University of Alberta
Mathematical and Statistical Sciences
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Math 667, Topics in Differential Equations
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Assignment 3, due March 07, 2005, 9 AM

Exercise 9: (2)

Find the spectrum and the corresponding eigenfunctions in $L^2([0,\pi])$ for the operators A and B with

$$A = -\frac{d^2}{dx^2}, \qquad \mathcal{D}(A) = \{f \in L^2(0,\pi); Af \in L^2(0,\pi), f(0)=0, f(\pi)=0\},$$
$$B = -\frac{d^2}{dx^2}, \qquad \mathcal{D}(B) = \{f \in L^2(0,\pi); Bf \in L^2(0,\pi), \frac{d}{dx}f(0)=0, \frac{d}{dx}f(\pi)=0\}.$$

Exercise 10: (2)

Let A be a symmetric linear operator on a Hilbert space H with $R(A) = H$ and with compact inverse A^{-1}. The natural domain of definition is

$$\mathcal{D}(A) = \{u \in H; Au \in H\}$$

Show that there exists an orthonormal basis $\{w_j\}$ of H such that

$$\mathcal{D}(A) = \left\{u; u = \sum c_j w_j, \qquad \sum |c_j|^2 \lambda_j^2 < \infty\right\}.$$

Exercise 11: (3)

For $A = -\Delta$ show that on a bounded domain the norm on $\mathcal{D}(A^{\frac{1}{2}})$ is equivalent to the norm on H^1. (Hint: If A is symmetric, $(Au,v) = (u,Av)$, then also $A^{\frac{1}{2}}$).

Exercise 12: (3)

Show that if Ω is bounded, then $L^2(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$.

Exercise 13: (3)

Let V be the subspace of $H^1(\Omega)$, consisting of functions with zero integral

$$V := \left\{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\right\}.$$

Arguing by contradiction, use Rellich-Kondrachov compactness to show that there exists a constant $C > 0$ such that we have a Poincaré inequality

$$\|u\|_2 \leq C\|\nabla u\|_2.$$

Exercise 14: (The Nagumo Model) (7)

FitzHugh and Nagumo derived a model for signal transduction in the axon:

$$u_t = u_{xx} + u(1-u)\,(u-\alpha)\,,$$

where u represents the membrane potential, and $0 < \alpha < 1$. We study this model on $0 \geq x \geq l$ with homogeneous Neumann boundary conditions,

$$u_x(t,0) = 0, \qquad u_x(t,l) = 0.$$

(a) Find a system of two ODEs which describe the steady states. We denote this system of ODE's by (*) for now.

(b) Study the steady states of (*) and their stability.

(c) Show that

$$H(u,v) = \frac{1}{2}(u_x)^2 - \frac{1}{4}u^4 + \frac{1+\alpha}{3}u^3 - \frac{\alpha}{2}u^2$$

is a Hamiltonian function for the system (*).

(d) Sketch the phase portrait of (*) in the (u, u_x)-plane. Distinguish the qualitatively different cases depending on $\alpha \in (0,1)$.

(e) Find the steady states which satisfy the Neumann boundary conditions, and plot the steady states u as a function of x.