University of Alberta
Mathematical and Statistical Sciences
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Math 667, Topics in Differential Equations
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Assignment 2, due Mo. Feb. 14, 2005, 9 AM

Exercise 5: (All those functions) (5)

1. Find a function $f \in C_c^\infty(\mathbb{R})$ with supp$f \subset [a,b]$, where $a < b \in \mathbb{R}$.
2. Let $\Omega \subset \mathbb{R}^n$ be bounded. Show that if $f \in L^2(\Omega)$ then it follows that $f \in L^1(\Omega)$.
3. If Ω is unbounded the above statement is not true. Show that $\rho(x) = \frac{1}{1+x}$ is contained in $L^2([0,\infty))$ but not in $L^1([0,\infty))$.
4. Show that $\rho(x) = e^{-x}x^{-\frac{2}{3}}$ is contained in $L^1([0,\infty))$ but not in $L^2([0,\infty))$.
5. Find a value $\gamma^* \in [0,1]$ such that the function $f(x) = x^{\frac{3}{2}}$ is element of the Hölder space $C^{1,\gamma}([-1,1])$ for $\gamma \le \gamma^*$ and $f(x)$ is not contained in $C^{1,\gamma}([-1,1])$ for $\gamma > \gamma^*$.

Exercise 6: (Mollifier) (4)

1. The mollification of a function can be written in two ways. Show that
$$\frac{1}{h^n}\int_{\mathbb{R}^n} \rho\left(\frac{x-z}{h}\right)u(z)dz = \frac{1}{h^n}\int_{\mathbb{R}^n} \rho\left(\frac{z}{h}\right)u(x-z)dz$$
2. Assume a Lipschitz continuous function $u \in C^{0,1}(\mathbb{R}^n)$ is uniformly Lipschitz continuous with constant K:
$$|u(x) - u(y)| \le K|x-y|.$$
Show that each mollification $u_h = \rho_h * u$ is uniformly Lipschitz continuous with the same constant K.

Exercise 7: (Interpolation Inequality) (3)

Use Hölder's inequality to show the *interpolation inequality*: Assume $1 \le p \le q \le r < \infty$ and consider $\lambda \in (0,1)$ such that $\frac{1}{q} = \lambda\frac{1}{p} + (1-\lambda)\frac{1}{r}$. Show
$$\|u\|_{L^q} \le \|u\|_{L^p}^{\lambda}\|u\|_{L^r}^{(1-\lambda)}.$$

Exercise 8: (Energy Method) (8)

We use the *energy method* to show that all solutions of the following reaction-diffusion equation approach 0 as $t \to \infty$: On $\Omega = [0,1]$ we consider
$$\begin{aligned} u_t &= 2u_{xx} - 3u \\ u_x(0,t) = 0, &\qquad u_x(1,t) = 0, \end{aligned}$$
where lower case indices denote the partial derivative with respect to that variable, e.g. $u_t = \frac{\partial}{\partial t}u(x,t)$.

1. Use Young's inequality (with $p = q = 2$) to show that the *energy*
$$E[u,u_x](t) := \frac{1}{2}\int_0^1 (|u|^2 + |u_x|^2)dx$$
satsfies the differential inequality
$$\frac{\partial}{\partial t}E(t) \le -\frac{1}{2}\int_0^1 |u_{xx}|^2dx - E(t).$$
2. Use Gronwall's inequality to show that $E(t) \to 0$ as $t \to \infty$.
3. Argue that $u(t)$ converges in $H^1([0,1])$ to 0 as $t \to \infty$. Does $u(t)$ also converge in $C^0([0,1])$?