounded on W. Without loss of generality, $W = W_1 \times W_2$, where W_1 and W_2 are open neighborhoods of a_1 and a_2, respectively.

We claim that $\big\{S(u_1, u_2) \,:\, S \in \mathfrak{S}\big\}$ is uniformly bounded on $K \times K$. First, show that it is bounded at every point of $K \times K$. Suppose not. Then there exists a point $(u_1, u_2) \in K \times K$ and a sequence $S_n \in \mathfrak{S}$ with $S_n(u_1, u_2) \geqslant n + 1$. Since $\mathfrak{S}$ is equicontinuous, we can find a neighborhood U of (u_1, u_2) such that $S_n \geqslant n$ on U. Again, without loss of generality, $U = U_1 \times U_2$, where U_1 and U_2 are open neighborhoods of u_1 and u_2, respectively.

Since $W_1 \times U_1$ is open, it is not contained in $\ker \mathfrak{S}$, so that there exists a pair $(b_1, v_1) \in W_1 \times U_1$ and $A \in \mathfrak{S}$ such that $A(b_1, v_1) > 0$. Similarly, there exist $b_2 \in W_2$, $v_2 \in U_2$, and $B \in \mathfrak{S}$ such that $B(v_2, b_2) > 0$. Since A and B are continuous, we can find $\varepsilon > 0$ and open neighborhoods V_1 of v_1 and V_2 of v_2 such that $V_1 \subseteq U_1$, $V_2 \subseteq U_2$,

$A(b_1,\cdot)>\varepsilon$ on V_1, and $B(\cdot,b_2)>\varepsilon$ on V_2. Then

$$\begin{aligned}(A*S_n*B)(b_1,b_2)&=\int_{K\times K}A(b_1,s)S_n(s,t)B(t,b_2)\,d\mu\otimes\mu\\&\geqslant\iint_{(s,t)\in V_1\times V_2}A(b_1,s)S_n(s,t)B(t,b_2)\,d\mu\otimes\mu\\&\geqslant\varepsilon^2 n\mu(V_1)\mu(V_2)\to+\infty.\end{aligned}$$

This contradicts the fact that $\mathfrak{S}$ is bounded at (b_1,b_2) because $(b_1,b_2)\in W$. Therefore, $\mathfrak{S}$ is bounded at every point of $K\times K$.

For every point $(u_1,u_2)\in K\times K$, put $T(u_1,u_2)=\sup\bigl\{S(u_1,u_2) : S\in\mathfrak{S}\bigr\}$. By the preceding claim, $T(u_1,u_2)$ is finite. Equicontinuity of $\mathfrak{S}$ implies that T is continuous. Since $K\times K$ is compact, there exists $M>0$ such that $T(u_1,u_2)\leqslant M$ for all $(u_1,u_2)\in K\times K$. Hence $S(u_1,u_2)\leqslant M$ for all $S\in\mathfrak{S}$ and all $(u_1,u_2)\in K\times K$. The result now follows by Arzela-Ascoli's Theorem. $\square$

Clearly, condition (i) in the preceding theorem is analogous to indecomposability of $\mathfrak{S}$. Since $\ker\mathfrak{S}$ is closed, (i) is equivalent to $\ker\mathfrak{S}$ being nowhere dense. In particular, it is satisfied when $\ker\mathfrak{S}$ has zero measure. On the other hand, viewing the elements of $C(K\times K)$ as kernels of integral operators on $L_p(\mu)$ for $1\leqslant p<\infty$ (under the assumption that μ is finite), we can consider the natural embedding of $C(K\times K)$ into the space $L\bigl(L_p(\mu)\bigr)$ of all bounded operators on $L_p(\mu)$. Moreover, the corresponding integral operators are Hilbert-Schmidt. Thus, $C(K\times K)$ embeds into the space of Hilbert-Schmidt integral operators on $L_2(\mu)$ equipped with the Hilbert-Schmidt norm. Since the two embeddings are clearly continuous, Theorem 23 guarantees that $\mathfrak{S}$ is relatively compact as a subset of $L\bigl(L_p(\mu)\bigr)$ and as a subset of the space of Hilbert-Schmidt integral operators.

The following example shows that the natural analogue of Corollary 10 or Theorems 17 or 19 fails in the case of convolution semigroups, where instead of $D^{-1}SD$ we consider $S(x,y)\frac{g(y)}{g(x)}$ for some $g\in C[0,1]$.

Example 24. *A convolution semigroup $\mathfrak{S}$ of non-negative functions in $C\bigl([0,1]^2\bigr)$ such that $S(x,x)\leqslant 1$ for all $S\in\mathfrak{S}$, but there is no $g\in C[0,1]$ with $\inf g>0$ such that $S(x,y)\frac{g(y)}{g(x)}\leqslant 1$ for all $S\in\mathfrak{S}$ and all $x,y\in[0,1]^2$.*

Let $S(x,y)=\frac{3}{2}(x-y)^2$, and let $\mathfrak{S}$ consist of the convolution powers of S. Observe that if $0\leqslant z\leqslant\frac{1}{2}$ then $S(x,z)=S(z,x)\leqslant S(z,1)$ for every $x\in[0,1]$, and if $\frac{1}{2}\leqslant z\leqslant 1$ then $S(x,z)=S(z,x)\leqslant S(z,0)$ for every $x\in[0,1]$. It follows that for every $(x,y)\in$

$[0,1]^2$ we have

$$(S*S)(x,y) \leqslant \int_0^{\frac{1}{2}} S(z,1)^2\,dz + \int_{\frac{1}{2}}^1 S(z,0)^2\,dz < 1.$$

Put $E(x,y)=1$ for every $(x,y)\in[0,1]^2$. Then $S*S\leqslant E$. Also,

$$(S*E)(x,y) = \int_0^1 \tfrac{3}{2}(x-t)^2\,dt = \tfrac{1}{2}(3x^2-3x+1) \leqslant \tfrac{1}{2}$$

for all $(x,y)\in[0,1]$. Hence, $S*E\leqslant\frac{1}{2}E$. Combining this with $S*S\leqslant E$, we get that $S^n\leqslant\frac{1}{2^{n-2}}E$ for all $n\geqslant 2$, where S^n is the n-th convolution power of S. In particular, $S^n(x,x)\leqslant 1$ for all $x\in[0,1]$ and all n.

On the other hand, suppose that $g\in C[0,1]$ with $\inf g>0$. Let $\alpha=S(1,0)\frac{g(0)}{g(1)}$ and $\beta=S(0,1)\frac{g(1)}{g(0)}$. Then $\alpha\beta=S(1,0)S(0,1)=\frac{9}{4}>1$. It follows that either $\alpha>1$ or $\beta>1$.

Hence, $\mathfrak{S}$ is indeed as we claimed above. Moreover, $0\leqslant S^n\leqslant\frac{1}{2^{n-2}}E$ implies that $S^n\to 0$ in the uniform topology, hence $\mathfrak{S}\cup\{0\}$ is a compact semigroup which still has (or, rather, fails) the desired property.

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