ON $CD_0(K)$-SPACES

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ABSTRACT. We present an elementary proof of the (known) fact that a $CD_0(K)$-space is a Banach lattice and is lattice isometrically isomorphic to a particular $C(\widetilde{K})$ for some compact space $\widetilde{K}$.

$CD_0(K)$-spaces were introduced in [AW91, AW93b] and further investigated in [AW93a, AE00, Erc04]. It is known [AW93b, AE00] that a $CD_0(K)$-space is a Banach lattice and a unital AM-space. In [Erc04] it was shown that $CD_0(K)$ is lattice isometrically isomorphic to $C\bigl(K \times \{0,1\}\bigr)$ with $K \times \{0,1\}$ equipped with a compact Hausdorff topology. In this note we present elementary proofs of these facts.

Throughout these notes, K stands for a compact Hausdorff topological space without isolated points. For $x \in K$, let $\mathcal{N}_x$ be a base of open neighborhoods of x in K. As usually, for a real-valued function f on K and $x_0 \in K$ we write $\lim\limits_{x \to x_0} f(x) = r$ if for every $\varepsilon > 0$ there exists $V \in \mathcal{N}_x$ such that $\bigl|f(x) - r\bigr| < \varepsilon$ for all $x \in V \setminus \{x_0\}$. Note that this notation is not vacuous for every $x_0 \in K$ because K has no isolated points.

We denote by $C(K)$ the Banach lattice of all continuous functions on X, equipped sup-norm and point-wise ordering. Denote by $c_0(K)$ the set of all real-valued functions f on K such that the set $\bigl\{|f| > \varepsilon\bigr\} = \bigl\{x \in K : \bigl|f(x)\bigr| > \varepsilon\bigr\}$ is finite for every $\varepsilon > 0$. Clearly, $c_0(K)$ is a vector subspace of $\ell_\infty(K)$, the space of all bounded functions on K equipped with sup-norm.

Lemma 1. *$f \in c_0(K)$ iff $\lim\limits_{x \to x_0} f(x) = 0$ for every $x_0 \in K$.*

Proof. Suppose that $f \in c_0(K)$. Fix $\varepsilon > 0$. The set $\bigl\{|f| > \varepsilon\bigr\}$ is finite; since K is Hausdorff there exists $V \in \mathcal{N}_{x_0}$ such that V doesn't contain any points of this set with the possible exception of x_0 itself. Thus, $\bigl|f(x)\bigr| \leqslant \varepsilon$ for all $x \in V \setminus \{x_0\}$. Therefore, $\lim\limits_{x \to x_0} f(x) = 0$.

Suppose now that $\lim\limits_{x \to x_0} f(x) = 0$ for every $x_0 \in K$ and assume that the set $\bigl\{|f| > \varepsilon\bigr\}$ is infinite for some $\varepsilon > 0$. Since K is compact, this set must have an accumulation point x_0, which contradicts $\lim\limits_{x \to x_0} f(x) = 0$. □

Date: March 30, 2004.

Vladikavkaz Math. J., **6**(1) (2004).

Lemma 2. *$c_0(K)$ is a closed subspace of $\ell_\infty(K)$.*

Proof. Suppose that a sequence of functions (f_n) in $c_0(K)$ converges in sup-norm to $f \in \ell_\infty(K)$. Fix $\varepsilon > 0$, then $\|f_n - f\| < \varepsilon/2$ for some n. It follows that $\bigl\{|f| > \varepsilon\bigr\} \subseteq \bigl\{|f_n| > \frac{\varepsilon}{2}\bigr\}$, hence is finite. $\square$

It follows that $c_0(K)$ equipped with the sup-norm is a Banach space. Define the space $CD_0(K)$ as follows: $f \in CD_0(K)$ if $f = g + h$ for some $g \in C(K)$ and $h \in c_0(K)$. Equipped with the sup-norm, $CD_0(K)$ is a normed space, a subspace of $\ell_\infty(K)$. We also equip $CD_0(K)$ with the pointwise order. We will see that $CD_0(K)$ is a Banach lattice, and, moreover, an AM-space.

Lemma 3. *If $f \in CD_0(K)$, namely, $f = g + h$ for some $g \in C(K)$ and $h \in c_0(K)$, then $g(x_0) = \lim\limits_{x\to x_0} f(x)$ for all $x_0 \in K$.*

Proof. By Lemma 1, $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) + \lim\limits_{x\to x_0} h(x) = g(x_0)$ for every $x_0 \in K$. $\square$

It follows that every f in $CD_0(K)$ has a unique decomposition into a continuous and a discrete part. Indeed, suppose that $f = g + h = g' + h'$ where $g, g' \in C(K)$ and $h, h' \in c_0(K)$, then for every $x_0 \in K$ Lemma 3 implies $g(x_0) = \lim\limits_{x\to x_0} f(x) = g'(x_0)$. Hence, $g = g'$ and, therefore, $h = h'$. In the rest of the paper, for $f \in CD_0(K)$ we will write f_c for the continuous component of f and f_d for the discrete component. The uniqueness of the decomposition also implies that $(f+g)_c = f_c + g_c$ and $(f+g)_d = f_d + g_d$ for $f, g \in CD_0(K)$ because $f + g = f_c + f_d + g_c + g_d = (f_c + g_c) + (f_d + g_d)$, and $f_c + g_c \in C(K)$ while $f_d + g_d \in c_0(K)$.

Proposition 4. *If $\lim\limits_{x\to x_0} f(x)$ exists for every $x_0 \in K$ then $f \in CD_0(K)$. In this case $f_c(x_0) = \lim\limits_{x\to x_0} f(x)$.*

Proof. For every $x_0 \in K$, put $g(x_0) = \lim\limits_{x\to x_0} f(x)$, and let $h = f - g$. Then $\lim\limits_{x\to x_0} h(x) = 0$ for every $x_0 \in K$, so that $h \in c_0(K)$ by Lemma 1. It remains to show that $g \in C(K)$. Fix $x_0 \in K$ and $\varepsilon > 0$, there exists $V \in \mathcal{N}_{x_0}$ such that $\bigl|f(x) - g(x_0)\bigr| < \varepsilon$ for all $x \in V \setminus \{x_0\}$. It follows that for every $y \in V$ we have

$$\bigl|g(y) - g(x_0)\bigr| = \bigl|\lim_{x\to y} f(x) - g(x_0)\bigr| < \varepsilon.$$

$\square$

Combining Lemma 3 and Proposition 4 we get the following result.

Corollary 5. *$f \in CD_0(K)$ if and only if $\lim\limits_{x\to x_0} f(x)$ exists for every $x_0 \in K$.*

Lemma 6. *For every $f \in CD_0(K)$ we have $\|f_c\| \leqslant \|f\| \leqslant \|f_c\| + \|f_d\|$.*

Proof. The first inequality follows from Proposition 4 while the second inequality is just the triangle inequality. □

Corollary 7. *$CD_0(K)$ is a Banach space.*

Proof. Suppose that a sequence (f_n) is Cauchy in $CD_0(K)$. It follows from Lemma 6 that the sequence of the continuous parts $(f_n)_c$ is Cauchy, and, therefore, the sequence of discrete parts $(f_n)_d$ is Cauchy. Since $C(K)$ and $c_0(K)$ are complete, $(f_n)_c$ converges to some $g \in C(K)$ and $(f_n)_d$ converges to some $h \in c_0(K)$. Hence (f_n) converges to $g+h$, which belongs to $CD_0(K)$. □

Next, we show that for this topology $CD_0(K)$ is order isometric to $C\big(K \times \{0,1\}\big)$, if the topology on $K \times \{0,1\}$ is defined as follows. We put discrete topology on $K \times \{1\}$, that is, we put $\mathcal{N}_{(x,1)} = \{(x,1)\}$ for each $x \in K$. Then all the points of $K \times \{1\}$ are isolated points of $K \times \{0,1\}$. For a point $(x,0)$ in $K \times \{0\}$ we take the basic open neighborhoods to be of the form $\widetilde{V} = \big(V \times \{0,1\}\big) \setminus \{(x,1)\}$, where $V \in \mathcal{N}_x$. One can easily verify that these sets indeed form a base of a Hausdorff topology. From now on we consider $K \times \{0,1\}$ equipped with this topology.

One can easily see that $K \times \{0\}$ is a closed subspace of $K \times \{0,1\}$, and the map $x \mapsto (x,0)$ is a homeomorphism between K and $K \times \{0\}$. It the future we will often identify $K \times \{0\}$ and K.

Lemma 8. *$K \times \{0,1\}$ is compact.*

Proof. Consider an open cover of $K \times \{0,1\}$. By replacing each set in the cover by a union of basic open neighborhoods of all the points in the set, we can assume that the cover is formed by basic open neighborhoods. Hence, the cover is of the form

$$\Big\{\big\{(x_\alpha,1)\big\}\Big\}_{\alpha\in\Lambda} \cup \big\{\widetilde{V}_\gamma\big\}_{\gamma\in\Gamma},$$

where $x_\alpha \in K$ and $V_\gamma \in \mathcal{N}_{x_\gamma}$ for some $x_\gamma \in K$. It is easy to see that $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of K, so that there is a finite sub-cover $V_1,\dots,V_n$. But then $\widetilde{V_1} \cup \dots \cup \widetilde{V_n}$ only misses finitely many points of $K \times \{0,1\}$, so that if we add the corresponding open sets from the original cover then we obtain a finite cover of the entire $K \times \{0,1\}$. □

Theorem 9. *$CD_0(K)$ is lattice isometrically isomorphic to $C\big(K \times \{0,1\}\big)$. In particular, $CD_0(K)$ is an AM-space.*

Proof. Define $T\colon CD_0(K)\to C\bigl(K\times\{0,1\}\bigr)$ via $(Tf)(x,r)=f_c(x)+rf_d(x)$. In other words, Tf agrees with f on $K\times\{1\}$ and with f_c on $K\times\{0\}$. It follows immediately that T is an isometry. It is obvious that $Tf\geqslant 0$ implies $f\geqslant 0$. On the other hand, if $f\geqslant 0$ then $f_c\geqslant 0$ by Proposition 4.

Observe that Tf is indeed a continuous function. Clearly, Tf is continuous on $K\times\{1\}$, as the later set consists of isolated points. Finally, it is left to show that $\lim\limits_{(x,r)\to(x_0,0)}(Tf)(x,r)=(Tf)(x_0,0)$ for every $x_0\in K$. Observe that $(x,r)\to(x_0,0)$ in $K\times\{0,1\}$ implies that $x\to x_0$ in K, so that $f_c(x)\to f_c(x_0)$ and $f_d(x)\to 0$ by Lemma 1. It follows that $(Tf)(x,r)=f_c(x)+rf_d(x)\to f_c(x_0)=(Tf)(x_0,0)$.

Show that T is onto. Let $F\in C\bigl(K\times\{0,1\}\bigr)$. For every $x\in K$ define $f(x)=F(x,1)$. Fix $x_0\in K$ and $\varepsilon>0$, there exists $V\in\mathcal{N}_{x_0}$ such that $\bigl|F(x,r)-F(x_0,0)\bigr|<\varepsilon$ for all $(x,r)\in\widetilde{V}$. In particular, for every $x\in V\setminus\{x_0\}$ we have $\bigl|f(x)-F(x_0,0)\bigr|=\bigl|F(x,1)-F(x_0,0)\bigr|<\varepsilon$, so that $\lim\limits_{x\to x_0}f(x)=F(x_0,0)$. It follows from Lemma 4 that $f\in CD_0(K)$ and $f_c(x)=F(x,0)$ for all $x\in K$, so that $F=Tf$. $\square$

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